PROCEEDINGS OF SCIENCE

Modelling the formation of structures after inflation

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The paper studies the growth of inhomogeneities of the inflaton field at the post-inflationary stage of the evolution of the Universe in realistic inflationary models and the dynamics of inhomogeneities at the nonlinear stage of evolution. At the beginning of this stage, the inflaton field can be considered almost homogeneous, and the expansion law of the Universe coincides with that at the dust-like stage. Over time, inhomogeneities of the inflaton field grow due to gravitational instability. If the stage of reheating is long enough, these inhomogeneities can give rise to the first gravitationally bound structures. For the study, a numerical code was written to simulate the evolution of inhomogeneities at the post-inflationary stage of expansion of the Universe.

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1. Introdution

After inflation, the inflation field forms structures whose evolution leads to the emission of gravitational waves. These waves are of great interest because they can provide valuable insight into the mechanism of inflation. In section 2 we derive equations describing the nonlinear evolution of the field perturbations inside the horizon. In section 3 we discuss the nonlinear phase and initial conditions for the inflaton field. In section 4 we perform several tests and outline the methods for computing the nonlinear evolution of the field, and also present preliminary results for the evolution of structures. Finally, in section 5 we describe methods for computing gravitational waves from the collapse of structures.

2. Non-linear equations for perturbations of the inflationary field under horizon

After inflation, the bulk of the energy is contained in a highly homogeneous massive scalar field, described by the action:

$$S = \int \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right)d^4x \tag{1}$$

We are interested in the post-inflationary period of the Universe's evolution where it enters the MD-stage. During this stage, the nonrelativistic approximation can be used. We use a non-relativistic approximation in the form $m\psi \gg \dot{\psi}$ when we obtain the Schrodinger-Poisson system (4), (5) and equation (11). Additionally, it is correct that $H \ll m$. The field can be split into two parts: the background $\varphi_0(t)$, which is considered homogeneous and dependent only on time, and the perturbations $\varphi(t,\vec{x})$, i.e.:

$$\phi = \varphi_0(t) + \varphi(t,\vec{x}) \tag{2}$$

The new variable ψ is defined as follows:

$$\varphi_{(0)} = \frac{e^{-imt}\psi_{(0)}}{\sqrt{2}a^{\frac{3}{2}}(t)} + h.c, \tag{3}$$

with the lower index used to distinguish between the background and perturbations. For a scalar field with a potential from (1), the approximation of an ideal liquid is valid, in which case it can be obtained from Einstein's equations that Bardeen's scalar potentials are equal [1], therefore the perturbed metric in a linear approximation will look like $ds^2 = (1+2\Phi)dt^2 - a^2(1-2\Phi)d\vec{x}^2$. Inserting the approximations described above and metric in the equations of motion of the field and averaging the double-frequency summands, we obtain the equations for the field ψ. The equation for the gravitational potential Φ follows from the Einstein equations within the same approximations:

$$i\dot{\psi} = -\frac{\Delta\psi}{2ma^2(t)} + m\Phi\psi \tag{4}$$

$$\Delta\Phi = 4\pi Gm^2\frac{|\psi|^2 - |\psi_0|^2}{a(t)} \tag{5}$$

Hereafter, $\dot{}$ is derivative with respect to physical time. Thus, the Schrödinger-Poisson (SP) system has been obtained, which describes the behavior of perturbations at the MD stage in the non-relativistic approximation. The formation and gravitational collapse of matter structures after inflation will be described by the evolution of the field ψ.

3. The evolution of the field perturbations before entering the horizon.

3.1 Initial conditions

Inflationary models predict a power spectrum for a value $\mathcal{R}$ which is defined as follows. Hereafter, $'$ is derivative with respect to conformal time:

$$\mathcal{R} = -\left(\Phi + \frac{a'}{a\phi_0'}\phi\right) \tag{6}$$

The Einstein equations for the 00 and 0i components can be expressed in terms of the Mukhanov-Sasaki variable, which defined as:

$$u = -\frac{a^2\phi_0'}{a'}\mathcal{R} \tag{7}$$

And the equations take the following form [1]:

$$\Delta\Phi = 4\pi G\frac{a}{a'}\phi_0'^2\left(\frac{a'u}{a^2\phi_0'}\right)', \tag{8}$$

$$\frac{a'}{a^2}\left(\frac{a^3}{a'}\Phi\right)' = 4\pi G\phi_0' u. \tag{9}$$

Thus, we can separate the solutions for the gravitational field and perturbations based on the predictions of the inflationary model for $\mathcal{R}$, using numerical solutions of the Einstein equations. These solutions can then be evolved to the stage where the Schrödinger-Poisson approximation is applicable. The equations for linear evolution are provided in the next section.

3.2 Linear evolution of the perturbations in the super- and subhorizon regimes

The system of equations for SP was derived using approximations that are only valid for the evolution of the subhorizon field perturbations in the non-relativistic regime. To set the initial conditions based on the spectrum predicted by various inflationary models, we obtained linearized equations that are accurate both in super- and subhorizon regimes. The equation for φ is as follows:

$$\ddot{\varphi} + \ddot{\varphi}_0 - 4\dot{\Phi}\dot{\varphi}_0 + 3H(\dot{\varphi} + \dot{\varphi}_0) + 2\Phi m^2\varphi_0 - \frac{\Delta\varphi}{a^2} + m^2(\varphi + \varphi_0) = 0 \tag{10}$$

Thus, there are three equations: one for the background, obtained from the action (1) assuming no perturbations; one for the perturbations; and Einstein's equation for the 00 component, which is used to determine the gravitational potential. These equations completely determine the linear evolution of the field perturbations at all wavelengths. By substituting the ansatz for the ψ field and using only the non-relativistic approximation, one again arrives at the Schrödinger equation:

$$\frac{e^{imt}}{\sqrt{2}a^{3/2}}\left(2m^2\Phi\psi_0 - 2im\dot{\psi} - \frac{\Delta\psi}{a^2}\right) + h.c + O(mt) = 0 \tag{11}$$

Thus, the non-relativistic Schrödinger equation may be applied both for super- and subhorizon regimes if $mt \gg 1$. The matching between the solutions for the ϕ field and the non-relativistic field ψ with respect to the parameter mt will be demonstrated in the section 4.

4. Numerical simulations

4.1 Equations in dimensionless variables

In order to minimize the impact of numerical rounding errors, we convert the equations to dimensionless form as follows:

$$\begin{cases} x = x_0 \cdot x_{pr} \\ t = t_0 \cdot t_{pr} \\ \psi = \psi_0 \cdot \psi_{pr} \\ \Phi = \Phi_0 \cdot \Phi_{pr} \\ a = a_0 \left(\dfrac{t_{pr}}{t_*} \right)^{2/3} \end{cases} \tag{12}$$

Then the Schrödinger-Poisson system takes the form:

$$\begin{cases} i\dfrac{d\psi_{pr}}{dt_{pr}} = -\dfrac{\Delta_{pr}\psi_{pr}}{2a_{pr}^2} + \Phi_{pr}\psi_{pr} \\ \Delta_{pr}\Phi_{pr} = \dfrac{4\pi}{a_{pr}}(|\psi_{pr}|^2 - 1) \end{cases} \tag{13}$$

In equation (12) we have six parameters. However, using the dimensional analysis, one can establish relations between them, revealing only two physical parameters: ψ_0, which is the average value of the field, and m, which is the mass of the inflaton field.

4.2 Numerical solutions for linearized equations

To assess the applicability of the nonrelativistic approximation, we solved the linear equations for ψ and φ for a set of initial times. We then examined the difference in contrasts δ, which is defined as:

$$\delta \equiv \frac{\delta T_0^0}{T_0^0} \tag{14}$$

where δT_0^0 represents the energy-momentum tensor associated with the perturbations, and T_0^0 represents the energy momentum tensor for the background field. To achieve this, we calculated the 00 components of the energy momentum tensor in terms of fields ψ and φ on the solutions of (10) and (11). Then we calculated the corresponding contrasts, which we denoted as $\delta(\psi)$ and $\delta(\varphi)$. In Fig.1 we plotted the ratio $\left|\dfrac{\delta(\psi) - \delta(\varphi)}{\delta(\psi)}\right|$ for the set of initial times. One can observe a linear decrease in the difference with increasing time.

4.3 Numerical methods for nonlinear equations

The system is solved jointly at each time step using the fourth-order symplectic method [2] to solve the Schrödinger equation. The Poisson equation is solved by decomposing the right-hand side into Fourier, dividing by k^2, and performing the inverse Fourier transform using Fast Fourier Transform algorithm. The complexity of this solution is $O(N^3)$ operations. Distributed memory computing (MPI) is employed to reduce the calculation time.

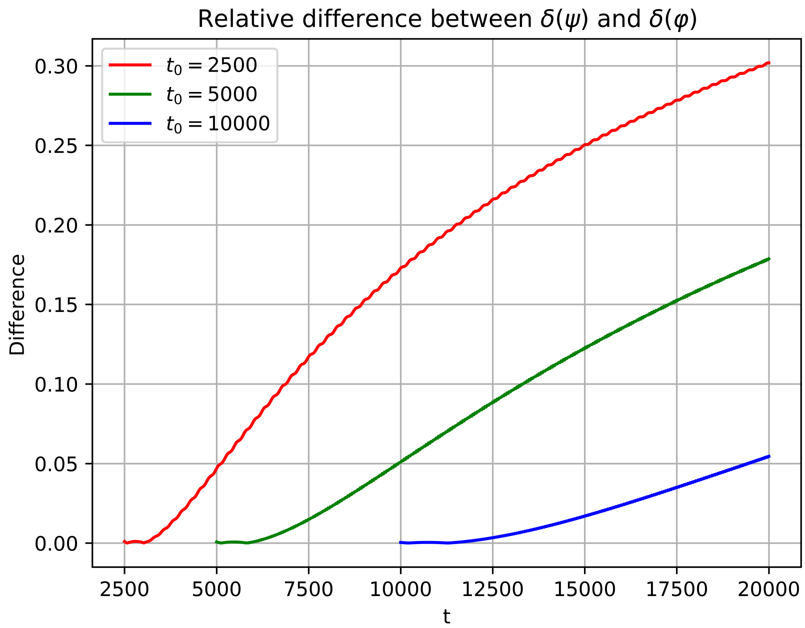

Figure 1: Relative difference of δ in terms of ψ and φ

To check the correctness of the modelling, we compare solutions of linear and non-linear equations for the small perturbations. The solutions obtained from linear and non-linear equations in a specific regime are plotted in figure 2. As can be seen, the solutions coincide at first, but they start to differ when approaching to the nonlinear regime.

Figure 3 shows the results for the relative difference $\left|\frac{\psi_{N=128}-\psi_{N=256}}{\psi_{N=256}}\right|$ between two solutions obtained on different lattices. We also look at the averaged value of the field at large momentum k to see the moment when our lattice stops accommodating structures: the results are shown in the figure 4. It is evident that at some point there has been a significant increase in values of modes, which momentum is close to the maximum momentum that fits to the lattice. This suggests that the collapsed structures no longer fit into the lattice.

Preliminary simulations were conducted to study the evolution and collapse of the structures. The results are presented in figure 5, where the colour represents the field density.

In order to evaluate the contribution of specific pulse ranges to the formation of structures, we will use the Press-Schechter formalism [1], introducing the smoothed density contrast $\delta_R(x,t)$ as:

$$\delta_R(x,t) = \int d^3y\delta(x+y,t)W_R(y) \tag{15}$$

$W_r(y)$ is defined as:

$$W_r(y) = \frac{3}{4\pi}\frac{a_0^3}{R^3}\theta(R - a_0|y|) \tag{16}$$

where R is the size of the collapsing region, a_0 defined by (12). Then, for the Fourier image $\delta_R(x,t)$,

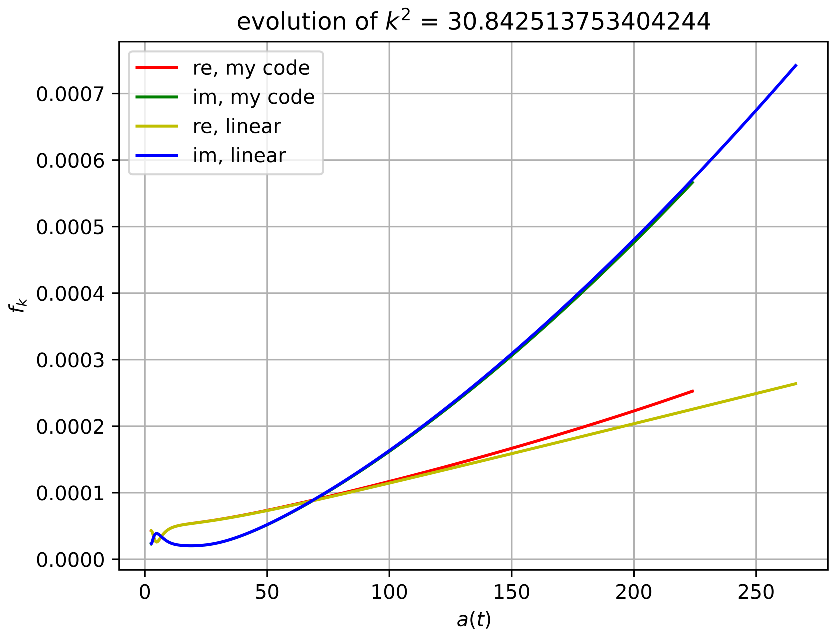

Figure 2: Solutions of linear and nonlinear equations

the following is true:

$$\delta_R(k,t) = \frac{3j_1(kR/a_0)}{kR/a_0}\delta(k,t) \tag{17}$$

where $\delta(k,t)$ is the matter density contrast. This formula shows that the main contribution to the smoothed density contrast comes from perturbations with conformal momenta $k < a_0/R$. In this way the programme will evaluate the impact of modes with different momenta to the formation of gravitationally bound structures.

5. Gravitational waves from structure formation

We simulate the evolution of the ψ field and at each time step we know the configuration of the field. The main gravitational wave signal is expected to come from the collapse of the inflaton field structures. The equations for the tensor modes have the form [3]:

$$\ddot{h}_{ij}(\vec{x},t) + 3H\dot{h}_{ij}(\vec{x},t) - \frac{\nabla^2}{a^2}h_{ij}(\vec{x},t) = 16\pi G\Pi_{ij}^{TT} \tag{18}$$

where Π_{ij}^{TT} is the transverse and traceless part of the energy-momentum tensor. The solution of such an equation for the quantity $\overline{h}$, defined as $\overline{h}_{ij}(\vec{k},\eta) = ah_{ij}(\vec{k},\eta)$ for the modes inside the horizon, assuming that no gravitational waves have been emitted up to the moment $\eta = \eta_i$ and after some moment $\eta = \eta_f$, is given by the Green's function and can be written as [4]:

$$\overline{h}_{ij}(\eta,\vec{k}) = -\frac{16\pi G}{k}\int_{\eta_i}^{\eta_f} d\eta' e^{k(\eta-\eta')}a(\eta')\Pi_{ij}^{TT}(\eta',\vec{k}) \tag{19}$$

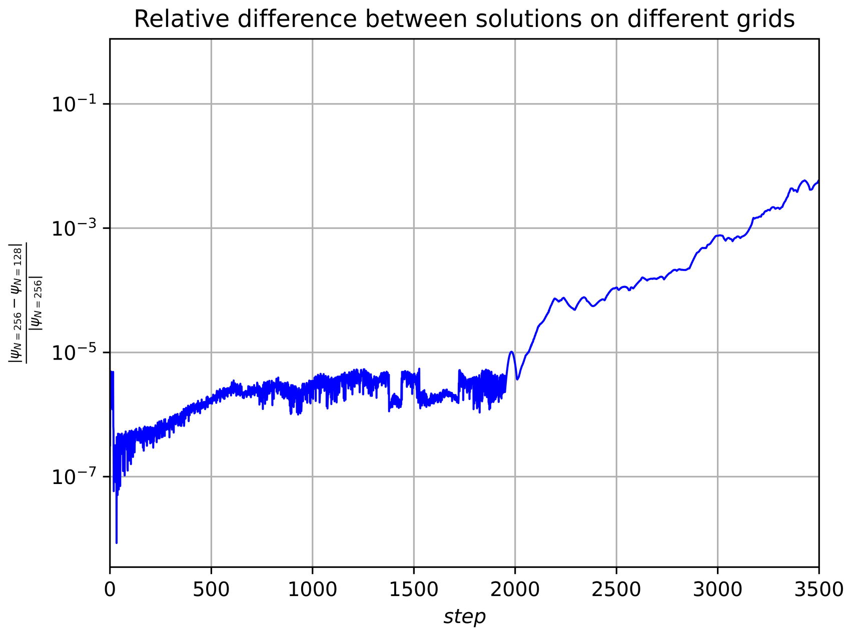

Figure 3: Relative difference between solutions on grids of 128/256 points in each dimention

We define gravitational waves energy density over a volume V as:

$$\rho_{gw} = \frac{1}{32\pi G}\langle \dot{h}_{ij}(x,t)\dot{h}_{ij}(x,t)\rangle_V \tag{20}$$

One can obtain the expression for the spectrum of the gravitational waves:

$$S_k(\eta) = \frac{8\pi G k^3}{V}\int d\Omega \sum_{p=+,\times}\left|\int_{\eta_i}^{\eta_f} d\eta' e^{ik(\eta_f-\eta')}a(\eta')\Pi_p(\eta',\vec{k})\right|^2 \tag{21}$$

which is defined as:

$$\left(\frac{d\rho_{gw}}{d\ln k}\right)_{\eta>\eta_f} = \frac{S_k(\eta_f)}{a^4(\eta)} \tag{22}$$

and the summation is based on the polarisations of the gravitational waves. Thus, knowing the field value at each step, we can numerically find the momentum energy tensor and calculate the contribution to the integral (21) at each time step.

6. Conclusions

In this work, expressions of the nonlinear and linear evolution of the field have been derived. An interesting result is that the evolution of the non-relativistic field can be described by the Schrödinger-Poisson system both in super- and subhorizon regimes. This fact was verified numerically. A code has also been written to compute the nonlinear evolution of structures, and in the future it is planned to compute the spectrum of gravitational waves for realistic inflationary models.

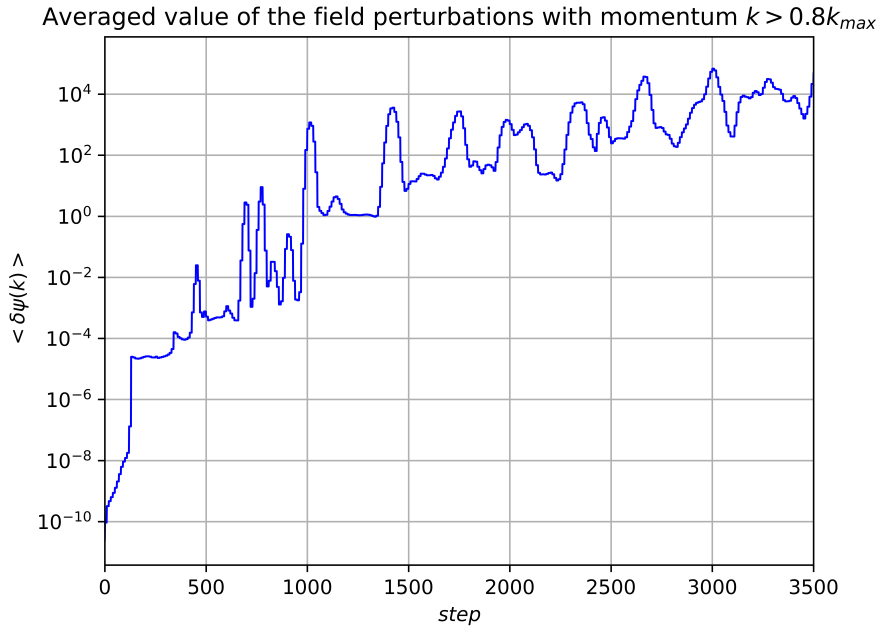

Figure 4: Averaged value of the field perturbations with momentum k>0.8k_{max}

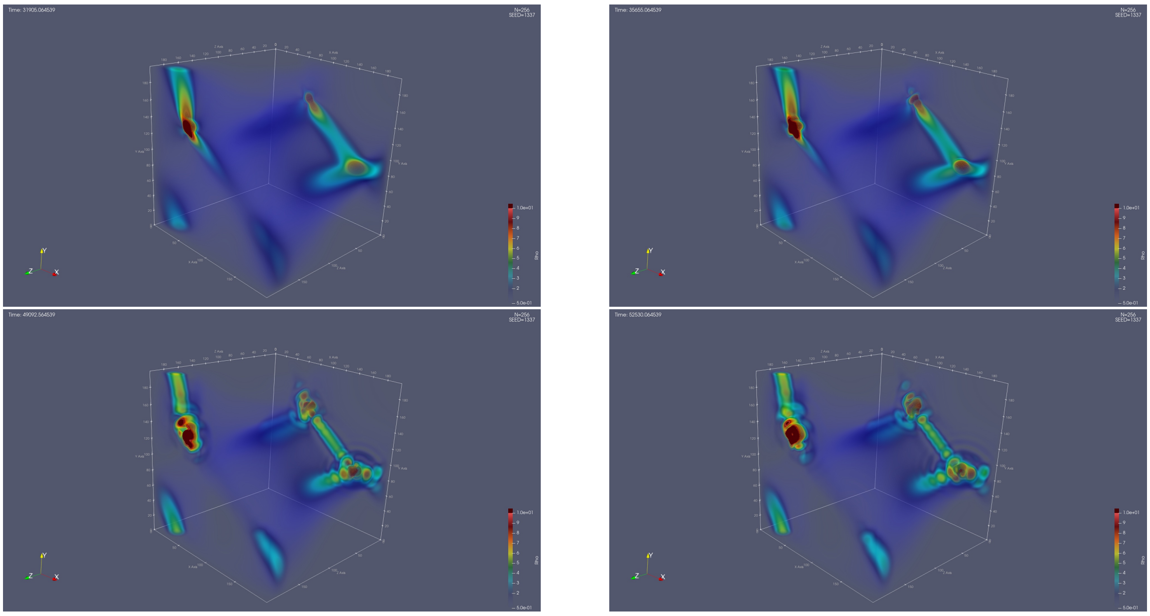

Figure 5: Evolution of two structures (preliminary)

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