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A pion decay constant in the multi-flavor Schwinger model

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The pion decay constant F_π plays an important role in QCD and in Chiral Perturbation Theory. It is hardly known, however, that a corresponding constant exists in the Schwinger model with $N_\mathrm{f} \geq 2$ degenerate fermion flavors. In this case, the "pion" does not decay and F_π is dimensionless. Still, F_π can be defined by 2d analogies to the Gell-Mann–Oakes–Renner relation, the Witten–Veneziano formula and the residual "pion" mass in the δ-regime. With suitable assumptions, and by inserting simulation data, these QCD-inspired relations are all compatible with $F_\pi \simeq 1/\sqrt{2\pi}$ at zero fermion mass, as we observe for $N_\mathrm{f} = 2, \ldots, 6$. We conclude that this is a meaningful constant in the multi-flavor Schwinger model.

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1. The multi-flavor Schwinger model

The Schwinger model [1], or 2d QED, shares important qualitative features with QCD, in particular confinement [2], chiral symmetry (breaking) and a topological structure of the gauge configurations. We are going to apply several aspects of the analogy between 2d QED and 4d QCD to define a "pion decay constant" F_π in the Schwinger model with $N_\mathrm{f} \geq 2$ degenerate fermion flavors.

Most analytic treatments are based on bosonization, which — for fermion mass $m = 0$ — leads to a boson with mass $M_\eta = g\sqrt{N_\mathrm{f}/\pi}$ (where g is the gauge coupling), plus $N_\mathrm{f} - 1$ massless "pions", see *e.g.* Refs. [3]. Other works, however, such as Ref. [4], assume $N_\mathrm{f}^2 - 1$ "pions", which matches the number of Nambu-Goldstone bosons in higher dimensions, when the chiral symmetry breaks spontaneously.

In this framework, we are interested in a quantity, which can be defined as a "pion decay constant" F_π, by invoking three different analogies to 4d QCD (although the "pion" in the Schwinger model does not decay). To the best of our knowledge, the only work which studied this constant before was carried out for $N_\mathrm{f} = 2$ with a light-cone formulation [5]. Referring to the divergence of the axial current J_μ^5, $\langle 0|\partial^\mu J_\mu^5(0)|\pi(p)\rangle$, that study obtained

$$F_\pi(m) = 0.394518(14) + 0.040(1)\, m/g \; . \tag{1}$$

Our results, to be summarized in the continuation, were presented in detail in Ref. [6], see also Refs. [7].

2. The 2d Gell-Mann–Oakes–Renner relation

In QCD, the Gell-Mann–Oakes–Renner relation is well-known [8],

$$F_\pi^2(m) = \frac{2m\Sigma}{M_\pi^2} \; , \tag{2}$$

where Σ is the chiral condensate. If we assume the same relation to hold in the multi-flavor Schwinger model, and combine it with the relation $\Sigma = M_\pi^2/4\pi m$ [9], we immediately arrive at

$$F_\pi = \frac{1}{\sqrt{2\pi}} \simeq 0.3989 \; . \tag{3}$$

without any mass-dependence.

Alternatively, we can numerically measure the terms in the Gell-Mann–Oakes–Renner relations. We do so on 24×24 lattices, by using overlap-hypercube fermions [10], which are treated by re-weighting quenched configurations, for $N_\mathrm{f} = 2, \dots, 6$. The results at $\beta = 4$ and 6 are very similar, as Figure 1 shows. We truncate the results at $m \leq 0.05$, to avoid strong finite-size effects, but we see that the chiral extrapolation $m \to 0$ is in all cases compatible with $F_\pi(0) \approx 0.4$.

3. Witten–Veneziano formula in the Schwinger model

According to Seiler and Stamatescu [11], the famous Witten–Veneziano formula [12]

$$M_\eta^2 = \frac{2N_\mathrm{f}}{F_\eta^2}\chi_\mathrm{t}^\mathrm{q} \tag{4}$$

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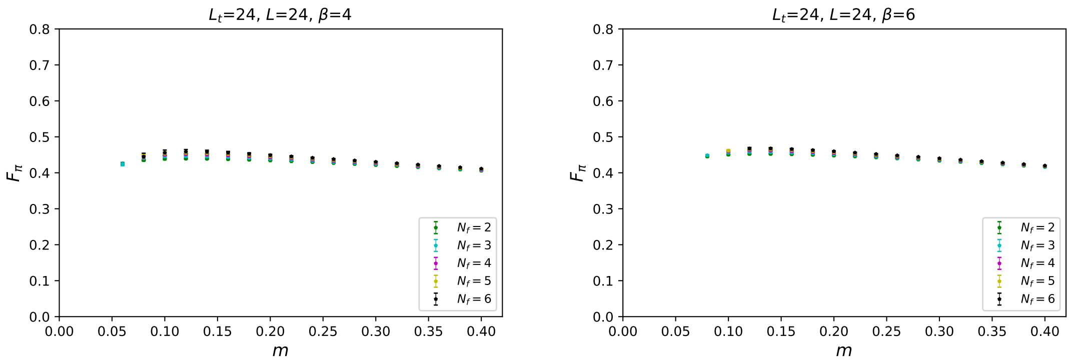

Figure 1: Results for $F_\pi(m)$ based on the Gell-Mann–Oakes–Renner relation (2). We are using overlap-hypercube fermions, so we can insert for m the bare fermion mass, while M_π and Σ are measured with quenched re-weighted configurations (for Σ we use the Dirac spectrum). The results at $\beta = 4$ (left) and $\beta = 6$ (right) are very similar, which shows that lattice artifacts are mild. Finite-size effects could be an issue, for this reason we exclude tiny fermion masses m. Still, we see that $F_\pi(m \to 0) \approx 0.4$.

is on particularly solid grounds in the multi-flavor Schwinger model at $m = 0$. M_η is given in Section 1, and the quenched topological susceptibility (in infinite volume) amounts to $\chi_t^q = g^2/4\pi^2$ [11], which is confirmed analytically and numerically by the continuum limits of different lattice formulations [6, 13]. This leads to the η-decay constant $F_\eta = 1/\sqrt{2\pi}$.

In large-N_c QCD, F_π and F_η coincide asymptotically. If we drive the analogy further and assume the same equivalence in the multi-flavor Schwinger model, we obtain

$$F_\pi(m = 0) = \frac{1}{\sqrt{2\pi}} , \tag{5}$$

in agreement with eq. (3), though here with the limitation to the chiral limit.

4. The 2d δ-regime

One formulation of Chiral Perturbation Theory refers to the δ-regime, *i.e.* to an anisotropic space-time volume with $L_t \gg L \approx M_\pi^{-1}$ [14]. Thus the system is quasi-1-dimensional and it can be approximated by a quantum mechanical rotor. In the chiral limit of zero quark masses, there is still a residual pion mass M_π^R, as a finite-size effect,

$$M_\pi^R = \frac{N_\pi}{2\Theta_{\rm eff}} , \tag{6}$$

where N_π is the number of pions (3 in QCD), and $\Theta_{\rm eff}$ is an effective moment of inertia. To leading order (LO), Leutwyler computed $\Theta_{\rm eff} = F_\pi^2 L^3$ [14]. For numerical studies with lattice QCD, we refer to Ref. [15]. Hasenfratz and Niedermayer extended this calculation to next-to-leading order (NLO) of an O(N) model in $d > 2$ [16],

$$\Theta_{\rm eff} = F_\pi^2 L^{d-1} \Big[1 + \frac{N_\pi - 1}{2\pi F_\pi^2 L^{d-2}} \Big(\frac{d-1}{d-2} + \dots \Big) + \dots \Big] . \tag{7}$$

They assumed spontaneous symmetry breaking $O(N) \to O(N-1)$, and therefore $N_\pi = N - 1$.

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In $d = 2$ this does not happen, and the NLO correction would diverge, so we can only conjecture that the LO remains applicable. If a numerical study in the multi-flavor Schwinger model confirms the behavior $M_\pi^{\rm R} \propto 1/L$, then eq. (6) provides another result for F_π.

We performed such simulations in two settings, with 10^4 configurations for each parameter set:

- Dynamical Wilson fermions, using the HMC algorithm, with $N_{\rm f} = 2$, $L_t = 64$, $L = 6, \dots, 12$ and $\beta \equiv 1/g^2 = 3$, 4 and 5.
- Overlap-hypercube fermions, with quenched configurations and re-weighting for $N_{\rm f} = 2, \dots, 6$, with $L_t = 32$ and $L = 4, \dots, 12$, at $\beta = 4$.

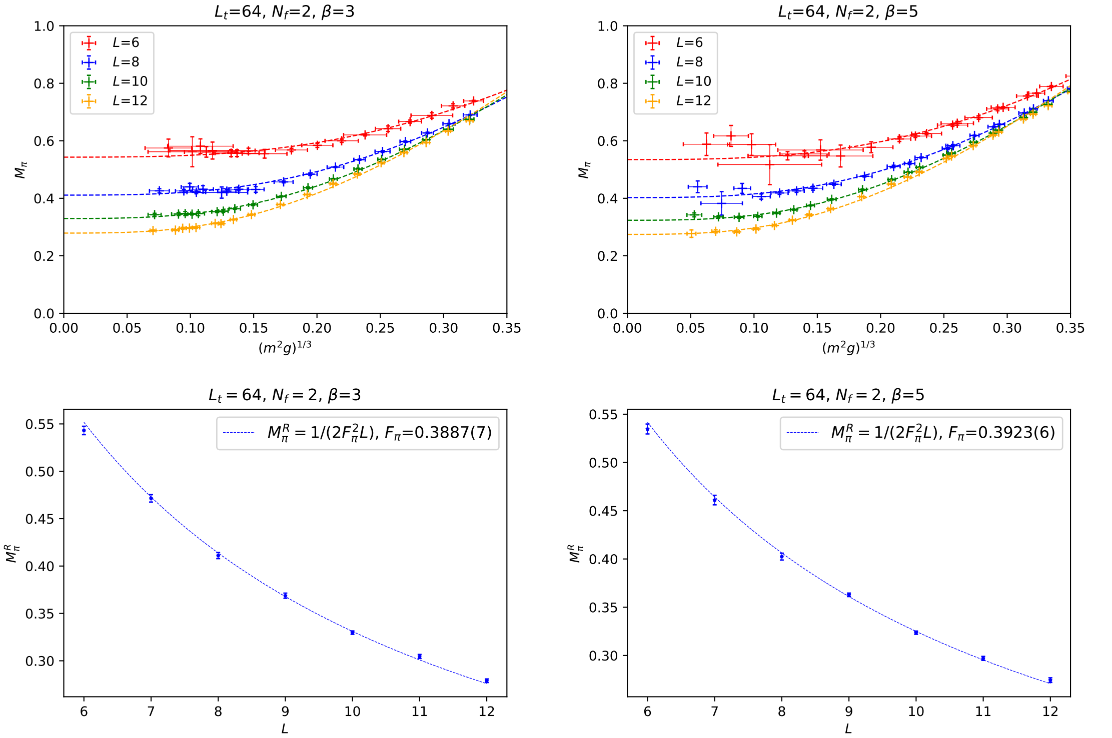

Figure 2: Results for M_π from simulations in the δ-regime, with Wilson fermions and $N_{\rm f} = 2$, at $\beta = 3$ and 5. Above we show M_π as a function of $(m^2 g)^{1/3}$, with the extrapolations to $M_\pi^{\rm R} = M_\pi(m = 0)$. The plots below confirm the behavior $M_\pi^{\rm R} \propto 1/L$, and the resulting values for $F_\pi(m = 0)$ are again close to $1/\sqrt{2\pi}$.

Figure 2 summarizes our results for $N_{\rm f} = 2$ with Wilson fermions. The plots above show M_π as a function of the parameter $(m^2 g)^{1/3}$ (in lattice units), at $\beta \equiv 1/g^2 = 3$ and 5. The (degenerate) fermion mass m is measured based on the PCAC relation, as it was done previously *e.g.* in Ref. [17]. As expected for Wilson fermions, the uncertainty becomes significant at small values of m, but there are smooth extrapolations to $M_\pi^{\rm R} = M_\pi(m = 0)$.

The plots below show how this residual pion mass $M_\pi^{\rm R}$ depends on L. The relation $M_\pi^{\rm R} \propto 1/L$ is well confirmed, which allows us to extract a value for F_π according to eq. (6). Here we insert $N_\pi = 1$, which matches the formula used in bosonization studies, as we mentioned in Section 1. This leads to results for $F_\pi(m = 0)$ which are again close to eq. (5). Our results obtained in this manner for $\beta = 3$, 4 and 5 are given in Table 1. They show that lattice artifacts are small, and they provide a picture of the chiral limit, which is consistent with Sections 2 and 3.

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$\beta = 1/g^2$	3	4	5
F_π	0.3887(7)	0.3877(11)	0.3923(6)

Table 1: Results for F_π, obtained from simulations of two flavors of Wilson fermions in the δ-regime, by fits to eq. (6), with $N_\pi = 1$, at three values of β.

We now extend our study in the δ-regime to $N_\mathrm{f} = 2, \dots, 6$ degenerate fermion flavors. Here we use quenched configurations, generated at $\beta = 4$, which are re-weighted with the fermion determinant which corresponds to overlap-hypercube fermions.

Figure 3 shows, for $N_\mathrm{f} = 2$, the "pion" mass $M_\pi((m^2 g)^{1/3})$ (again we can use the bare mass m thanks to the chiral symmetry of Ginsparg-Wilson fermions) and $M_\pi^\mathrm{R}(L)$, as in Figure 2.

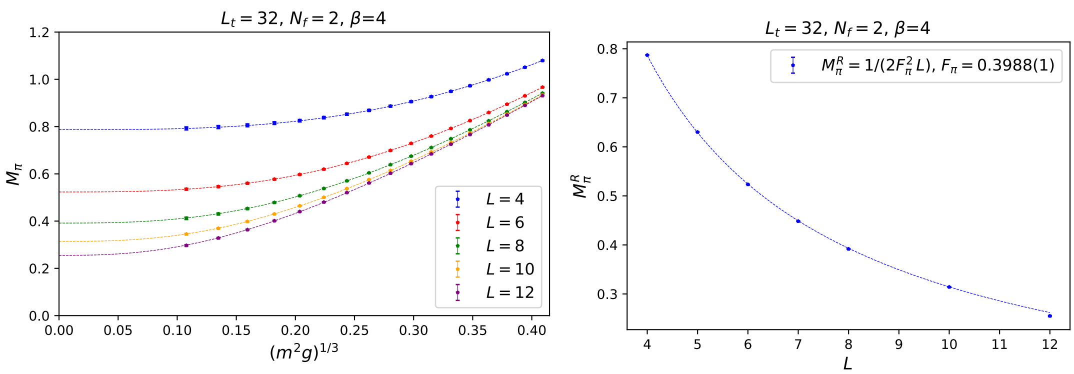

Figure 3: Results for M_π from simulations in the δ-regime, with overlap-hypercube fermions and $N_\mathrm{f} = 2$, at $\beta = 4$. On the left we show M_π as a function of $(m^2 g)^{1/3}$, with the extrapolations to $M_\pi^\mathrm{R} = M_\pi(m = 0)$. The plot on the right confirms the behavior $M_\pi^\mathrm{R} \propto 1/L$, and the resulting value for $F_\pi(m = 0)$ is once more close to $1/\sqrt{2\pi}$.

Finally, Figure 4 presents the corresponding results if the re-weighting is performed for $N_\mathrm{f} = 2, \dots, 6$. We see that increasing N_f reduces the range in L where the proportionality relation $M_\pi^\mathrm{R} \propto 1/L$ is well approximated. We perform fits within this range for each N_f.

Now the question is what value for N_π should be inserted in eq. (6). The bosonization formula $N_\pi = N_\mathrm{f} - 1$ fails for $N_\mathrm{f} > 2$. However, we obtain values for $F_\pi(m = 0)$ which are well consistent with the previous considerations, if we apply the effective formula

$$N_\pi = \frac{N_\mathrm{f} - 1}{N_\mathrm{f}} , \tag{8}$$

although these values for N_π are non-integers for $N_\mathrm{f} > 2$. The results for $F_\pi(m = 0)$ are displayed inside Figure 4.

5. Conclusions

We have attracted attention to a constant, which plays an important role in the multi-flavor Schwinger model (with $N_\mathrm{f} \geq 2$), but which has been ignored in the literature, with the exception of the light-cone study in Ref. [5].

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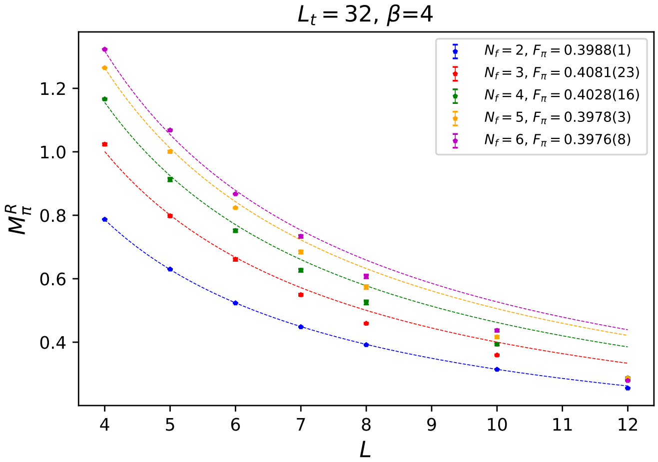

Figure 4: Results for M_π from simulations in the δ-regime, with overlap-hypercube fermions and $N_\mathrm{f} = 2, \ldots, 6$, at $\beta = 4$. We see that increasing N_f reduces the range where the relation $M_\pi^\mathrm{R} \propto 1/L$ is well approximated. Fits within this range, along with the application of the effective formula (8), lead to values for $F_\pi(m = 0)$, which are again close to $1/\sqrt{2\pi}$.

In several respects, this constant is analogous to the pion decay constant in QCD and Chiral Perturbation Theory, hence we denote it as F_π. It is, however, dimensionless in $d = 2$.

We presented three ways to introduce this constant, which all involve some analogy between the Schwinger model and QCD. They refer to the Gell-Mann–Oakes–Renner relation, to the Witten–Veneziano formula and to the residual pion mass in the δ-regime of a small spatial box, but a large extent in Euclidean time. For details, we refer to Ref. [6].

All three considerations lead to results close to $F_\pi(m = 0) = 1/\sqrt{2\pi} \simeq 0.3989$, which is also close to the value obtained in Ref. [5]. We conclude that this constant is physically significant. Further aspects of its meaning remain to be explored.

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