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Can superfluid stars be mistaken for black holes in astronomical observations?

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We consider a general relativistic model of a self-interacting complex scalar field with logarithmic nonlinearity motivated by studies of laboratory superfluids and Bose-Einstein condensates. Spherically-symmetric gravitational equilibria are shown in this model, which do not have event horizons but which are regular, singularity-free and asymptotically flat. They can be thus interpreted as compact stars whose stability against gravitational collapse is enhanced not only by the Heisenberg uncertainty principle but also by the property of superfluidity itself, their "darkness" comes naturally as a result of suppressed dissipative excitations. Such objects do not obey any absolute upper mass limit of a Tolman-Oppenheimer-Volkoff type, while their relativisticity and effective compactness values are comparable to those of black holes. Their spatial density distribution drops abruptly (at the Gaussian-like rate), which can be mistaken in realistic astronomical observations for the presence of an exact material surface. We therefore present logarithmic superfluid stars as dark compact objects and black hole mimickers.

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Probably the most important problem in modern astrophysics, both theoretical and experimental, is the final evolution stage of supermassive compact objects (SMO) with the core mass of more than three solar masses, and the nature of dark compact objects (DCO). It is usually believed that black holes, i.e., objects with event horizons, are a strong candidate for the final stage, but are they the only possible one? The related question is how to prove in practice the presence of an event horizon, while being at an astronomically large distance from it. Yet another related, even more general, question is how to define black hole horizons, assuming the underlying quantumness of our world. These questions are still open nowadays, for a number of reasons:

First, from a practical observational point of view, the presence of a horizon (or any null surface in general) is difficult to establish with complete certainty, due to the large complexity of high-energy phenomena surrounding SMO's cores, and because of significant observational errors which occur [1–3]. Therefore, one usually relies on indirect arguments such as theoretical upper mass limits for horizonless massive compact objects; Tolman-Oppenheimer-Volkoff (TOV) being one of them [4], assuming that any object whose mass goes above this limit collapses under an event horizon. This however causes interpretations of the SMO/DCO observational data to be model-biased, which can be risky, because those objects exist in extreme conditions where new physics and hitherto theoretically unaccounted effects can pop up without any notice. On the other hand, from a theoretical point of view, all models are necessarily based on assumptions and approximations. For example, the above-mentioned TOV limit assumes that stars are equilibrium configurations of the classical matter modelled by barotropic perfect fluid [5]. This is a rather strong assumption because it largely ignores quantum nature of all matter. On the other hand, quantum effects are likely to occur as the density grows – because inter-particle distances decrease, therefore, particles' de Broglie waves inevitably begin to overlap at some stage.

Moreover, a deeper layer of theoretical issues exists when it comes to the fundamental (quantum-mechanical) description of black holes. The definition of a black hole *per se* is based on the notion of the event horizon, which is a special case of a null hypersurface. The latter is a four-dimensional hypersurface that has null tangent vectors, i.e., vectors that lie completely within it. In the classical physics approach, one defines this surface as

$$n_\mu n^\mu = 0, \tag{1}$$

where n^μ is a tangent four-vector defined on a pseudo-Riemannian four-dimensional manifold. However, in (quantum) reality the Heisenberg uncertainty principle is known to disallow "exact" particle trajectories with simultaneously measurable values of dynamically conjugated variables, but allows only the mean values thereof. This occurs due the quantum-mechanical requirement that dynamical variables must be replaced by their operators acting in a suitably defined Hilbert space. Using arguments of such kind, formula (1) must be changed to its averaged version:

$$\langle \hat{n}_\mu \rangle \langle \hat{n}^\mu \rangle = 0, \tag{2}$$

where $\hat{n}^\mu$ is a quantum operator corresponding to the tangent vector, and the angle brackets denote a mean value computed with respect to a quantum state vector $|\Psi\rangle$.

The validity of formula (2) requires conjugated operators $\hat{n}^\mu$ and $\hat{n}_\nu$ to be simultaneously measurable, which means that their commutator $\left[\hat{n}_\mu, \hat{n}^\nu\right] \sim [\hat{n}^\mu, \hat{n}^\nu]$ must be zero, at all times and in all local inertial frames of reference. But is it so in general?

To clarify this, let us work in the position representation of quantum mechanics where the operators corresponding to tangent vectors are represented by derivatives. Local Lorentz symmetry requires those derivatives to be covariant in four-dimensional spacetime, therefore $\hat{n}_\mu \sim \nabla_\mu$, $\hat{n}^\mu \sim \nabla^\mu$, where ∇_μ is a covariant derivative operator. The question of commutativity of tangent vector operators thus reduces to the commutativity of covariant derivatives. It is known that the commutator of covariant derivatives is a function of spacetime curvature, therefore

$$\left[\hat{n}_\mu, \hat{n}^\mu\right] |\Psi\rangle \sim \left[\nabla_\mu, \nabla^\mu\right] |\Psi\rangle = f(\mathrm{R}; \Psi), \tag{3}$$

where R is a Riemann curvature tensor. The right hand side of this formula equals zero in any inertial frame of reference if and only if it is a four-dimensional Lorentz scalar. The latter, however, is not allowed by probabilistic interpretation of a quantum-mechanical wavefunction – because the wavefunction must be normalizable on a three-dimensional space-like surface, therefore it must transform adjointly to a three-dimensional spatial volume.

In other words, the commutator $\left[\hat{n}_\mu, \hat{n}^\nu\right]$ does not vanish in a general case. According to the generalized Heisenberg principle,

$$\mathrm{Var}(n_\mu)\, \mathrm{Var}(n^\nu) \geq \frac{1}{4}\left|\left[\hat{n}_\mu, \hat{n}^\nu\right]\right|^2 > 0, \tag{4}$$

where $\mathrm{Var}(X) \equiv \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2$ is variance of a physical variable described by the operator $\hat{X}$, this implies that four-vectors n_μ and n^ν cannot be measured simultaneously. In (quantum) reality, it means that not only null worldlines but also null hypersurfaces, black hole horizons being a special case thereof, become observationally ill-defined objects: even under conditions of an ideal measurement, one cannot measure their exact location in space and time. This fundamental uncertainty supplements the observational errors mentioned above.

In view of the above-mentioned theoretical and/or observational issues with black holes, it is not surprising that there are alternative answers to the questions asked at the beginning of this paper. These horizonless alternatives, usually referred to as black hole mimickers (BHM) models [6, 7], do not exclude the existence of black holes *per se*, but they establish a perspective on the problem, which is more compatible with the laws of quantum physics.

One of the popular examples of DCO/BHM models are boson stars, which are equilibrium self-gravitating configurations of self-interacting scalar fields, to mention just a very few works on the theme [8–12]. Their advantage is that they *a priori* obey the Heisenberg uncertainty principle enhancing their stability against the gravitational collapse and formation of null hypersurfaces, which can be understood by analogy with the (absence of) classical trajectories in quantum mechanics. The drawback of conventional boson star models is that their effective masses and radii are estimated as $R^{(\mathrm{BS})} \sim 1/m$, $M^{(\mathrm{BS})} \sim M_p^2/m$, where m is the rest mass of a constituent boson and $M_p = \sqrt{\hbar c/G}$ is the Planck mass, which seriously restricts astrophysical applications: in order to obtain the value of mass of a typical star, not to mention supermassive objects in the centers of galaxies, one must assume m to be less than an electronvolt. However, long-lived scalar particles of such mass are neither vetted by the modern Standard Model of elementary particles, nor observed in the Earth's collider experiments known to date.

This means that one needs to chose a different quantum object – which would not only obey the Heisenberg uncertainty but also be free from conflicts with the Standard Model. Such an object has been suggested for SMO/DCO/BHM modelling purposes relatively recently [13]. It is based on the idea of the superfluid (SF), which is a collective quantum state endowed with the suppression mechanism of dissipative fluctuations due to the Landau(-Bogoliubov) "roton" excitation spectrum.

To begin with, superfluid star models automatically account for the Heisenberg uncertainty principle, due to their dynamics described by a Schrödinger-type quantum equation. Furthermore, their stability against gravitational collapse is enhanced not only by quantum uncertainty, but also by the property of superfluidity itself: an absence of viscosity and friction prevents fluid parcels from decreasing their relative momenta and adhering to each other. The "darkness" of SF stars comes naturally as a result of suppressed dissipative excitations.

Moreover, unlike boson star models, superfluid models are *ab initio* free from conflicts with the Standard Model because they do not require any exotic particles: superfluids can be formed from almost any bosonized system of known particles. For example, the superfluid phase can occur inside the cores of neutron stars, as soon as neutrons become bosonized by gravitational attraction when it overtakes the Pauli repulsion [14, 15].

Let us thus define the relativistic SF star model through the Lagrangian (in units $c = 1$):

$$\mathcal{L} = \frac{1}{16\pi G}\mathcal{R} - \frac{1}{2}\nabla_\mu \phi^* \nabla^\mu \phi - V(\phi,\phi^*), \quad V(\phi,\phi^*) \equiv -b\,|\phi|^2 \left[\ln\left(|\phi|^2/a\right) - 1\right], \tag{5}$$

where $\mathcal{R}$ is a Ricci scalar, a star denotes the complex conjugate, and minimally coupled complex scalar field ϕ represents the relativistic effect of superfluidity, because the scalar field potential in eq. (5) is chosen in such a way as to obtain the logarithmic nonlinearity in the corresponding field equation, $\nabla_\mu \nabla^\mu \phi = 2\frac{\partial}{\partial \phi^*} V(\phi,\phi^*) = -2b \ln(|\phi|^2/a)\,\phi$, which is a Lorentz-covariant analogue of the logarithmic Schrödinger equation for the condensate function. This kind of nonlinearity is motivated by models of laboratory superfluids and Bose-Einstein condensates where it takes into account vacuum effects and multi-body interactions [16–19], while going beyond the two-body approximation and even the perturbative approach itself [20]. It should be emphasized that the potential in eq. (5) is the simplest possible one, because condensate equations for the quantum liquids we know of contain not only logarithmic but also polynomial nonlinear terms [17, 19].

The logarithmic Schrödinger equations themselves have been extensively studied in the mathematical literature, to mention only very recent reports [21–33]. The relativistic logarithmic wave equations in the fixed Minkowski spacetime have been examined in some details too [34–38].

Furthermore, the physical meaning of parameters a and b differs from the conventional relativistic scalar field theories, such as ϕ^4 [11]. Their values do not come from the physics of point relativistic particles, but are determined by the dynamics of quantum Bose liquids, the latter being the macroscopic extended objects described by collective degrees of freedom. For example, coupling b comes about a linear function of the wave-mechanical temperature T_Ψ, which is defined as a thermodynamic conjugate of the Everett-Hirschman's quantum information entropy [39]:

$$S_\Psi = -\langle \Psi | \ln(|\Psi|^2/\bar{\rho}) | \Psi \rangle = -\int |\Psi|^2 \ln(|\Psi|^2/\bar{\rho})\, d^3\vec{x}, \tag{6}$$

where $\Psi = \Psi(\vec{x}, t)$ is a condensate wavefunction, constant $\bar{\rho}$ is the decoupling density value, the integral is taken over the volume occupied by the liquid; in this formula and below, we adopt the units where the Boltzmann constant equals one. In other words, b is not an *ab initio* fixed parameter, but it is related to a quantum-mechanical notion:

$$b \sim T_\Psi - T_\Psi^{(0)}, \tag{7}$$

where $T_\Psi^{(0)}$ is some reference value.

Let us assume that the simplest equilibrium configurations of the system (5) are a subset of spherically symmetric and time independent solutions of Einstein field equations. For such solutions, the spacetime interval can be assumed in a static-observer spherically symmetric form. Due to the symmetry of the problem, we assume our scalar to be spherically symmetric and stationary $\phi(r,t) = e^{-i\omega t}\Phi(r)$, where $\Phi(r)$ is a real-valued function. Correspondingly, our field equations result in a set of ordinary differential equations for the dimensionless spatial variable $x = r/L$, with parameters $L = 1/\sqrt{b}$, $k = 4\pi G a$, and $\Omega = \omega/\sqrt{2b}$. Using the shooting method adopted from [9, 11], these differential equations can be searched for regular (singularity-free) finite-mass solutions, assuming that SF-associated scalar field ϕ vanishes at spatial infinity and has no nodes. The total mass is derived from the asymptotic value of the mass function at large x: $M = \mathcal{M}(\infty)L/G = \mathcal{M}(\infty)/(G\sqrt{b})$.

The outcome of the above-mentioned numerical searches is that equilibrium configurations of relativistic superfluid coupled to gravity do exist, which are localized, horizonless, have finite mass and no spacetime singularities. The SF-associated scalar field has no nodes and no singularity points, and it rapidly decays at spatial infinity (in the Minkowski limit it has the Gaussian form [34]).

From the computations one can deduce that most of mass (99 per cent or more) is effectively contained inside the radius

$$R = \alpha/\sqrt{b}, \tag{8}$$

where α is a dimensionless number of order one (within the range of conducted numerical calculations, its value varied between 1/2 and 2, depending on initial conditions and value of k).

Furthermore, asymptotic values of $\mathcal{M}(\infty)$ suggest the approximate formula for the maximum of a relativistic superfluid star

$$M_{\text{max}} \approx (4G\sqrt{b})^{-1}, \tag{9}$$

assuming that the numerically found maximum is single and finite.

The effective compactness of found equilibria can be computed as the ratio $C = \mathcal{M}_{99}/x_{99}$, where $\mathcal{M}_{99} = 0.99\mathcal{M}(\infty)$, and x_{99} is the dimensionless effective radius containing $\mathcal{M}_{99}$, to be regarded as the effective radius of the superfluid star. The compactness profile is similar to that of boson stars [12] and comparable to black holes (0.5) by an order of magnitude: $C \lesssim 0.15$.

Formulae for R, M_{max} and C indicate that logarithmic superfluid tends to form lumps whose size and mass scale as $b^{-1/2}$. More precisely, their effective radius $R_{99} = (2C)^{-1}R_S(M_{99})$ can be as small as $3R_S(M_{99})$, where $R_S(m) \equiv 2Gm$ is the Schwarzschild radius for a given mass m. This suggests that stationary superfluid stars are dense compact objects with sizes comparable to those of black holes of same mass, but without any horizon.

Notice here that coupling b does not obey any known conditions other than being positive-definite, therefore both mass and size of logarithmic superfluid stars have no upper and lower bounds. This scale independence also agrees with the dilatation symmetry of logarithmically nonlinear field equations [35]. This symmetry can also be seen directly from the scalar field equation where changing the value of the rest mass term is equivalent to rescaling the SF-associated scalar field. From eq. (7) and formula for M_{max}, one can deduce that

$$M \sim \left(T_\Psi - T_\Psi^{(0)}\right)^{-1/2}, \tag{10}$$

which indicates that mass of a superfluid star grows (unrestrictedly *akin* to black holes) as the star's quantum temperature approaches its "zero" – a reference value $T_\Psi^{(0)}$, although its infinite value *per se* is likely to be prevented by a quantum analogue of the third law of thermodynamics applied to the entropy (6): it is impossible for any process to reach the isotherm $T_\Psi = T_\Psi^{(0)}$ in a finite number of steps.

Finally, according to eq. (7), this coupling is not related to any constituent particle's mass because superfluid is a collective state which can be formed from (almost) any bosonic or bosonized particles, or quasi-particles, subject to certain conditions. The most important of those conditions is the strong interaction regime when interparticle potentials dominate over kinetic energies. This condition can naturally occur not only at low temperatures (such as those for laboratory superfluids and condensates) but also at high densities and in effectively lower-dimensional systems [40]. The latter two conditions do not require temperature to be close to absolute zero.

To conclude, superfluid stars described by equilibria of relativistic logarithmic liquid models are macroscopic quantum objects of finite mass without horizons and singularities, whose stability against gravitational collapse is enhanced not only by the Heisenberg uncertainty principle but also by superfluidity. We propose that these objects are viable SMO/DCO candidates or mimickers, due to the following reasons: (i) they are compact with effective radii which can be as small as the Schwarzschild radius by an order of magnitude, (ii) their spatial density distribution drops abruptly, at the Gaussian-like rate, which can be mistaken in realistic astronomical observations for the presence of a material surface, (iii) their "darkness" occurs as a result of suppressed friction and dissipation compared to surrounding accreting matter; and, additionally, (iv) they do not have an absolute upper mass limit of a TOV type, or any other types known, (v) they do not require any exotic elementary relativistic particles unknown in the current Standard Model of particle physics.

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References

[1] M. A. Abramowicz, W. Kluzniak, and J.-P. Lasota, *Astron. Astrophys.* **396** (2002) L31-L34.

[2] Y.-F. Yuan, R. Narayan, and M. J. Rees, *Astrophys. J.* **606** (2004) 1112-1124.

[3] J. E. McClintock, J. Narayan, and G. B. Rybicki, *Astrophys. J.* **615** (2004) 402-415.

[4] J. R. Oppenheimer and G. M. Volkoff, *Phys. Rev.* **55** (1939) 374-381.

[5] R. C. Tolman, *Phys. Rev.* **55** (1939) 364-373.

[6] J. P. S. Lemos and O. B. Zaslavskii, *Phys. Rev. D* **78** (2008) 024040.

[7] F. S. Guzmán and J. M. Rueda-Becerril, *Phys. Rev. D* **80** (2009) 084023.

[8] D. J. Kaup, *Phys. Rev.* **172** (1968) 1331-1342.

[9] R. Ruffini and S. Bonazzola, *Phys. Rev.* **187** (1969) 1767.

[10] E. Takasugi and M. Yoshimura, *Z. Phys. C* **26** (1984) 241.

[11] M. Colpi, S. L. Shapiro, and I. Wasserman, *Phys. Rev. Lett.* **57** (1986) 2485.

[12] P. Amaro-Seoane, J. Barranco, A. Bernal, and L. Rezzolla, *J. Cosmol. Astropart. Phys.* **11** (2010) 002.

[13] K. G. Zloshchastiev, *Low Temp. Phys.* **47** (2021) 89-95.

[14] A. B. Migdal, *Nucl. Phys.* **13** (1959) 655.

[15] U. Lombardo and H.-J. Schulze, *Lect. Notes Phys.* **578** (2001) 30.

[16] K. G. Zloshchastiev, *Eur. Phys. J. B* **85** (2012) 273.

[17] T. C. Scott and K. G. Zloshchastiev, *Low Temp. Phys.* **45** (2019) 1231.

[18] K. G. Zloshchastiev, *Int. J. Mod. Phys. B* **33** (2019) 1950184.

[19] K. G. Zloshchastiev, *Int. J. Mod. Phys. B* **36** (2022) 2250121.

[20] K. G. Zloshchastiev, *Fluids* **7** (2022) 358.

[21] X. Qian, H. Zhang, J. Yan, and S. Song, *Numer. Math. Theor. Meth. Appl.* **16** (2023) 993-1012.

[22] L. Cely, *Electron. J. Differ. Equ.* **2023 (76)** (2023) 1-20.

[23] C. O. Alves and I. S. da Silva, *Anal. Appl.* **21** (2023) 1477-1516.

[24] F. C. E. Lima and C. A. S. Almeida, *Fortschr. Phys.* **2023** (2023) 2300051.

[25] J. Wang, *Front. Math.* **18** (2023) 999-1024.

[26] J. Cui and L. Sun, *SIAM J. Math. Anal.* **55** (2023) 3044-3080.

[27] X. An and X. Yang, *Discrete Contin. Dyn. Syst.* **43** (2023) 3940-3968.

[28] M.B. Chrouda and K. Hassine, *Bull. Malays. Math. Sci. Soc.* **47** (2024) 3.

[29] L.-L. Wang, J. Yan, and X. Zhang, *SIAM J. Numer. Anal.* **62** (2024) 119-137.

[30] L. Cui and A. Mao, *Z. Angew. Math. Phys.* **75** (2024) 30.

[31] R. Carles and G. Ferriere, *Commun. Partial Differ. Equ.* **49** (2024) 88-120.

[32] H. Fan, Y. Wang, and L. Zhao, *J. Math. Phys.* **65** (2024) 031509.

[33] T. Huang and Y.-Y. Shang, *Complex Anal. Oper. Theory* **18** (2024) 49.

[34] G. Rosen, *J. Math. Phys.* **9** (1968) 996-998.

[35] G. Rosen, *Phys. Rev.* **183** (1969) 1186-1188.

[36] K. G. Zloshchastiev, *Acta Phys. Polon.* **42** (2011) 261-292.

[37] V. Dzhunushaliev and K. G. Zloshchastiev, *Open Phys.* **11** (2013) 325-335.

[38] E. Belendryasova, V. A. Gani, and K. G. Zloshchastiev, *Phys. Lett. B* **823** (2021) 136776.

[39] K. G. Zloshchastiev, *Z. Naturforsch. A* **73** (2018) 619-628.

[40] K. G. Zloshchastiev, *Europhys. Lett. (EPL)* **122** (2018) 39001.