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Lattice QCD Calculation of TMD Physics

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Recently years have seen significant progress in the first-principles calculation of TMD physics from lattice QCD. We will describe the theoretical method for calculating both quark and gluon TMDs, which has been developed under the framework of large-momentum effective theory. Then we review its most recent applications to the non-perturbative quark TMDs and their rapidity evolution anomalous dimension, i.e., the Collins-Soper kernel, and discuss the control and improvement of systematic uncertainties in such calculations.

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1. Introduction

The transverse-momentum-dependent parton distributions (TMDs) are the key observables for a 3D tomography of the nucleon in the momentum space. They are among the top targets for high-energy scattering experiments at Fermilab, CERN, Jefferson Lab, RHIC and the future Electron-Ion Collider. Over the past two decades, significant progress has been made in the global fitting of quark TMDs from the semi-inclusive deep inelastic scattering (SIDIS) and Drell-Yan processes. See the review in Ref. [1]. The main goal of these experiments is to extract the intrinsic non-perturbative TMDs at parton transverse momentum $|k_\perp| \sim \Lambda_{\rm QCD}$, the region that is most relevant for nucleon structure, but the uncertainties in this domain still remains.

In recent years, there have been growing efforts in the first-principles calculation of TMDs from lattice quantum chromodynamics (QCD). Since TMDs are defined by quark and gluon correlators involving staple-shaped Wilson lines on the light-cone, it is impossible to directly compute their matrix elements on the Euclidean lattice due to the real-time dependence. Hence, initial efforts concentrated on the ratio of TMD moments, which are weighted averages in the longitudinal momentum fraction x space and remain independent of time [2–7]. Then, a breakthrough was made by Large-Momentum Effective Theory (LaMET) [8–10] to calculate the x-dependence of parton distribution functions (PDFs), which has undergone profound development to make precision controlled calculations [11, 12] nowadays. The LaMET approach has also motivated the study of TMDs from lattice QCD [13–30], leading to the calculations of the TMD evolution kernel [31–39], the TMD soft function [32, 33, 38], and their $(x, k_\perp)$ dependence [40, 41].

In this review, I will introduce the LaMET formalism for TMD calculation, and discuss its recent applications.

2. TMD Definition

The TMDs in SIDIS and Drell-Yan processes involve a collinear part (beam function B) and soft part (soft function S). For example, the quark TMD can be schematically defined as:

$$f_i(x, \mathbf{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z_{\rm UV}(\epsilon, \mu, \zeta) \lim_{\tau\to 0} \frac{B_i(x, \mathbf{b}_T, \epsilon, xP^+, \tau)}{\sqrt{S^q(\mathbf{b}_T, \epsilon, \tau)}}, \tag{1}$$

where i is the parton flavor index, x is the longitudinal momentum fraction, $\mathbf{b}_T$, is the Fourier conjugate to the transverse momentum $\mathbf{k}_T$, and P^+ is the light-cone momentum of the target nucleon. ϵ and τ are the regulators for the ultraviolet (UV) and rapidity divergences, and μ and ζ are the corresponding renormalization scales, with ζ also being called the Collins-Soper scale.

The beam function B is defined from the hadronic matrix element of a staple-shaped quark Wilson line correlator, shown in Fig. 1, while a soft function is defined from the vacuum matrix elements of a Wilson loop operator that involves two lightlike directions. Both the beam and soft functions include the so-called rapidity divergences, which can be regulated by τ. The choice of τ can be the large rapidity y_B of a spacelike Wilson line that is close to the light-cone, whose direction is given by

$$n_B^\mu = (n_B^+, n_B^-, n_B^\perp) = (-e^{2y_B}, 1, 0_\perp), \tag{2}$$

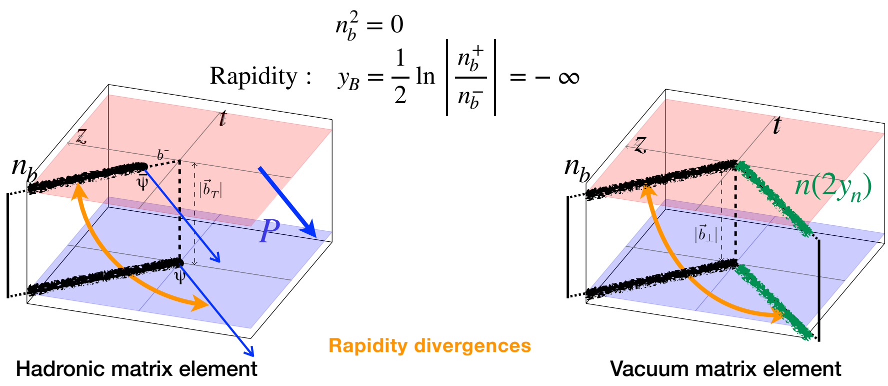

Figure 1: Beam and soft functions on the light-cone.

and $y_B \to -\infty$ corresponds to the light-cone limit or $\tau \to 0$.

Due to the real-time dependence of light-cone, neither the beam or soft function can be directly simulated on the lattice. However, LaMET has provided a framework to relate the light-cone PDFs from time-independent lattice observables [42–44]. Thanks to years of development, the lattice calculation of PDFs has entered the era of precision calculation [11, 12, 45].

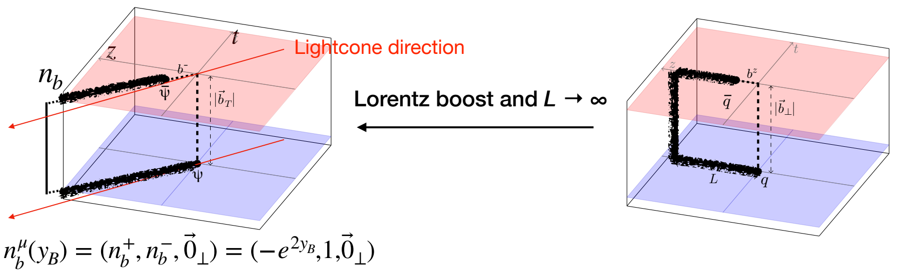

Figure 2: Spacelike beam function and the (static) quasi beam function.

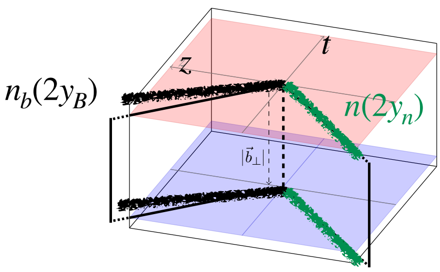

Figure 3: Spacelike soft function.

Within the LaMET framework, the TMDs can also be calculated through a factorization formula [16, 18, 19, 24]. The beam function with the off-the-light-cone rapidity regulator can be approximated by a static staple-shaped quark correlator in a highly boosted hadron state, see Fig. 2, enabled by the principle of Lorentz invariance [24]. However, since the soft function involves two close-to-the-light-cone directions, see Fig. 3, they cannot be related to any static Wilson loop operator on the lattice through Lorentz transformation. Nevertheless, at large rapidity difference $y_B - y_n$, the soft function can be expanded as

$$\lim_{y_n - y_B \to \infty} S_q(b_T, \mu, 2(y_n - y_B))) = S_r(b_T, \mu)\, e^{-2(y_n - y_B)\gamma_\zeta(b_T, \mu)} + O(e^{-2(y_n - y_B)}), \tag{3}$$

where γ_ζ is the rapidity anomalous dimension, and S_r is called a reduced soft function [18] that can be extracted from a meson form factor defined as

$$F_\pi(P^z, b_T) = \langle \pi(-P) | j_1(b_T) j_2(0) | \pi(P) \rangle, \tag{4}$$

where $\pi(P)$ is a pion state with momentum P, and $j_{1,2} = \bar\psi \Gamma_{1,2} \psi$ are light-quark currents. At large momentum $P^z \gg \Lambda_{\rm QCD}$, the form factor can be factorized as

$$F_\pi(P^z, b_T) = S_r(b_T, \mu) \int dx dx'\; H(x, x', \mu) \Phi^\dagger(x, b_T, P^z, \mu) \Phi(x', b_T, P^z, \mu), \tag{5}$$

where Φ is a quasi TMD wave function that is defined by a pion to vacuum matrix element and can be directly simulated on the lattice.

With the quasi TMD and reduced soft functions, we can establish the factorization formula that relate them to the light-cone TMD [16, 18, 19, 24],

$$\begin{aligned} \frac{\tilde f^{[s]}_{i/p}(x, \mathbf{b}_T, \mu, \tilde P^z)}{\sqrt{S_r(b_T, \mu)}} &= C\big(\mu, x\tilde P^z\big)\; \exp\left[\frac{1}{2}\gamma_\zeta(\mu, b_T) \ln \frac{(2x\tilde P^z)^2}{\zeta}\right] \\ &\quad \times f^{[s]}_{i/p}(x, \mathbf{b}_T, \mu, \zeta) \left\{1 + O\left[\frac{1}{(x\tilde P^z b_T)^2}, \frac{\Lambda^2_{\rm QCD}}{(x\tilde P^z)^2}\right]\right\}, \end{aligned} \tag{6}$$

where s stands for the Dirac spin structure, and C is a perturbative matching coefficient which is free from mixing with the gluon or other quark flavors.

The above factorization formula allows us to compute:

- The Collins-Soper kernel [13, 15];
- The flavor separation of TMDs;
- The spin-dependence of TMDs [21];
- The full TMD and TMD wave function dependence on x and b_T;
- Twist-3 PDFs from the small-b_T expansion of TMDs [22].
- Sub-leading TMDs [25].

3. Lattice applications

Since its proposal, the quasi-TMD approach within the LaMET framework has been applied to the lattice calculations of the Collins-Soper kernel, the TMD soft function, and the full (x, b_T) dependence of TMD PDF and wave function. Among them, the Collins-Soper kernel has been the most studied by several lattice groups [31–39].

The master formula for the lattice extraction of the Collins-Soper kernel is [15]

$$\gamma_\zeta^q(\mu, b_T) = \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{C(\mu, xP_2^z) \int db^z\, e^{ib^z xP_1^z}\, \tilde{Z}'(b^z,\mu,\tilde{\mu})\tilde{Z}_{\rm UV}(b^z,\tilde{\mu},a)\tilde{W}(b^z,\mathbf{b}_T,a,\eta,P_1^z)}{C(\mu, xP_1^z) \int db^z\, e^{ib^z xP_2^z}\, \tilde{Z}'(b^z,\mu,\tilde{\mu})\tilde{Z}_{\rm UV}(b^z,\tilde{\mu},a)\tilde{W}(b^z,\mathbf{b}_T,a,\eta,P_2^z)} \times \left\{1 + O\left[\frac{1}{(x\tilde{P}^z b_T)^2}, \frac{\Lambda_{\rm QCD}^2}{(x\tilde{P}^z)^2}, \frac{1}{((1-x)\tilde{P}^z b_T)^2}, \frac{\Lambda_{\rm QCD}^2}{((1-x)\tilde{P}^z)^2}\right]\right\}, \tag{7}$$

where $\tilde{W}$ is the bare lattice matrix element of the quasi-beam or wave function, $\tilde{Z}_{\rm UV}$ is the lattice renormalization factor, and $\tilde{Z}'$ converts the lattice renormalization scheme to the continuum $\overline{\rm MS}$ scheme.

The sources of systematic uncertainties include: 1) unphysical quark masses; 2) lattice renormalization; 3) operator mixing under lattice regularization; 4) Fourier transform; 5) perturbative matching; 6) extraction of the Collins-Soper kernel from the quasi-TMD ratios. Recently, a lattice QCD calculation of the kernel [39] used quark masses corresponding to a close-to-physical value of the pion mass, with next-to-next-to-leading logarithmic (NNLL) matching to TMDs from the corresponding quasi-TMD, and includes a complete analysis of systematic uncertainties arising from operator mixing. The simulation of quasi-TMD wave function matrix elements is much less expensive than the quasi-TMD PDF, as it only involves two-point correlation functions. Thus, using the same statistics one achieve better statistical precision and have more stable Fourier transform. Besides, the physical pion mass helps better suppress the power corrections in the factorization formula.

The perturbative matching correction is an important source of error. It can be derived as

$$\delta\gamma_q(x, P_1^z, P_2^z, \mu) = \frac{1}{\ln(P_1^z/P_2^z)} \left[\ln \frac{C(xP_2^z,\mu)}{C(xP_1^z,\mu)} + x \to \bar{x}\right], \tag{8}$$

where $\bar{x} = 1 - x$. The matching coefficient can be resummed as

$$C(xP^z, \mu) = C(xP^z, 2xP^z) \exp\left[K(\mu, 2xP^z)\right] . \tag{9}$$

At NNLL, the matching $C(xP^z, 2xP^z)$ is truncated at $O(\alpha_s)$ and K is at $O(\alpha_s^2 \ln(\mu/(2xP^z))$.

In addition, it was found that the power correction is significant when the condition $xP^z b_T \gg 1$ is not satisfied, which is required for TMD factorization. When $xP^z \gg \Lambda_{\rm QCD}$ and $\bar{x}P^z \gg \Lambda_{\rm QCD}$, we have

- If $xP^z b_T \gg 1$ and $\bar{x}P^z b_T \gg 1$, we have TMD factorization;
- If $xP^z b_T \ll 1$ and $\bar{x}P^z b_T \ll 1$, we have collinear factorization;
- If $xP^z b_T \sim 1$ and $\bar{x}P^z b_T \sim 1$, we have collinear factorization but with calculable power corrections. In this region, we can compute the fixed-order expansion of the matching coefficient;

Under this argument, it was proposed to use the unexpanded matching coefficient [39],

$$C^{\rm uNNLL}(p^z, b_T, \mu) = C^{\rm uNLO}(p^z, b_T, 2p^z) \exp\left[K^{\rm NNLL}(p^z, 2p^z\right], \tag{10}$$

where

$$C^{\mathrm{uNLO}}(p^z, b_T, \mu) = C(p^z, \mu) + \delta C(p^z, b_T), \qquad \lim_{p^z b_T \to \infty} \delta C(p^z, b_T) = 0\,. \tag{11}$$

In this way, when $p^z b_T \gg 1$, the matching coefficient smoothly approaches the TMD limit. The b_T dependent matching coefficient can be extracted from Refs. [16, 27], which is plotted in Fig. 4. The unexpanded power correction shows a good cancellation of the unphysical imaginary part of the kernel in the small-b_T region.

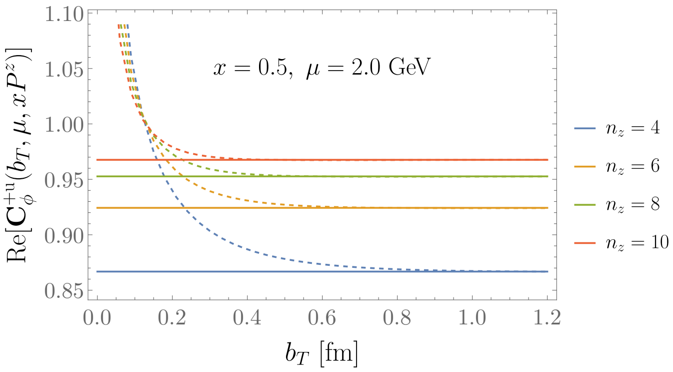

Figure 4: The unexpanded matching coefficient in $xP^z b_T$.

By taking into account of the uNNLL matching coefficient, the Collins-Soper kernel is obtained from the ratio of quasi-TMD wave functions, which is shown in Fig. 5. Since the kernel is independent of x, the desired result should be constant in x in the moderate region. However, as shown in Fig. 5, the slight dependence indicates the higher-order effects and power corrections. Nevertheless, within our statistical errors, the bands are quite flat, which allows us to have a reliable extraction of the Collins-Soper kernel.

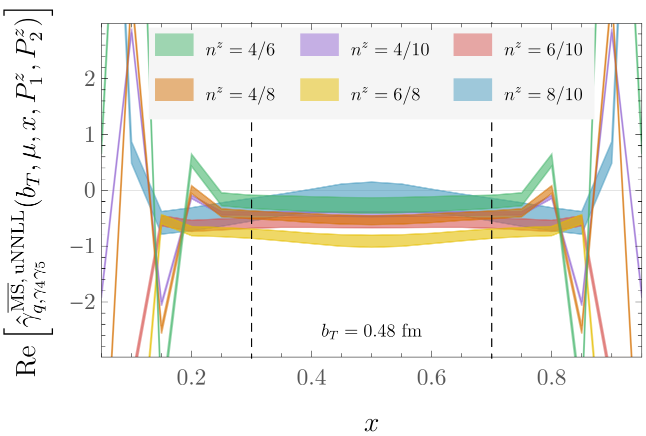

Figure 5: The Collins-Soper kernel extracted as a function of x.

The final result is compared to the global analysis as Fig. 6. As one can see, the lattice results have reached a precision that can begin to differentiate the global fits. This is an encouraging step towards systematic control to have a greater impact on the experiments.

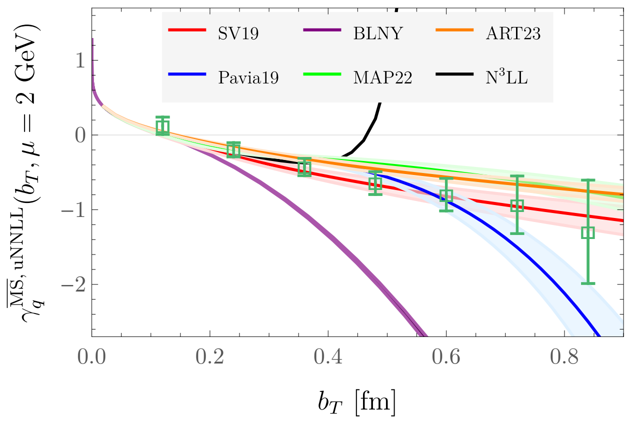

Figure 6: Collins-Soper kernel with uNNLL matching in b_T space (green squares) compared to phenomenological parameterizations of experimental data in Refs. [46–50] labeled BLNY, SV19, Pavia19, MAP22, and ART23, respectively, as well as perturbative results from Refs. [51–53] labeled N^{3}LL.

Another development is the lattice calculation of the reduced soft function [32, 33, 38], using the meson form factor method. This effort has been making progress in the past few years, and the most recent calculation was done using multiple lattice ensembles at unphysical valence quark masses, at next-to-leading order (NLO) accuracy, which is shown in Fig. 7.

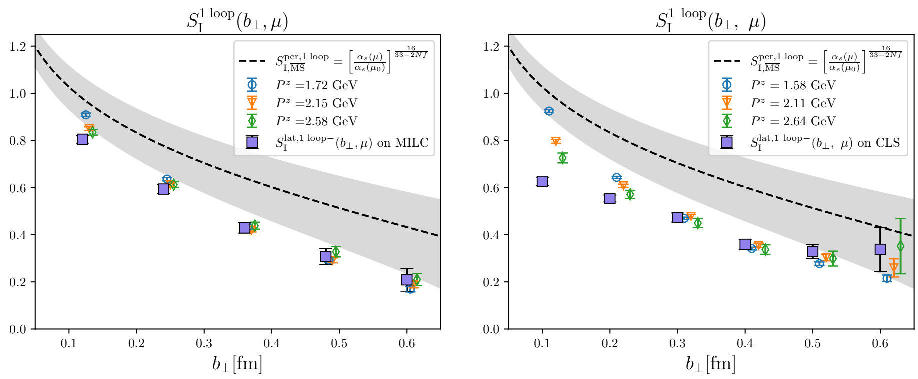

Figure 7: A recent lattice QCD calculation of the reduced TMD soft function at NLO accuracy [38].

Since there is no experimental result on the soft function, the only comparison that can be done is with the perturabtive prediction which is only valid in the small-b_T region. However, due to the enhanced power correction and discretization effects, the systematics in this region is underestimated, and it is not surprising that the lattice results are not consistent with perturbation theory. Future efforts should focus on using finer lattice that allows for a window where one can find the agreement between lattice and perturbation theory.

Finally, with the quasi beam function and reduced soft factor, one can eventually obtain the full kinematic dependence of the TMDs. The first such attempt was made in a recent calculation [40] for the isovector unpolarized proton TMD, which is shown in Fig. 8. The lattice results show some qualitative agreement with the recent global analysis, but the systematics still need to be under better control.

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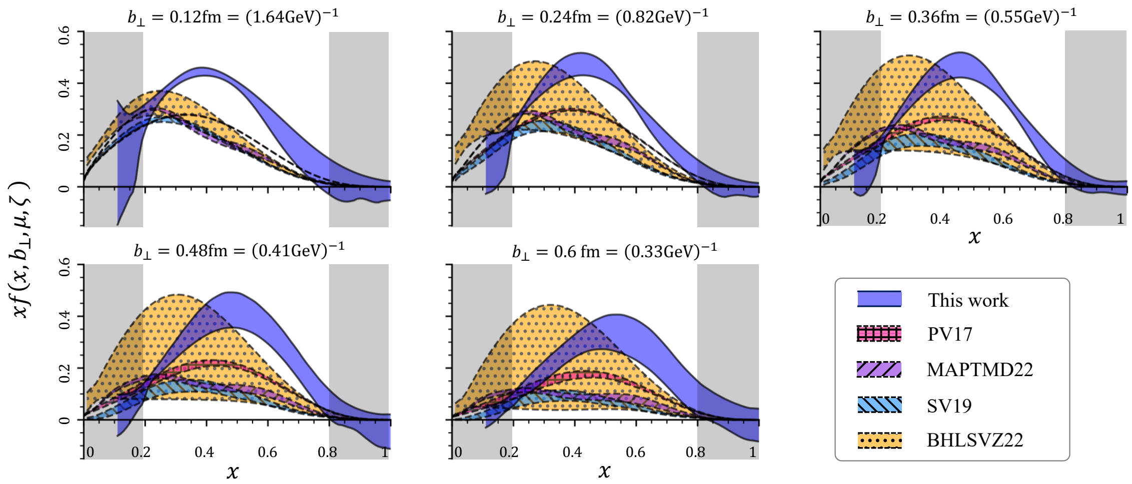

Figure 8: A recent lattice QCD calculation of the isovector unpolarized proton TMD at NLO accuracy [40].

4. Conclusion

In summary, we have reviewed the LaMET framework for the lattice QCD calculation of TMDs, which should cover all leading-power TMDs of all spin structures. This method has been applied to the calculation of the Collins-Soper kernel, which has undergone significant development over the past four years, with promising improvement of the systematic uncertainties. There are also first calculations of the soft function and the full kinematic dependence of the TMDs, leading us to one step closer to the complete 3D tomography of the nucleon from lattice QCD.

Last but not the least, it is worth mentioning that very recently, there is a new proposal to calculate the PDFs and TMDs from pure correlators fixed in the Coulomb gauge [54, 55], which can significantly improve the signal-to-noise ratio and simplify the lattice renormalization procedure.

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