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Quantum corrections to effective potentials of simplest α-attractors

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In this paper, we consider examples of applying the formalism of generalised renormalisation-group equations to obtain effective potentials for the simplest alpha-attractor models. We show how one-loop quantum corrections and RG-summed corrections affect the behaviour of the classical potential.

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1. Introduction

The formalism of effective potentials has long been firmly established in quantum field theory and has served as a significant boost for understanding various interactions. The pioneering work of Coleman-Weinberg [1] became a driver for the study of the mechanisms of spontaneous symmetry breaking and found applications in elementary particle physics, cosmology, and condensed matter physics [2–4]. However, in spite of significant simplicity of the Coleman-Weinberg formalism, the area of its applicability was limited only by renormalisable potentials [5].

Nevertheless, it turns out that the Bogoliubov-Parasyuk theorem [6] allows us to extend the scope of the study of scalar models and, in general, to study scalar potentials of arbitrary kind (even non-renormalisable) [7]. Indeed, in the case of the renormalisation-group summation, it is also required to satisfy the convergence conditions of perturbation theory so that unitarity is not violated. The generalised renormalisation-group equation turns out to be more complicated than the usual renormalisation-group equations due to the absence of the usual beta function, but in the renormalisable limit, it reproduces the Coleman-Weinberg results. Due to the application of the generalised renormalisation group (RG), it is possible to look a little further into quantum theory, which gives excellent opportunities for studying the effective potentials in various fields of physics [8].

We will here attempt to apply the formalism of generalised effective potentials to cosmological potentials, which are known as α-attractors [9–11]. These types of models have flat potentials which exponentially quickly reach a plateau at large values of the inflation field and can be used to investigate early time acceleration as well as dark energy dominated times [11, 12]. Originated from supergravity, these models provide model-independent universal predictions, satisfying the observed data from PLANCK and BICEP/Keck [13, 14]. The simplest forms of α-attractors can be represented as

$$V(\phi) = g \tanh\left(\frac{\phi}{\sqrt{6\alpha}M}\right)^n = gV_T, \tag{1}$$

(as we consider the $n = 2, 4$ cases, we will refer to them as T^2, T^4 models, respectively) and

$$V(\phi) = g\left(1 - e^{-\sqrt{\frac{2}{3\alpha}}\frac{\phi}{M}}\right)^n = gV_E \tag{2}$$

we call them E-models (we call the $n = 2, 4$ cases as E^2 and E^4 models, respectively) [11]. Here ϕ is the inflation scalar field, $M = (8\pi G)^{-1/2}$ is the reduced Planck mass, g is a scale of inflation, and α is a positive number. One can apply the formalism of the generalised RG to these kinds of potentials and study the changes occurring due to all-loop quantum corrections in the leading logarithmic approximation [7].

The first part of this work is devoted to describing the formalism of the effective potential. The method of obtaining the generalised RG-equation is given as well as transformation of its solution to an effective potential. In the second part, we apply the derived equations to evaluate the behaviour of one-loop effective potentials and the result of the numerical solution of the RG equation. We also analyse the obtained solutions from the point of view of inflationary cosmology and compare qualitatively the predictions of different T and E-models in their simplest forms. We compare the predictions of the T and E-models as in the classical case they give the same observations but can be modified due to quantum corrections.

2. Effective potential for general scalar model.

The effective potential is defined as the constant part of the effective action, i.e., as the part without derivatives. The direct way to find the effective potential $V_{eff}(\phi)$ is to sum all one-particle irreducible (1PI) vacuum diagrams using Feynman rules derived from the shifted action $S[\phi+\widehat{\phi}]$ [5]. Here ϕ is the classical field obeying the equation of motion and $\widehat{\phi}(x)$ is the quantum field. From the point of view of Feynman rules, this means that one must consider 1PI vacuum diagrams with propagators containing an infinite number of inserts $v_2(\phi) \equiv \frac{d^2V(\phi)}{d\phi^2}$, which after full summation act as a mass term depending on the field ϕ: $m^2(\phi) = gv_2(\phi)$. Feynman's rules for vertices can be found from the expansion of the potential $V(\phi+\widehat{\phi})$ in terms of the quantum field $\widehat{\phi}$. As a result, the effective potential is constructed as a perturbation expansion by the coupling constant g

$$V_{eff} = g\sum_{n=0}^{\infty}(-g)^n V_n, \tag{3}$$

where $V_0 = V$ is the initial classical potential.

We can calculate V_n contributions using the BPHZ renormalizaton procedure and Bogoliubov-Parasyuk theorem [6, 15, 16]. This theorem allows one to obtain recurrence relations connecting the leading divergences in subsequent loops. Due to the features of the $\mathcal{R}$-operations, the leading contributions are determined by one-loop diagrams. This obviously follows from the local structure of $\mathcal{R}$-operations [7]. Hence, first of all, we calculate the one-loop diagram which corresponds to V_1. Thus, for convenience, we choose dimensional regularisation $D = 4-2\epsilon$, so the one-loop quantum correction [7] is given by

$$V_1 = \frac{1}{16\pi^2}\frac{1}{4}\frac{v_2^2}{\epsilon}\left(\frac{\mu^2}{m^2}\right)^{\epsilon} \to \frac{1}{16\pi^2}\frac{v_2^2}{4}\left(\frac{1}{\epsilon}+\log\frac{\mu^2}{m^2}\right), \quad m^2 = gv_2(\phi). \tag{4}$$

An obvious but important property for the following is that the coefficient of the singular term $\sim 1/\epsilon$ coincides with the coefficient of the logarithm $\sim \log(gv_2(\phi))$ which gives the contribution to the effective potential. Consequently, this feature will allow us to obtain the whole sum of leading logarithms if we calculate the whole sum of pole terms.

Denoting the singular part of the effective potential (coefficient of the leading pole $1/\epsilon^n$) in the n-th order of perturbation theory by ΔV_n, one can obtain the following recurrence relation [7]:

$$n\Delta V_n = \frac{1}{4}\sum_{k=0}^{n-1} D_2\Delta V_k D_2\Delta V_{n-1-k}, \quad n \geq 1, \quad \Delta V_0 = V_0, \tag{5}$$

where D_2 is the second derivative by the field ϕ. Using this recurrence equation one can compute all ΔV_n algebraically.

To sum up the leading divergences, we pass to the differential equation for the sum of the following series:

$$\Sigma(z,\phi) = \sum_{n=0}^{\infty}(-z)^n \Delta V_n(\phi), \tag{6}$$

where $z = g/\epsilon$. Multiplying Eq.(5) by the factor $(-z)^n$ and summing over n from $n=2$ to ∞, we obtain the differential equation for the function $\Sigma(z,\phi)$,

$$\frac{d\Sigma}{dz} = -\frac{1}{4}\left(\frac{\partial^2}{\partial\phi^2}\Sigma\right)^2, \quad \Sigma(0,\phi) = V_0(\phi). \tag{7}$$

The resulting partial differential equation generalises the well-known RG-equation in the renormalisable case [7]. Substituting the pole term with the corresponding logarithm gives us the equation for the effective potential

$$V_{eff}(g,\phi) = \Sigma(z,\phi)|_{z\to -\frac{g}{16\pi^2}\log g v_2/\mu^2}. \tag{8}$$

This equation looks simple, but it is a complex nonlinear partial differential equation (PDE) and therefore we need to investigate it numerically. Although the equation is universal, it will look different for different interaction models because of the explicit dependence of the solution on the form of the initial potential.

3. Generalized RG-equations vs one-loop corrections

3.1 T-model

Consider the theory with the potential (1) which corresponds to the T-model. Further, to simplify equation (7), it is convenient to represent the function Σ in dimensionless variables $x = z/M^4$ and $y = \tanh^n(\phi/\sqrt{6\alpha}M)$, because the loop decomposition of this function can be represented as polynomials on tangents so that we identify

$$\Sigma(z/M^4, \tanh^n(\phi/\sqrt{6\alpha}M)) \equiv S(x,y).$$

The obtained function is in some way similar to an arbitrary function $F(\tanh(\frac{\phi}{\sqrt{6\alpha}M}))$ in the theory of chaotic inflation [9]. However, in our case this function is restricted by the RG-equation (7) and it contains an additional regularisation parameter μ^2.

Changing the variables and functions in the original equation (7), the generalised RG-equation for such potentials can be written as for $n = 2$

$$S_x = -\frac{(y-1)^2\left((3y-1)S_y + 2(y-1)yS_{yy}\right)^2}{36\alpha^2}, \tag{9}$$

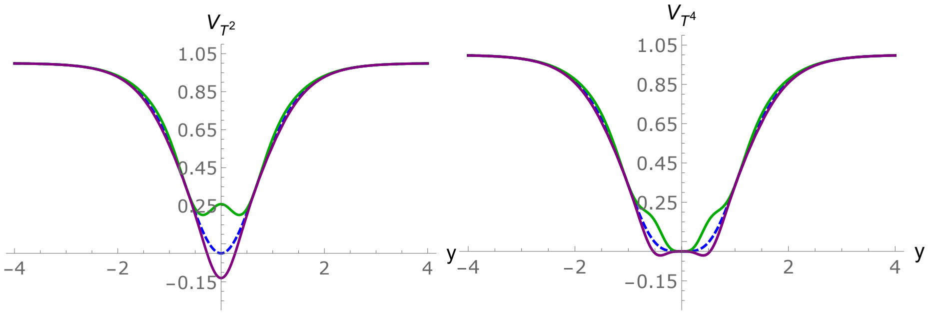
and for $n = 4$

$$S_x = -\frac{y\left(4y\left(\sqrt{y}-1\right)^2 S_{yy} + \left(5y - 8\sqrt{y} + 3\right)S_y\right)^2}{9\alpha^2}. \tag{10}$$

It can be seen that the parameter α during the calculations affects g and the transmutation parameter μ, so α can be used for some fine-tuning purposes (in this case, for convenience, we set it equal to one). The initial conditions for these PDEs are set according to the fact that the effective potential at $g = 0$ must satisfy the classical potential as well as the asymptotic conditions:

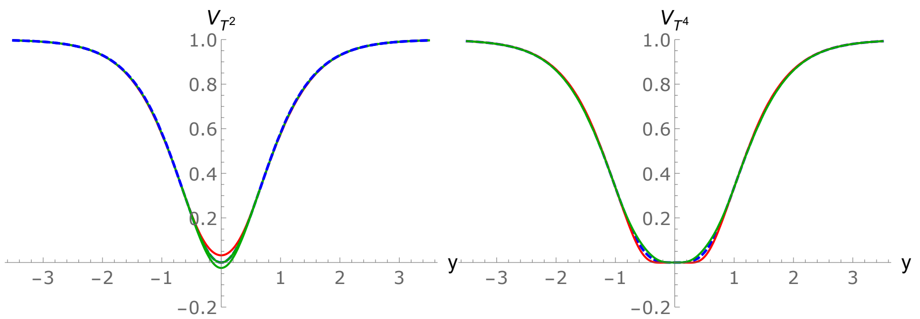
$$S(0,y) = y,\ S(x,0) = 1,\ S^{(0,1)}(x,0) = 0. \tag{11}$$

Unfortunately, these equations cannot be solved analytically (moreover, one can notice that such a system of PDE is stiff [17]) but one can use numerical methods to find solutions. We present our comparisons of classical potentials, one-loop effective potentials and all-loop effective potentials on Figures 1 and 2. One can notice that one-loop corrections distort the initial classical potential near $\phi = 0$ causing spontaneous symmetry breaking meanwhile all-loop corrections smooth of the effect



(a) T^2-model **(b)** T^4-model

Figure 1: T-models (blue line) and their one-loop corrections with $\mu \ll 1$ (green line) and $\mu \gg 1$ (purple line) and $g \sim 1$. Here y denotes a normalized scalar field.



(a) T^2-model **(b)** T^4-model

Figure 2: T-models (blue line) and their all-loop corrections with $\mu \ll 1$ (green line) and $\mu \gg 1$ (purple line) and $g \sim 1$ (all values are the same as on Fig.1). Here y denotes a normalized scalar field. All-loop potential behaviour is smoother than one-loop effective potential.

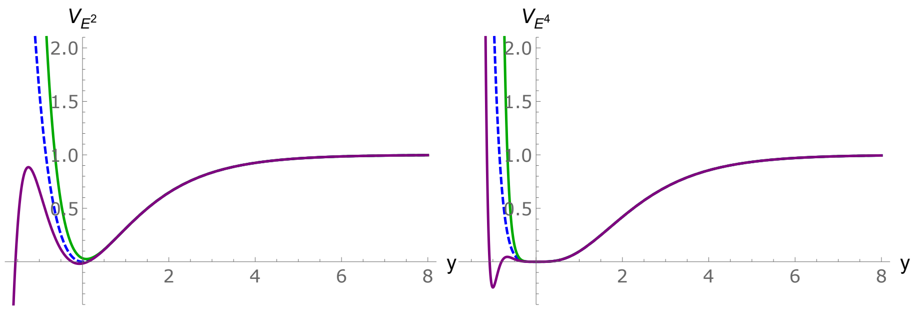
in both considered T-model cases. The plateau-like behaviour in quantum-corrected potentials is preserved as expected [10]. It can also be seen that when the parameter μ increases with respect to the Planck mass, an anti-de Sitter landscape arises (in the opposite case, de Sitter behaviour arises) at $V(\phi = 0)$.

3.2 E-model

Here we perform in the same way as above for the theory with potential (2), which corresponds to the E-model. As before, to simplify equation (7), represent the function Σ in dimensionless variables $x = z/M^4$ and $y = (1 - e^{\sqrt{\frac{2}{3\alpha}}\frac{\phi}{M}})^n$. Then

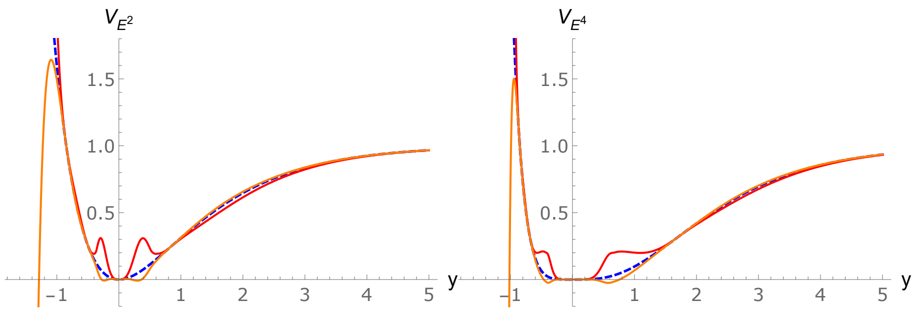
$$\Sigma\left(z/M^4, (1 - e^{\sqrt{\frac{2}{3\alpha}}\frac{\phi}{M}})^n\right) \equiv S(x, y).$$

Changing the variables and functions in the original equation (7), the generalised RG-equation for such potentials can be written as for $n = 2$



(a) E^2-model **(b)** E^4-model

Figure 3: E-models (blue line) and their one-loop corrections with $\mu \ll 1$ (green line) and $\mu \gg 1$ (purple line) and $g \sim 1$. Here y denotes a normalized scalar field.



(a) E^2-model **(b)** E^4-model

Figure 4: E-models (blue dashed line) and their all-loop corrections with $\mu \ll 1$ (orange line) and $\mu \gg 1$ (red line) and $g \sim 1$. Here y denotes a normalized scalar field. All-loop potential behaviour is smoother than one-loop effective potential.

$$S_x = -\frac{4}{9\alpha^2}\left(2y\left(\sqrt{y}+1\right)^2 S_{yy} + \left(2y+3\sqrt{y}+1\right) S_y\right)^2 \tag{12}$$

and for $n = 4$:

$$S_x = -\frac{16}{9\alpha^2}\left(\sqrt[4]{y}+1\right)^2 y\left(\left(4\sqrt[4]{y}+3\right) S_y + 4y\left(\sqrt[4]{y}+1\right) S_{yy}\right)^2, \tag{13}$$

so the initial conditions are given by

$$S(0, y) = y, S(x, 0) = 0, S^{(0,1)}(x, 0) = 0. \tag{14}$$

Here lower indices denote corresponding derivative. These equations are also difficult to solve using analytical tools due to nonlinearity of PDEs, but numerical methods are available.

The results of numerical solutions of the equations are shown in Figures 3 and 4. In the case of the E-model, we notice the appearance of additional minima also near the zero value of the field. The type of the potential barrier for negative fields is also modified: if in the case of the usual E-model it was an infinite barrier, in the case of the RG-summed effective potential it turns into a discontinuity (this kind of behaviour of the exponential function was found in [7]). We can observe that the behaviour of the plateau at infinity is not affected by quantum corrections. Among other things, for a wide range of values of the dimensional transmutation parameter the slow-roll behaviour of the inflaton field does not change.

4. Conclusion

In this paper, we have succeeded in obtaining generalised renormalisation-group equations for a class of simplest potentials of α-attractor type often encountered in inflationary cosmology. We showed that the plateau-type behaviour is preserved in E and T-models, but the shape of the potentials near zero can change significantly. For example, for T-models symmetric with respect to the scalar field sign change, we observe spontaneous symmetry breaking from a certain value of the dimensional transmutation parameter and/or an uplifting of the potential near the zero value of the scalar field. For asymmetric E-models, one can also observe a rise of the potential near zero and the appearance of minima separated by a sharp rise. The infinite potential barrier for such models is replaced by a field maximum at certain values of the dimensional transmutation parameter. In spite of such a significant modification of the initial potential, for a wide region of values of the parameter μ one can expect that the Hubble flow does not change much.

The RG-equation can be used to consider numerous types of potentials and with its help it is possible, for example, to consider field theories and study effects related to spontaneous symmetry breaking in various physical models. For instance, it would be interesting to study multi-field non-renormalisable theories to explore hybrid inflation, which are quite interesting from the point of view of inflationary physics [18].

Acknowledgements

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