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Higher-group symmetry in lattice gauge theories with restricted topological sectors

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In this paper, we give a brief overview of generalized symmetries from the point of view of the lattice regularization as a fully regularized framework. At first, we illustrate the generalization of 't Hooft anomaly matching for higher-form symmetries. Furthermore the main interest goes to the higher-group symmetry. In particular, we find that the so-called 4-group appears in the lattice Yang–Mills theory under modification of instanton sum.

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1. Introduction and summary

Every physicist has cultivated intuition about symmetry. Having established a paradigm in triumph, we understand more aspects of physics, such as conservation laws, phase transitions, and fundamental forces in the nature. Anomalous or spontaneous breaking of symmetries also plays an essential role to clarify theoretical (and experimental) consistency, and/or to classify the phase structure since the Landau theory. The 't Hooft anomaly [1], a renormalization-group-invariant quantum anomaly arising from gauging a global symmetry, tells us quite nontrivial restrictions on the low-energy dynamics and vacuum structure. This idea of anomaly matching is applicable to the recent generalization of symmetry [2, 3], and thus can improve our insight about nonperturbative phenomena.[1]

First of all, in Sect. 2, let us review the basic concept of generalized symmetries: higher-form symmetry which leads us to a generalized 't Hooft anomaly in the $SU(N)$ Yang–Mills theory with the θ term. To aim at transparent understanding of it, we make remarks on topology within a fully regularized framework given in Refs. [6, 7]; its description enjoys the topological structure *even on a lattice* [8]. In this paper, we focus on higher-group symmetry (see Sect. 3). For instance, on background Abelian gauge fields A_μ and $B_{\mu\nu}$, the gauge transformation acts as $A \mapsto A + d\omega$, $B \mapsto B + d\lambda + \omega dA$, and then such a mixture of symmetries is called the 2-group; as we know, this is similar to the Green–Schwarz mechanism.[2] From our lattice viewpoint of generalized symmetries, as in Refs. [9, 10], we can construct a 4-group structure in the *lattice* $SU(N)$ Yang–Mills theory with restricted topological sectors [11].

Other kinds of generalized symmetries are still developing. We hope to apply our approach to recent developments as non-invertible symmetry, subsystem symmetry and so on.

2. Understanding of generalized symmetries within a fully regularized framework

2.1 Higher-form symmetry and 't Hooft anomaly

The basic notion of generalized symmetries is as follows:

1. The concept of symmetry is regarded as a topological defect. As depicted in Fig. 1, for an ordinary (0-form) symmetry we have the symmetry defect operator $U_\alpha(\Sigma)$ on the codimension-1 space Σ with fixed time,
$$Q \equiv \int_{\Sigma_{D-1}} j_0 dx_1 \wedge \cdots \wedge dx_{D-1}, \qquad U_\alpha(\Sigma_{D-1}) \equiv e^{i\alpha Q}, \tag{1}$$
while a generic symmetry defect on the codimension-$(p+1)$ surface is given by
$$Q \equiv \int_{\Sigma_{D-p-1}} \star j^{(p+1)}, \qquad U_\alpha(\Sigma_{D-p-1}) \equiv e^{i\alpha Q}. \tag{2}$$

2. The above p-form symmetry with $p \geq 1$ is Abelian. In particular we are interested in discrete global symmetries. For instance, the $SU(N)$ Yang–Mills theory possesses the $\mathbb{Z}_N$ 1-form ($\mathbb{Z}_N^{[1]}$) center symmetry; Fig. 2 shows its defect from the lattice viewpoint.

[1] For studies of gauge theories, see Refs. [4, 5] and references cited therein.
[2] In the case of the Green–Schwarz mechanism, both gauge fields are dynamical rather than background.

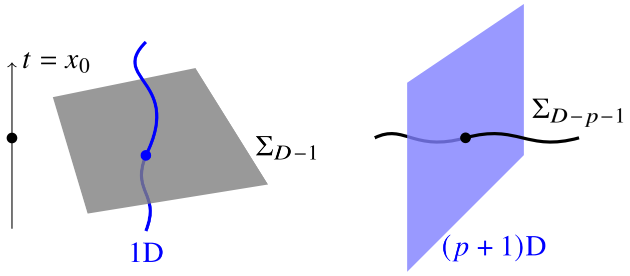

Figure 1: Generalization of symmetry as a topological defect. Symmetry operator (or charge) is defined on the codimension-$(p+1)$ defect Σ, as $U_\alpha(\Sigma)$ in eq. (2).

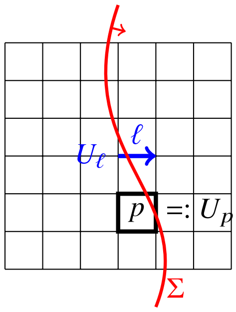

Figure 2: Center of gauge group as an example of 1-form symmetry. The lattice Λ divides a four-dimensional torus into hypercubes, and Σ is a codimension-1 oriented surface such that $\Sigma \cup \Lambda = \emptyset$. One may regard Σ as a network on the dual lattice. Under a center transformation on Σ, any plaquette is invariant, whereas $\partial\Sigma$ can provide the codimension-2 symmetry defect.

3. A charged object $W(C)$ is defined as a "loop" operator, and transforms under action of the symmetry operator by

$$\langle W(C)\rangle \mapsto \langle U(\Sigma)W(C)\rangle = \left\langle e^{i\alpha\,\mathrm{Link}(\Sigma,C)}W(C)\right\rangle, \tag{3}$$

where $\mathrm{Link}(\Sigma, C)$ denotes the intersection number between the surfaces Σ and C. The center transformation in Fig. 2 is then given by $U_\ell \mapsto e^{\frac{2\pi i}{N}k\,\mathrm{Link}(\Sigma,\ell)}U_\ell$ with any bond ℓ and an integer k. Note that any plaquette (or lattice action) is invariant; $U_p \mapsto U_p$.

4. To gauge higher-form global symmetries, for example, let us construct the $\mathbb{Z}_N^{[1]}$ gauge symmetry from the $SU(N)$ gauge theory. The lattice action,

$$S[U_\ell, B_p] = \sum_p \beta\left[\mathrm{tr}\left(1 - e^{-\frac{2\pi i}{N}B_p}U_p\right) + \mathrm{c.c.}\right], \tag{4}$$

where B_p is a 2-form gauge field associated with $\mathbb{Z}_N^{[1]}$,[3] is invariant under the gauge transformation

$$U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell}U_\ell, \qquad B_p \mapsto B_p + (d\lambda)_p \bmod N. \tag{6}$$

[3]Some people are familiar with the 't Hooft twisted boundary condition. For $\forall\ell = (n,\mu)$, gauge functions obey

$$U_{n+L\hat{\nu},\mu} = g_{n,\nu}^{-1}U_{n,\mu}g_{n+\hat{\mu},\nu}, \qquad g_{n+L\hat{\nu},\mu}^{-1}g_{n,\nu}^{-1}g_{n,\mu}g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N}z_{\mu\nu}} \in \mathbb{Z}_N \tag{5}$$

It is convenient to consider the above idea from the lattice viewpoint. The strategy through the use of 't Hooft anomaly matching is, however, closely related to topology of gauge fields and based on cohomological operations. From now, one would have a tendency to refrain the lattice regularized framework. To do this, as a formal standpoint in the continuum theory,

1. The "$\mathbb{Z}_N$ 2-form gauge field" is described by $U(1)$ 2-form gauge field $B^{(2)}$ and charge-N Higgs field; $U(1)$ should be broken to $\mathbb{Z}_N$.

2. The topological charge $Q \pmod 1$ is formally given as $-\frac{N}{8\pi^2}\int B^{(2)} \wedge B^{(2)} \in \frac{1}{N}\mathbb{Z}$. Note that this notation is not always correct because the de Rham cohomology (or the $\wedge$ product) may miss the information of discrete gauge group. Therefore, we would replace $\wedge$ by the cohomological operations: $Q = -\frac{1}{N}\int_X \frac{1}{2}\mathfrak{P}_2(B^{(2)}) \in \frac{1}{N}\mathbb{Z}$, where the Pontryagin square defined by $\mathfrak{P}_2(f) \equiv f \cup f + f \cup_1 \delta f \in H^{2q}(X, \mathbb{Z}_{2r})$ with respect to $f \in H^q(X, \mathbb{Z}_r)$ possesses the graded commutativity for the cup product of simplicial cochains [13]. Note that we introduced the higher-cup product (e.g., $\cup_1$) because of the manifest $\mathbb{Z}_N^{[1]}$ gauge invariance at the cochain level.

3. Finally, in the partition function with the θ term, we observe the violation of the θ periodicity

$$\mathcal{Z}_{\theta+2\pi k}[B] = e^{2\pi i k Q}\mathcal{Z}_\theta[B], \qquad \text{with } Q \in \frac{1}{N}\mathbb{Z},\ k \in \mathbb{Z}. \tag{7}$$

   This is called the mixed 't Hooft anomaly between the $\mathbb{Z}_N^{[1]}$ symmetry and θ periodicity.

2.2 Topology of lattice gauge fields

Do we miss the topological structure on the lattice because spacetime discretization breaks continuity? The answer is no; Lüscher [8] constructed the $SU(N)$ principal bundle from lattice $SU(N)$ gauge fields under the *admissibility* condition such that

$$|1 - U_p| < \varepsilon \qquad \text{for } \exists\varepsilon > 0. \tag{8}$$

This implies that U_p should be sufficiently close to the classical continuum limit ~ 1 because of its well-defined-ness; see Fig. 3. Then, he proved the presence of topological sectors and explicitly defined the topological charge on the lattice, which takes an integral value.[4]

Quite recently, in Refs. [6, 7], starting from the lattice action (4) with the lattice $\mathbb{Z}_N$ 2-form gauge field B_p, Lüscher's construction was generalized; we can find the fractional topological charge and the mixed 't Hooft anomaly (7) on the lattice. This generalized method provides a fully

where the 't Hooft flux $z_{\mu\nu}$ is identical to $\sum B_p \mod N$. The topological charge has a fractional shift $\sim \frac{1}{N}$ due to $z_{\mu\nu}$ [12]. This expression looks to be written by using the global data, but it can be written in terms of local operations to B_p from a modern perspective as we will see.

[4] It is known that the index can be computed on the lattice thanks to the overlap Dirac operator D_{ov}, so that

$$\text{Index}(D) = -\frac{1}{2}\operatorname{Tr}\gamma_5 D_{\text{ov}} = n_+ - n_- \in \mathbb{Z}. \tag{9}$$

Here $n_\pm$ is the number of positive/negative chiral modes. The index theorem states that the index is identical to the topological charge so constructed.

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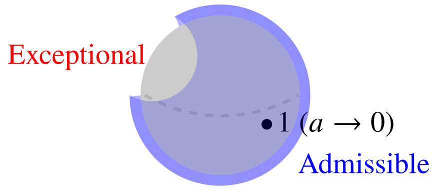

Figure 3: Admissible or exceptional configuration space on $SU(2) \cong S^3$. By removing one point on $SU(2)$ (in general a region), we have a well-defined configuration space depicted as the blue region, which is homeomorphic to a disk and called admissible.

regularized framework for studies of generalized symmetries and 't Hooft anomaly matching. We are ready to understand some studies more transparently and deeply.

To obtain an expression of Q, we can use the Pontryagin square which is defined *on the hypercubic lattice* [14]. On the other hand, after some calculations on an appropriate bundle structure,

$$Q[U_\ell, B_p] = \sum_n q(n), \qquad q(n) = -\frac{1}{8N} \sum_{\mu,\nu,\rho,\sigma} \epsilon_{\mu\nu\rho\sigma} B_{\mu\nu}(n) B_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) + \check{q}(n), \tag{10}$$

where we have used $B_{\mu\nu}(n)$ instead of B_p with $p = (n, \mu, \nu)$ and a unit vector of lattice $\hat{\mu}$ in the μ direction. We note that since $\sum_n \check{q}(n) \in \mathbb{Z}$ the first term written in terms of $B_{\mu\nu}(n)$ gives rise to a fractional contribution. For notational simplicity, let us introduce a product on the hypercubic lattice, $\cup_{\mathrm{H}}$, as $q(n) - \check{q}(n) = -\frac{1}{2N}(B \cup_{\mathrm{H}} B)_n$ and so $\sum_n \frac{1}{2}(B \cup_{\mathrm{H}} B)_n \in \mathbb{Z}$.[5] We can obtain the mixed 't Hooft anomaly (7) within the lattice regularized framework.

3. Instanton-sum modification and higher-group structure in lattice gauge theories

Now, we add the term, $\sum_n i\chi(n)\,[q(n) - pc(n)]$, in the Yang–Mills action (4) with the θ term, $i\theta Q$.[6] The equation of motion for the compact scalar χ, which is a Lagrange multiplier field, implies that the topological sectors are restricted to instanton numbers as $Q = p \sum_n c(n) \in p\mathbb{Z}$ for the $U(1)$ 4-form field strength $c(n)$. It is known that for any $p \in \mathbb{Z}$ this restriction provides a local and unitary quantum field theory. Obviously, nontrivial configurations of B_p are forbidden so that eq. (10) takes a multiple of p. As we discuss from now, gauging not only the $\mathbb{Z}_N^{[1]}$ symmetry (i.e., B_p) but also the $\mathbb{Z}_p^{[3]}$ symmetry at the same time, we can avoid this obstruction and observe the higher-group structure, that is, 4-group on the lattice [9, 10].

[5] The definition of $\check{q}(n)$ in Ref. [10] is quite complicated, while we do not necessarily have to define it in order to prove the fractionality [7]. Actually, we can find

$$Q[U_\ell, B_p] \in -\frac{1}{8N} \epsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} + \mathbb{Z}, \tag{11}$$

in terms of the 't Hooft flux $z_{\mu\nu}$ (5).

[6] For the $U(1)/\mathbb{Z}_q$ gauge theory, the Witten effect for dyon plays an essential role for the presence of not only higher-group but also higher-form symmetries. Then, we should multiply θ by q so that the θ term is given by $iq\theta Q$. However it is not known under the admissibility condition how to observe the monopole or dyon as dynamical degrees of freedom. See recent works [15, 16].

The fractional part of Q in the equation of motion is compensated by a replacement as $c - \frac{1}{Np}\Omega$,

$$q(n) - pc(n) + \frac{1}{N}\Omega(n) = 0. \tag{12}$$

We find that, assuming $\sum_n \Omega(n) \in \mathbb{Z}$, all fractional contributions from B_p can be absorbed into $\Omega(n)$. There are alternative options as follows:

- Strict structure: The minimal compensation is given by $\Omega(n) = w(n) \equiv -N[q(n) - pc(n)] \in \mathbb{Z}$. The $\mathbb{Z}_N^{[1]}$ gauge transformation acts as eq. (6), and the $\mathbb{Z}_{Np}^{[3]}$ gauge transformation acts as

$$w(n) \mapsto w(n) + (d\omega_{\mathrm{s}})_n \bmod Np, \qquad c(n) \mapsto c(n) + \frac{1}{Np}(d\omega_{\mathrm{s}})_n \bmod 1, \tag{13}$$

  where $\omega_{\mathrm{s}} \in \mathbb{Z}$. Note that $w(n)$ is not transformed by $\mathbb{Z}_N^{[1]}$; the 3-form symmetry is not $\mathbb{Z}_p$ but $\mathbb{Z}_{Np}$ because the strict 4-group mixes the $\mathbb{Z}_N^{[1]}$ symmetry into the *physical* $\mathbb{Z}_p^{[3]}$ symmetry.

- Weak structure: The simpler equation of motion is realized by redefining $\Omega(n)$ as

$$\check{q}(n) - pc(n) + \tilde{\Omega}(n) = 0, \qquad \tilde{\Omega}(n) \equiv \frac{1}{N}\Omega(n) - \frac{1}{2N}(B \cup_{\mathrm{H}} B)_n. \tag{14}$$

  In the left hand side of the equation of motion, all terms contribute as integral values after the summation over n. Then, we can see the explicit mixture of gauge transformations such that, thanks to eq. (6) and

$$\frac{1}{N}\Omega(n) \mapsto \frac{1}{N}\Omega(n) + (d\omega_{\mathrm{w}})_n \bmod p, \qquad c(n) \mapsto c(n) + \frac{1}{p}(d\omega_{\mathrm{w}})_n \bmod 1, \tag{15}$$

  where $\omega_{\mathrm{w}} \in \mathbb{R}$, $\tilde{\Omega}(n)$ transforms under the $\mathbb{Z}_N^{[1]}$ and 3-form gauge symmetries as

$$\tilde{\Omega}(n) \mapsto \tilde{\Omega}(n) + (d\omega_{\mathrm{w}})_n - \frac{1}{2N}\left[(B \cup_{\mathrm{H}} d\lambda)_n + (d\lambda \cup_{\mathrm{H}} B)_n + (d\lambda \cup_{\mathrm{H}} d\lambda)_n\right], \tag{16}$$

  where we have omitted the modulo operations for simplicity. By using the $\mathbb{Z}_N^{[1]}$ and *continuum* 3-form gauge transformations, we suppose that $\tilde{\Omega}(n) = \tilde{w}(n) \in \mathbb{Z}$; that is, the genuine 3-form symmetry is *discrete* $\mathbb{Z}_p^{[3]}$ as $\omega_{\mathrm{w}} \in \mathbb{R} \to \mathbb{Z}$. The weak 4-group is described in the local way by the mixture of the $\mathbb{Z}_N^{[1]}$ and continuum 3-form gauge symmetries, and the discrete $\mathbb{Z}_p^{[3]}$ gauge symmetry.

Finally, we make a remark on the case of the continuum theory. As discussed in the previous section, we have to use some more $U(1)$ gauge fields to describe this phenomenon. Its explanation should be done in a careful way, which may be quite wise but not transparent, since there are some subtleties about the observation of periodicity and the presence of higher-form symmetries. On the other hand, on the lattice, all we need to do is just taking a count of (fractional) numbers.

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