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Spin-taste structure of minimally doubled fermions

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Minimally doubled fermions realize one degenerate pair of Dirac fermions on the lattice. Similarities to staggered fermions exist, namely, spin and taste degrees of freedom become intertwined, and a remnant, non-singlet chiral symmetry and ultralocality are maintained. However, charge conjugation, isotropy and some space-time reflection symmetries are broken by the cutoff. For two variants, i.e., Karsten-Wilczek (KW) or Borici-Creutz (BC) fermions, a tasted charge conjugation symmetry can be identified, and the respective representations of the spin-taste algebra can be constructed explicitly. In the case of BC fermions, the tasted symmetry indicates that amendments to the published counterterms are necessary. The spin-taste representation on the quark level permits construction of local or extended hadron interpolating operators for any spin-taste combination, albeit with contamination by parity partners and taste-symmetry violation. The few available numerical results for KW fermions are in line with expectations.

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1. Introduction

Minimally doubled fermions (MDF) realize a specific taste-vector chiral symmetry at finite lattice cutoff with a strictly local operator and the minimal number of Dirac fermions, namely two, for arbitrary even dimension D in compliance with the Nielsen-Ninomiya No-Go theorem [1]. On the one hand, the Karsten-Wilczek (KW) variant, which has its tastes located on one axis of the Brillouin zone, has been proposed in the early 80s [2, 3]. On the other hand, the Borici-Creutz (BC) variant has its tastes located on a hypercubic diagonal and has been proposed in the late 2000s [4, 5]. Both BC or KW variants have been shown (for $D = 2$) to perceive the gauge field topology correctly [6] and yield renormalizable QFTs that require analogous patterns of counterterms, which are known in perturbation theory [7, 8], or–in some cases–non-perturbatively [9, 10][1]. Further variants going by the names of twisted-ordering or dropped twisted-ordering exist [11], but become indistinguishable from BC or KW variants for $D = 2$ [12], respectively.

The construction of MDF is similar to Wilson fermions insofar as operators of lower mass dimension are added to the naive fermion operator. At the tree-level, these cancel at their leading mass dimension for a subset of tastes that play the role of surviving Dirac fermions with opposite chirality in the continuum limit. The necessary fine-tuning in the interacting theory gives rise to the relevant counterterm. For MDF such extra operators are γ_5 hermitian products of γ matrices and discretized Laplacian components, times an r parameter (some restrictions apply to realize minimal doubling). Thus, symmetries under charge conjugation and some subset of space-time reflections are broken, while the symmetry under the product $\mathcal{CPT}$ remains intact. For the KW variant one may preserve the symmetry under either time reflection or parity [13], while neither can be preserved for the BC variant [14]. In order to understand the spin-taste (ST) structure, one might try ST diagonalization that would reduce naive fermions to $2^{D/2}$ identical copies of Kogut-Susskind (KS) fermions [15]. However, this does not produce a diagonal subgroup when applied to BC or KW variants, but instead identical ST structures in the two separate taste $U(2^{D/2})$ groups [16]. Since the ST structures differ between the single-site and site-split contributions to the Laplacian terms, ST diagonalization is generally impossible for MDF. While ad-hoc site-splitting prescriptions could approximately filter the two tastes, e.g. as suggested for KW fermions [17] or BC fermions [18], they cannot accommodate the two different non-trivial ST structures in each of the two variants, and cannot bring forth the exact ST representations of $\mathfrak{su}(2)$ for these discretizations. In the interacting theory the taste symmetry is approximate, and the eigenvalues arrange into taste multiplets [19].

Fortunately, on the one hand, the ST structure of the KW variant is straightforward to derive from first principles, and emerges naturally in the chiral representation [20]. In Sec. 2, we revisit the KW representation of the taste $\mathfrak{su}(2)$ algebra, and derive a ST basis of meson-interpolating operators as our first new result. We relate these to published results on spin-zero meson correlation functions [21]. Unfortunately, on the other hand, the ST structure of the BC variant is not straightforward at all. Building on the same ideas we can construct a ST representation for the BC variant, too, if the corresponding r parameter is appropriately restricted. This derivation is our second new result and discussed in Sec. 3, where we correct the published result for the marginal fermionic counterterm, our third new result. We close in Sec. 4 with a comment on extensions

[1]Non-perturbative tuning of dynamical KW fermions or in the valence sector has been discussed at this conference.

of these ST representations to different r or to twisted-ordering operators, and an outlook towards phenomenologically relevant applications of MDF.

2. Karsten-Wilczek fermions

The standard KW action (in the conventions and notation of [12]) reads

$$S_{\mathrm{KW}}[\psi,\bar{\psi}] = a^D \sum_{n,m\in\Lambda} \bar{\psi}(n)\Big[D_{\mathrm{nai}}[U] \;+\; m_0 \;-\; \frac{ra}{2}\mathrm{i}\gamma_D \sum_{j=1}^{D-1} \Delta_j[U]\Big](n,m)\,\psi(m)\,, \tag{1}$$

where we define the naive Dirac operator and discretized Laplacian (components) with gauge-covariant translation operators $t_{\pm\mu}[U](n,m) \equiv U_{\pm\mu}(n)\delta(n\pm\hat{\mu},m)$, $U_{-\mu}(n) \equiv U_\mu^\dagger(n-\hat{\mu})$ as

$$D_{\mathrm{nai}}[U](n,m) \equiv \sum_{\mu=1}^{D} \gamma_\mu \frac{s_{+\mu}[U](n,m)}{a} = \sum_{\mu=1}^{D} \gamma_\mu \frac{t_{+\mu}[U](n,m) - t_{-\mu}[U](n,m)}{2a}\,, \tag{2}$$

$$\Delta_\mu[U](n,m) \equiv \frac{2c_\mu[U](n,m) - 2\delta(n,m)}{a^2} = \frac{t_{+\mu}[U](n,m) + t_{-\mu}[U](n,m) - 2\delta(n,m)}{a^2}\,. \tag{3}$$

The KW action as described in Eq. (1) is obviously invariant under discrete translations (for periodic boundary conditions), discrete spatial rotation-reflections (W_3, and in particular, parity $\mathcal{P}$), and the product of (Euclidean) time reflection ($\mathcal{T}$) and charge conjugation (C). Furthermore, it is γ_5 hermitian and satisfies a chiral symmetry with (unmodified) γ_5 for $m_0 \to 0$. Anisotropy between temporal (D) and spatial ($1 \le j \le D-1$) directions is a consequence of the individually broken symmetry under $\mathcal{T}$ that maps the r-parameter as $r \to -r$. For $r^2 > 1/4$, the KW action is minimally doubled with the survivors located at $ak_D = 0, \pi$ and $ak_j = 0$ in the Brillouin zone. Thus, it seems natural to define a KW fermion hyper-site as two boson sites one step apart in the D-direction. Necessarily, a translation $t_{\pm D}[U](n,m)$ within that hyper-site is a ST vector transform; thus, any translation by an odd number of steps in the D-direction must be a ST vector transform, too.

The KW action inherits the $\mu = D$ mirror fermion symmetry [13] of the naive operator in Eq. (2), which is a symmetry under $ak_\mu \to \pi - ak_\mu$ in momentum space. In position space this is understood as being due to the product of a (Euclidean) reflection of the μ-direction ($\mathcal{R}_\mu$), where $\mathcal{R}_D = \mathcal{T}$ is just (Euclidean) time reflection, and a ST rotation ($\tau_\mu \equiv \tau_\mu(n) = \tau_\mu^\dagger(n) = \gamma_{\mu 5}(-1)^{n_\mu}$, with hermitian matrices $\gamma_{\mu\nu} \equiv \mathrm{i}\gamma_\mu\gamma_\nu$ for $1 \le \mu < \nu \le 5$)

$$\mathcal{R}_\mu\psi(n_\nu,n_\mu) = \gamma_\mu\gamma_5\psi(n_\nu,-n_\mu)\,, \qquad \bar{\psi}(n_\nu,n_\mu)\mathcal{R}_\mu^\dagger = \bar{\psi}(n_\nu,-n_\mu)\gamma_5\gamma_\mu\,, \tag{4}$$

$$\tau_\mu(n)\psi(n_\nu,n_\mu) = \gamma_{\mu 5}(-1)^{n_\mu}\psi(n_\nu,n_\mu)\,, \qquad \bar{\psi}(n_\nu,n_\mu)\tau_\mu^\dagger(n) = \bar{\psi}(n_\nu,n_\mu)(-1)^{n_\mu}\gamma_{\mu 5}\,, \tag{5}$$

where we have omitted the standard transformation of the gauge fields under $\mathcal{R}_\mu$; see e.g. [22]. Obviously, this mirror fermion symmetry (under $\tau_D\mathcal{T}$) implies that the ST rotation τ_D flips the sign of the r parameter, too, such that the KW action has a ST charge conjugation symmetry. There is a KW representation of the generators $\{\rho_i(n)\} \equiv \{\rho[\sigma_i]\}$ of the $\mathfrak{su}(2)$ algebra, namely

$[\rho_i, \rho_j] = 2\mathrm{i}\epsilon_{ijk}\rho_k$, yielding well-defined transformation patterns for all parts of Eq. (1), commuting with the continuum limit of $D_{\mathrm{nai}}[U](n,m)$, and with $\tau_D(n)$ as a generator. Specifically[2]

$$\begin{aligned} \rho_1(n) &= \gamma_D(-1)^{p_S(n)} , \\ \rho_2(n) &= \mathrm{i}\gamma_D\gamma_5(-1)^{p_D(n)} \equiv \tau_D(n) , \\ \rho_3(n) &= \gamma_5(-1)^{p(n)} = \gamma_5\epsilon(n) \equiv \tau_5(n) , \end{aligned} \tag{6}$$

where we have introduced site parities restricted to spatial, temporal, or all components ($p_S(n) = (n_1 + \ldots + n_{D-1}) \mod 2$, $p_D(n) = n_D \mod 2$, $p(n) = (n_1 + \ldots + n_D) \mod 2$). The staggered $\epsilon(n) = (-1)^{p(n)}$ maps between survivors at $ak_j = 0$ and lifted doublers at $ak_j = \pi$. Site-split or extended operators with translations transform non-trivially under the $\rho_i(n)$, such that we conclude

$$\begin{aligned} t_{\pm D}[U](n,m) &\sim \rho_1(n,m) , \\ t_{\pm j}[U](n,m) &\sim \rho_2(n,m) , \\ t_{\pm j}[U](n,l)t_{\pm D}[U](l,m) &\sim \rho_3(n,m) , \end{aligned} \tag{7}$$

where the symbol "$\sim$" means that site-split path transforms under $\rho_j(n)$ as $\rho_j(n)\rho_i(n,m)\rho_j(m) = (-1)^{\delta_{ij}-1}\rho_i(n,m)$. While an even number of steps in any one direction is a ST singlet transform for KW fermions, naive site-splitting procedures [17] fail to implement the ST structure, since the $\rho_i(n,m)$ do not satisfy the $\mathfrak{su}(2)$ algebra. Because translations in different directions do not commute in the interacting theory, $\rho_\mu(n,l)\rho_\nu(l,m) \neq \rho_\nu(n,l)\rho_\mu(l,m)$, we see that a ST vector mass term $\sim \rho_3(n,m)$ must renormalize differently from a ST singlet mass term.

$\otimes$	Γ_5	Γ_D	Γ_j	Γ_{jk}	Γ_{jD}	Γ_{j5}	Γ_{D5}	$\mathbb{1}$
ρ_1	(γ_D)	γ_D	$[\gamma_{j5}\epsilon]$	(γ_j)	(γ_{j5})	γ_{j5}	$[\gamma_D\epsilon]$	–
ρ_2	–	$[\gamma_{D5}\epsilon]$	γ_j	–	–	$[\gamma_j\epsilon]$	γ_{D5}	(γ_{D5})
ρ_3	γ_5	(γ_5)	(γ_{jk})	$[\gamma_{jD}\epsilon]$	γ_{jD}	(γ_{jD})	–	$[\gamma_5\epsilon]$
$\mathbb{1}$	$[\epsilon]$	–	–	γ_{jk}	$[\gamma_{jk}\epsilon]$	–	$(\mathbb{1})$	$\mathbb{1}$
ρ_1	$c_j\gamma_5$	$(c_j\gamma_5)$	$(c_D\gamma_{jk})$	$c_D\gamma_{jk}$	$c_j\gamma_{jD}$	$(c_j\gamma_{jD})$	$(c_D\mathbb{1})$	$c_D\mathbb{1}$
ρ_2	$c_D\gamma_5$	$(c_D\gamma_5)$	$(c_j\gamma_{jk})$	$c_j\gamma_{jk}$	$c_D\gamma_{jD}$	$(c_D\gamma_{jD})$	$(c_j\mathbb{1})$	$c_j\mathbb{1}$
ρ_3	$(c_j\gamma_D)$	$c_j\gamma_D$	$c_D\gamma_j$	$(c_D\gamma_j)$	$(c_j\gamma_{j5})$	$c_j\gamma_{j5}$	$c_D\gamma_{D5}$	$(c_D\gamma_{D5})$
$\mathbb{1}$	$(c_D\gamma_D)$	$c_D\gamma_D$	$c_j\gamma_j$	$(c_j\gamma_j)$	$(c_D\gamma_{j5})$	$c_D\gamma_{j5}$	$c_j\gamma_{D5}$	$(c_j\gamma_{D5})$

Table 1: Spin-taste assignment of meson-interpolating operators for KW fermions. Columns/rows identify spin(Γ)/taste(ρ) structures; entries indicate $M(n,m)$ of $\bar{\psi}(n)M(n,m)\psi(m)$. M represents continuum/Wilson fermion spin assignment, while (M) indicates the assignment for the parity partner. Top: Single-site/zero-link operators. $[M]$ indicates a state of desired quantum numbers at the three-momentum cutoff; two-link operators c_jc_DM achieve the same assignment near zero momentum. Bottom: One-link operators $c_\mu M$.

When we split the Laplacian according to Eq. (3), the different ST structures of the terms in the KW action emerge, which are the same as in Table 1 of Ref. [20]. Since odd powers in the cutoff are combined with odd powers in r, conclusions regarding automatic cancellation of odd powers of r and a in the KW determinant can be taken over[3]. Thus, one may average both signs of r in

[2] In the chiral representation of the Euclidean gamma matrices $\rho_i(n) \propto \sigma_i \otimes \mathbb{1}$ suggest a natural identification.
[3] This is a stronger statement than automatic $O(a)$ improvement, since it applies to any odd powers.

the valence sector and enforce explicit cancellation of any odd powers in the valence sector without extra fine tuning. For meson-interpolating operators, we only need to classify between zero- or one-link operators, shown in Table 1, since two-link operators can be related—up to the major subtlety, which Fermi points are connected—to the zero-link ones. Symmetrization is optional, but reduces noise, i.e. $t_{\pm\mu}$ instead of c_μ yields the same ST structures.

We studied spin-zero single-site operators on pure gauge backgrounds [21] and found for naive fermions perfect degeneracy between the respective parity partners $M = \gamma_5$ vs γ_D or $M = \gamma_{D5}$ vs $\mathbb{1}$. For KW fermions, however, there is no degeneracy between $M = \gamma_5$ or γ_D (while $M = \gamma_{D5}$ vs $\mathbb{1}$ remains inconclusive due to large statistical errors). The splitting between $M = \gamma_5$ vs γ_{5D} is similar for both. Thus, two different sources of taste-symmetry violation affect the KW variant.

3. Borici-Creutz fermions

The standard BC action (in the conventions of [12]) and in a convenient notation reads

$$S_{\mathrm{BC}}[\psi,\bar{\psi}] = a^D \sum_{n,m\in\Lambda} \bar{\psi}(n)\Big[D_{\mathrm{nai}}[U] \;+\; m_0 - \frac{ra}{2}\sum_{\mu=1}^{D} \mathrm{i}\gamma'_\mu c_\mu[U] + \frac{r}{a}\mathrm{i}\Gamma\Big](n,m)\,\psi(m)\,, \tag{8}$$

The dual gamma matrices γ'_μ are generated by selecting one hypercubic-diagonal direction via Γ,

$$\gamma'_\mu = \Gamma\gamma_\mu\Gamma\,, \quad \Gamma \equiv \frac{1}{\sqrt{d}}\sum_{\mu=1}^{d}\gamma_\mu = \frac{1}{\sqrt{d}}\sum_{\mu=1}^{d}\gamma'_\mu\,. \tag{9}$$

The BC action as described in Eqs. (8) and (9) is obviously invariant under discrete translations (for periodic boundary conditions), discrete rotations around the hypercubic diagonal or axis exchanges (leaving Γ invariant), and the product $\mathcal{CPT}$. Furthermore, it is γ_5 hermitian and satisfies a chiral symmetry with (unmodified) γ_5 for $m_0 \to 0$. Anisotropy between the hypercubic-diagonal direction and those orthogonal to it is a consequence of the broken symmetry under $\mathcal{PT}$ (or C) that maps the r-parameter as $r \to -r$. For $r^2 > 1/2$ (in $D = 4$), the BC action is minimally doubled with the survivors located at $ak_\mu = 0$ or $-2\arctan(1/r)$ (all k_μ equal) in the Brillouin zone. The forward/backward symmetry along the hypercubic-diagonal direction is broken; cf. Eq. 9.

In order to identify the BC representation of taste $\mathfrak{su}(2)$ we first define a unitary ST rotation

$$\psi(n) \xrightarrow{T_\Gamma} T_\Gamma(n)\,\psi(n)\,, \qquad \bar{\psi}(n) \xrightarrow{T_\Gamma} \bar{\psi}(n)\,T^\dagger_\Gamma(n)\,, \tag{10}$$

$$T_\Gamma(n) \equiv \mathrm{i}\Gamma\gamma_5\xi_r^{q(n)} \quad , \quad \xi_r \equiv \mathrm{i}s_r \equiv \mathrm{i}\,\mathrm{sign}(r) \quad , \quad q(n) = (n_1 + \ldots + n_D) \mod 4\,, \tag{11}$$

$$T_\Gamma(n)\,\gamma_\mu\,T^\dagger_\Gamma(m) = -\gamma'_\mu\xi_r^{q(n-m)} \quad \Rightarrow \quad T_\Gamma(n)\,\Gamma\,T^\dagger_\Gamma(m) = -\Gamma\xi_r^{q(n-m)}\,, \tag{12}$$

which converts Eq. (8) into

$$S_{\mathrm{BC}}[\psi,\bar{\psi}] = a^D \sum_{n,m\in\Lambda} \bar{\psi}(n)\Big[|r|D_{\mathrm{nai}}[U] \;+\; m_0 \;+ \frac{s_r a}{2}\sum_{\mu=1}^{D} \mathrm{i}\gamma'_\mu c_\mu[U] - \frac{r}{a}\mathrm{i}\Gamma\Big](n,m)\,\psi(m) \tag{13}$$

in terms of the transformed fields. Note that an odd power in $|r|$ is swapped between the derivative and Laplacian terms. For $r^2 = 1$ (and thus $|r| = 1$, $s_r = r$)–to which we restrict hereafter–the form of

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Eq. (8) is restored with $r \longrightarrow -r$ under C (or $\mathcal{PT}$). Thus, we have identified an additional ST charge conjugation symmetry seemingly similar to the one of the KW variant. A certain awkwardness follows from the unitary nature of $T_\Gamma(n)$–hermitian/antihermitian for even/odd n– cf. Eq. (11). The block decomposition of Γ,

$$\Gamma = \begin{pmatrix} 0 & R \\ R^\dagger & 0 \end{pmatrix} , \quad R = \sqrt{\frac{2}{d}} \begin{pmatrix} \varrho^{-1} & \varrho^{-3} \\ \varrho^{-1} & \varrho^{+1} \end{pmatrix} , \quad \varrho = \frac{1+\mathrm{i}}{\sqrt{2}} = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}} , \tag{14}$$

permits us to write down a BC representation of the generators of taste $\mathfrak{su}(2)$ in all their glory,

$$\begin{aligned}
\rho_1(n) &= \begin{pmatrix} 0 & (-)^{p(n)} \quad R \\ R^\dagger & 0 \end{pmatrix} \xi_r^{(2p(n)-1)q(n)} &&= \Gamma\gamma_5^{p(n)} \quad \xi_r^{(2p(n)-1)q(n)} &&= [-\mathrm{i}T_\Gamma T_5^{p(n)-1}](n) , \\
\rho_2(n) &= \mathrm{i} \begin{pmatrix} 0 & (-)^{p(n)+1} R \\ R^\dagger & 0 \end{pmatrix} \xi_r^{(2p(n)+1)q(n)} &&= \mathrm{i}\Gamma\gamma_5^{p(n)+1} \xi_r^{(2p(n)+1)q(n)} &&= [T_\Gamma T_5^{p(n)}](n) , \\
\rho_3(n) &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \xi_r^{2q(n)} &&= \quad \gamma_5 \qquad \xi_r^{2q(n)} &&\equiv T_5(n) ,
\end{aligned} \tag{15}$$

which indeed satisfy the usual $\mathfrak{su}(2)$ algebra $[\rho_i, \rho_j] = 2\mathrm{i}\epsilon_{ijk}\rho_k$ for arbitrary n. We note that $\rho_3(n)$ is the same real operator as in Eq. (6) for KW fermions, i.e. $T_5(n) = \tau_5(n)$, since $\xi_r^{2q(n)} = \epsilon(n)$. All terms in Eq. (8) have well-defined transformation behaviors under the generators in Eq. (15): on the one hand, the site-split ones acquire extra factors of $(-\gamma_5) = \rho_3(\pm\hat{\mu})$, while the taste-vector single-site one acquires only powers of (-1). On the other hand, the combinations $T_\Gamma(n)$–which we used to obtain Eq. (13)–(or $-\mathrm{i}[T_\Gamma T_5](n)$) do not introduce factors of γ_5. The unitary operators $T_\Gamma(n)$ (or $-\mathrm{i}[T_\Gamma T_5](n)$) correspond to $\rho_{2/1}(n)$ (or $\rho_{1/2}(n)$, respectively) for even/odd sites (up to phases), such that the generators in the ST charge conjugation symmetry combine to one power of $\rho_3(n)$ for both site-split terms (up to phases).

With the taste generators of Eq. (15) in hand, it is straightforward to write down two-link operators that transform non-trivially under the $\rho_i(n)$ (or $T_\Gamma(n)$ or $-\mathrm{i}[T_\Gamma T_5](n)$, respectively), such that we conclude (for arbitrary $1 \le \mu, \nu \le D$)

$$t_{\pm(\mu+\nu)}[U](n,m) \sim \rho_3(n,m) , \quad t_{\pm(\mu-\nu)}[U](n,m) \sim \mathbb{1}\delta(n,m) . \tag{16}$$

Thus, any forward/backward hops from even/odd sites have the same ST structure, and combine with any forward/backward hops from odd/even sites to $\rho_3(n,m)$ (independent of μ,ν). Most importantly, we can write down a ST vector mass term via Eq. (16), which commutes both with $\mathcal{PT}$ or C such that there is a simultaneous eigenbasis. Each hop individually contains a factor of the chirality matrix γ_5 and a forward/backward dependent phase factor $\xi_r^{\pm 1}$:

$$\begin{aligned}
\rho_1(n)t_{\pm\mu}[U](n,m)\rho_1(m) &\sim -\gamma_5\xi_r^{\pm 1}t_{\pm\mu}[U](n,m) , \\
\rho_2(n)t_{\pm\mu}[U](n,m)\rho_2(m) &\sim +\gamma_5\xi_r^{\pm 1}t_{\pm\mu}[U](n,m) .
\end{aligned} \tag{17}$$

Eq. (16) follows trivially from Eq. (17). However, the one-link operators in Eq. (17) do not permit a simultaneous eigenbasis with $\mathcal{PT}$ or C, and one-link site-splitting [18] does not access the tastes.

	D_{nai}	$\mathrm{i}r\gamma'_\mu c_\mu$	$\mathrm{i}r\Gamma$	$\mathbb{1}$	$c_{\mu+\nu}$	Γs_μ	$\mathrm{i}r\Gamma c_\mu$
$\mathcal{PT}$	+	-	-	+	+	+	-
C	+	-	-	+	+	+	-
$-\mathrm{i}[T_\Gamma T_5]$	$+\mathrm{i}s_r\gamma'_\mu c_\mu$	$-\lvert r\rvert D_{\mathrm{nai}}$	+	+	-	$+\mathrm{i}s_r\Gamma c_\mu$	$-\lvert r\rvert\Gamma s_\mu$
T_Γ	$+\mathrm{i}s_r\gamma'_\mu c_\mu$	$-\lvert r\rvert D_{\mathrm{nai}}$	-	+	-	$+\mathrm{i}s_r\Gamma c_\mu$	$-\lvert r\rvert\Gamma s_\mu$
T_5	+	+	-	+	+	+	+
γ_5	-	-	-	+	+	+	+

Table 2: Spin-taste structure of the renormalized BC action including the marginal fermionic counterterm.

We collect the ST structure of all fermionic terms of the renormalized BC action in Table 2, and see yet another important consequence of the ST charge conjugation symmetry. The 2nd to last column is the marginal fermionic counterterm as suggested in Ref. [7], which is mapped onto itself by $\mathcal{PT}$ or C, yet mapped onto the last column by $T_\Gamma(n)$ (or $-\mathrm{i}[T_\Gamma T_5](n)$). Thus, the correct counterterm combines both columns, propped up by the single-site term, as

$$a^D \sum_{n,m\in\Lambda} \bar{\psi}(n)\Gamma \sum_{\mu=1}^{D} \left[s_\mu[U] - \frac{\mathrm{i}r}{a}\left(c_\mu[U]-1\right)\right](n,m)\,\psi(m)\,, \tag{18}$$

where the $c_\mu[U](n,m)$ structure makes the counterterm compliant with the ST charge conjugation symmetry, while the constant cancels the relevant contribution due to the $c_\mu[U](n,m)$ structure. Eq. (18) supersedes Ref. [7]. Attempts to tune with the wrong counterterm produced explicit $\mathcal{T}$-symmetry violation in pion correlators for any non-zero coefficient [18], while a correct counterterm respecting the symmetries as in Eq. (18) will not induce it despite mistuning.

Ultimately, it is the different taste structures of the site-split and the single-site terms in the BC action that are responsible for automatic cancellation of odd powers of r and a in the BC determinant. The argument is based on symmetry under C, T_5 and γ_5, analogous to the one for the KW variant [20], and does not depend on the condition $r^2 = 1$. Automatic cancellation of odd powers is possible in the valence sector similar to the KW variant, too. However, since parity partners appear as complex oscillating contributions (with period $4a$) in any directions that have non-zero projection on the hypercubic diagonal (cf. Eq. 9), identification of the ST structures is much more complicated in the case of the BC variant. A similar clarity as in Table 1 cannot be achieved unless the hypercubic diagonal is used as the time direction. Because there is no real incentive, we do not compose a similar table here. Zero- or two-link operators could be worked out from Eq. (16), while one-link operators need substantially more care than naively applying Eq. (17).

4. Summary

We have clarified the spin-taste structure of BC and KW variants. Studying the KW variant is far easier [20], and we classified zero-, one-, and two-link meson interpolating operators taking into account the parity partner contributions. We also derived the ST structure for the BC variant relying on an ST charge conjugation symmetry for $r^2 = 1$. We used this symmetry to correct the marginal fermionic counterterm [7]. While we expect that the BC representation of the taste $\mathfrak{su}(2)$

algebra and the form of the counterterm hold for general r or for twisted-ordering operators, there is no substitute for the ST charge conjugation symmetry. Yet the properties of the determinant and automatic cancellation of odd powers in a can be generalized. We advise to avoid BC or twisted-ordering variants with their extreme complexities and drawbacks in favor of the KW variant.

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