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Quantum field theory at finite time and neutrino oscillations

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We discuss the role of finite time and energy uncertainty in the quantum field theory description of neutrino oscillations. In order to to achieve this goal, we review the flavor Fock-space approach and the time-energy uncertainty relation in the Mandelstam–Tamm form, expressed as a flavor-energy uncertainty relation. Such relation, together with the inequivalence of mass and flavor neutrinos Fock spaces, puts a lower bound on neutrino energy uncertainty. Similar considerations can be derived by a perturbative computation of flavor transitions, which employs the Dirac picture to compute flavor transition probability. Remarkably, both flavor Fock space and interaction picture approach lead to the same oscillation probability, within the approximation adopted in the perturbative calculation.

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1. Introduction

The proposal of neutrino oscillations originated with Pontecorvo and collaborators [1–4], and was later validated by numerous experiments [5–8].

Although many aspects of neutrino mixing and oscillations are comprehensively understood [9–11], a consensus regarding their definitive description within quantum field theory (QFT) remains elusive. Over the past three decades, various theoretical frameworks have been proposed [12–22]. In the flavor Fock space approach [14], one starts from the observation of the unitary inequivalence of the representations of equal-time anticommutation relations for flavor and mass neutrino fields [23–27]. Then, one needs to construct a Fock space where flavor field operators are defined. Here, oscillation probabilities are computed by taking the expectation value of lepton currents/charges on one-particle neutrino states at a reference time. This approach modifies the classic Pontecorvo result in two significant ways [28]: (i) introducing a fast-oscillation term dependent on the sum of frequencies alongside the usual oscillation term dependent on the difference of energies/frequencies, and (ii) including energy-dependent oscillation amplitudes as coefficients of a Bogoliubov transformation [14].

Within such framework it has been pointed out that an intrinsic energy uncertainty for neutrinos is implied by the inequivalence of flavor and mass representations [29, 30]. This result was derived by computing the Mandelstam–Tamm time energy uncertainty relation (TEUR) and using the flavor/lepton charges as "clock observables". Similar results were also found in curved spacetimes [31] (see Ref.[32] for a review).

In Ref.[33] we introduced a new approach to deal with neutrino oscillations in QFT, akin to the treatment of unstable particles [34, 35], by employing the interaction (Dirac) picture. The interaction Lagrangian in the Dyson series solely incorporates the mixing term between different flavor fields. We computed amplitudes for various decay channels at the first order, describing both flavor-changing and survival processes. Remarkably, within the adopted approximation, we found non-trivial agreement between the neutrino flavor-transition formula derived with that perturbative approach and the non-perturbative formula of the flavor Fock-space approach.

In the present work, we both review TEUR in the flavor Fock space approach to neutrino flavor oscillations and the interaction picture approach. Such different angles of view converge to evidentiate the importance of inherently having an energy uncertainty, which means that a finite-time formulation of QFT must be employed: TEUR explicitly relates time and energy uncertainty, while interaction pictures leads to non-trivial results only when one abandons the S-matrix in favor of the time evolution operator U.

The paper is structured as follows: Section 2, basic notions on neutrino mixing in QFT are presented. Then in Section 3, flavor Fock space and TEUR are briefly reviewed, while the interaction picture approach is reported in Section 4. Finally, Section 5 is devoted to conclusions.

2. Neutrino mixing: basic facts

Consider the weak decay of a W^+ boson, $W^+ \to e^+ + \nu_e$. This process can be described by the Lagrangian

$$\mathcal{L} = \sum_{\sigma=e,\mu} \left[\overline{\nu}_\sigma \left(i\partial\!\!\!/ - m_\sigma\right) \nu_\sigma + \bar{l}_\sigma \left(i\partial\!\!\!/ - \tilde{m}_\sigma\right) l_\sigma\right] + \mathcal{L}_{mix} + \mathcal{L}_{wint} , \tag{1}$$

with

$$\mathcal{L}_{mix} = -m_{e\mu}\left(\overline{\nu}_e\nu_\mu + \overline{\nu}_\mu\nu_e\right) , \tag{2}$$

$$\mathcal{L}_{wint} = -\frac{g}{2\sqrt{2}} \sum_{\sigma=e,\mu} \left[W^+_\mu\, \overline{\nu}_\sigma\, \gamma^\mu\, (1-\gamma^5)\, l_\sigma + h.c.\right] \tag{3}$$

The neutrino kinetic term (including $\mathcal{L}_{mix}$) can be diagonalized by the *mixing transformation* [36, 37]

$$\nu_\sigma = \sum_{j=1,2} U^*_{\sigma j}\nu_j , \tag{4}$$

U is the *mixing matrix*. In the two flavor case, here analyzed, the matrix can be parametrized as

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} , \tag{5}$$

with $\tan 2\theta = 2m_{e\mu}/(m_\mu - m_e)$.

In order to perform perturbative computations, one is led to decompose the above Lagrangian into a free and an interaction part. A typical choice one finds in literature is to work in the *mass basis*

$$\mathcal{L} = \mathcal{L}^m_0 + \mathcal{L}^m_{int} , \tag{6}$$

with

$$\mathcal{L}^m_0 = \sum_j \overline{\nu}_j \left(i\gamma_\mu\partial^\mu - m_j\right) \nu_j + \sum_\sigma \bar{l} \left(i\gamma_\mu\partial^\mu - \tilde{m}_\sigma\right) l , \tag{7}$$

$$\mathcal{L}^m_{int} = -\frac{g}{2\sqrt{2}} \sum_{\sigma,j} \left[W^+_\mu\, \overline{\nu}_j\, U^*_{j\sigma}\, \gamma^\mu\, (1-\gamma^5)\, l_\sigma + h.c.\right] . \tag{8}$$

In such a case the effect of mixing is entirely included in the weak-interaction vertex. However, one should find an appropriate definition of flavor states [13]: in fact, in charged current weak interaction processes as the one we mentioned, neutrinos are produced with a definite flavor. An escape is represented by the *external wave-packets approach* [38–40], where neutrinos are only treated as internal lines of macroscopic Feynman diagrams.

Another possibility is to take the following split

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}^g_{int} , \tag{9}$$

with

$$\mathcal{L}_0 = \sum_{\sigma=e,\mu} \overline{\nu}_\sigma \left(i\partial\!\!\!/ - m_\sigma\right) \nu_\sigma + \sum_{\sigma=e,\mu} \bar{l}_\sigma \left(i\partial\!\!\!/ - \tilde{m}_\sigma\right) l_\sigma , \tag{10}$$

$$\mathcal{L}^g_{int} = \mathcal{L}_{mix} + \mathcal{L}_{wint} . \tag{11}$$

In this approach, $\mathcal{L}_{wint}$ is diagonal in the asymptotic fields appearing in Eq.(10): with this choice, we can give a natural definition of flavor/lepton charges and then of flavor states. In fact, one can easily see that the Lagrangian $\mathcal{L}$ is invariant under the action of the global $U(1)$ transformations $\nu \to e^{i\alpha}\nu$ and $l \to e^{i\alpha} l$. From Nother's theorem, this symmetry leads to the conservation of the total flavor charge Q_l^{tot}, which can be physically interpreted as the total lepton number [37]. The total charge can be thus written in terms of the flavor charges for neutrinos and charged leptons

$$Q_l^{tot} = \sum_{\sigma=e,\mu} Q_\sigma^{tot}(t)\,, \qquad Q_\sigma^{tot}(t) = Q_{\nu_\sigma}(t) + Q_\sigma\,, \tag{12}$$

with

$$\begin{aligned} Q_e &= \int d^3\mathbf{x}\, e^\dagger(x) e(x)\,, \qquad Q_{\nu_e}(t) = \int d^3\mathbf{x}\, \nu_e^\dagger(x)\nu_e(x)\,, \\ Q_\mu &= \int d^3\mathbf{x}\, \mu^\dagger(x)\mu(x)\,, \qquad Q_{\nu_\mu}(t) = \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\nu_\mu(x)\,. \end{aligned} \tag{13}$$

Because $[\mathcal{L}_{wint}(\mathbf{x},t), Q_{\nu_\sigma}(t)] = 0$, it is clear that neutrinos are produced and detected with a definite flavor, i.e. as eigenstates of Q_{ν_σ}, at some reference time[1]. However, $[\mathcal{L}_{mix}, Q_{\nu_\sigma}(t)] \neq 0$, leading to the flavor oscillation phenomenon. In the next Section we will show how to build the eigenstates of $Q_{\nu_\sigma}(t)$. From now we will disregard $\mathcal{L}_{wint}$ (zeroth-order in g), so that the charged-lepton part also decouples.

3. The flavor Fock space approach and the flavor-energy uncertainty

A key observation is that the field mixing transformation (4) can be exactly rewritten as [14]

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t)\nu_j^\alpha(x) G_\theta(t)\,, \qquad (\sigma, j) = (e,1), (\mu,2) \tag{14}$$

where the *mixing generator* reads

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2 - \nu_2^\dagger(x)\nu_1(x)\right)\right]. \tag{15}$$

The mass operator can be Fourier expanded as usual

$$\nu_j(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \left[u_{\mathbf{k},j}^r(t)\,\alpha_{\mathbf{k},j}^r(t) + v_{-\mathbf{k},j}^r(t)\,\beta_{-\mathbf{k},j}^{r\dagger}(t)\right], \quad j = 1,2\,, \tag{16}$$

with $u_{\mathbf{k},j}^r(t) = e^{-i\omega_{\mathbf{k},j}t}\, u_{\mathbf{k},j}^r$, $v_{\mathbf{k},j}^r(t) = e^{i\omega_{\mathbf{k},j}t}\, v_{\mathbf{k},j}^r$, $\omega_{\mathbf{k},j} = \sqrt{|\mathbf{k}|^2 + m_j^2}$.

An important issue is that the vacuum $|0\rangle_{1,2}$, which is annihilated by mass neutrino ladder operators $\alpha_{\mathbf{k},j}^r(t)|0\rangle_{1,2} = \beta_{\mathbf{k},j}^r(t)|0\rangle_{1,2} = 0$, is not invariant under the action of the mixing generator $G_\theta(t)$. In fact, one has

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\; |0\rangle_{1,2}\,. \tag{17}$$

[1]The above considerations are exact at tree level. Of course, loop corrections can spoil flavor conservation in the vertices, but such corrections are negligible for the present discussion.

The state (17) is known as *flavor vacuum* because it is annihilated by the flavor ladder operators $\alpha_\sigma(t)$ and $\beta_\sigma(t)$ defined by

$$\alpha_e(t)|0(t)\rangle_{e,\mu} \;\equiv\; G^{-1}_\theta(t)\alpha_1 G_\theta(t)\; G^{-1}_\theta(t)|0\rangle_{1,2} \;=\; 0\,, \tag{18}$$

and similarly for $\beta_\sigma(t)$. Their explicit form is

$$\alpha^r_{\mathbf{k},e}(t) \;=\; \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right), \tag{19}$$

$$\alpha^r_{\mathbf{k},\mu}(t) \;=\; \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right), \tag{20}$$

$$\beta^r_{-\mathbf{k},e}(t) \;=\; \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right), \tag{21}$$

$$\beta^r_{-\mathbf{k},\mu}(t) \;=\; \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right). \tag{22}$$

In these equations we have defined $\epsilon^r \equiv (-1)^r$, while $U_{\mathbf{k}}$ and $V_{\mathbf{k}}$ are the *Bogoliubov coefficients*

$$U_{\mathbf{k}}(t) \;\equiv\; u^{r\dagger}_{\mathbf{k},2}u^r_{\mathbf{k},1}\; e^{i(\omega_{\mathbf{k},2}-\omega_{\mathbf{k},1})t} \;=\; |U_{\mathbf{k}}|\,e^{i(\omega_{\mathbf{k},2}-\omega_{\mathbf{k},1})t}\,, \tag{23}$$

$$V_{\mathbf{k}}(t) \;\equiv\; \epsilon^r\; u^{r\dagger}_{\mathbf{k},1}v^r_{-\mathbf{k},2}\; e^{i(\omega_{\mathbf{k},2}+\omega_{\mathbf{k},1})t} \;=\; |V_{\mathbf{k}}|\,e^{i(\omega_{\mathbf{k},2}+\omega_{\mathbf{k},1})t}\,, \tag{24}$$

and the time-independent part of the coefficients is given by

$$|U_{\mathbf{k}}| \;\equiv\; u^{r\dagger}_{\mathbf{k},2}\,u^r_{\mathbf{k},1} \;=\; v^{r\dagger}_{-\mathbf{k},1}\,v^r_{-\mathbf{k},2}$$
$$= \left(\frac{\omega_{\mathbf{k},1}+m_1}{2\omega_{\mathbf{k},1}}\right)^{\frac12}\left(\frac{\omega_{\mathbf{k},2}+m_2}{2\omega_{\mathbf{k},2}}\right)^{\frac12}\left(1+\frac{\mathbf{k}^2}{(\omega_{\mathbf{k},1}+m_1)(\omega_{\mathbf{k},2}+m_2)}\right), \tag{25}$$

$$|V_{\mathbf{k}}| \;=\; \epsilon^r\; u^{r\dagger}_{\mathbf{k},1}\,v^r_{-\mathbf{k},2} \;=\; -\epsilon^r\; u^{r\dagger}_{\mathbf{k},2}\,v^r_{-\mathbf{k},1}$$
$$= \frac{|\mathbf{k}|}{\sqrt{4\omega_{\mathbf{k},1}\omega_{\mathbf{k},1}}}\left(\sqrt{\frac{\omega_{\mathbf{k},2}+m_2}{\omega_{\mathbf{k},1}+m_1}} - \sqrt{\frac{\omega_{\mathbf{k},1}+m_1}{\omega_{\mathbf{k},2}+m_2}}\right). \tag{26}$$

Notice that $|U_{\mathbf{k}}|^2+|V_{\mathbf{k}}|^2 = 1$. It is straightforward to check that in the relativistic limit $\omega_{\mathbf{k},j} \approx |\mathbf{k}|$, $|U_{\mathbf{k}}| \to 1$ and $|V_{\mathbf{k}}| \to 0$. Also, $|V_{\mathbf{k}}| = 0$ when $m_1 = m_2$ and/or $\theta = 0$, i.e. when no mixing occurs. $|V_{\mathbf{k}}|^2$ has the maximum at $|\mathbf{k}| = \sqrt{m_1 m_2}$ with $|V_{\mathbf{k}}|^2_{max} \to 1/2$ for $\frac{(m_2-m_1)^2}{m_1 m_2} \to \infty$, and $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2-m_1)^2}{4|\mathbf{k}|^2}$ for $|\mathbf{k}| \gg \sqrt{m_1 m_2}$ at the first non-vanishing order.

The flavor fields can be thus expanded as

$$\nu_\sigma(x) \;=\; \sum_{\mathbf{k},r}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}}\left[u^r_{\mathbf{k},j}(t)\,\alpha^r_{\mathbf{k},\sigma}(t) \;+\; v^r_{-\mathbf{k},j}(t)\,\beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\right], \quad (\sigma,j) = (e,1),(\mu,2)\,. \tag{27}$$

A *flavor Fock space* (at some reference time, say $t = 0$) is defined as $\mathcal{H}_{e,\mu} = \left\{\alpha^\dagger_{e,\mu}\,,\;\beta^\dagger_{e,\mu}\,,\;|0\rangle_{e,\mu}\right\}$, with $|0\rangle_{e,\mu} \equiv |0(t=0)\rangle_{e,\mu}$. Such Hilbert space is different with respect to the mass-neutrino Fock space, spanned by mass-neutrino ladder operators on the vacuum $|0\rangle_{1,2}$. In fact, one can verify that

$$\lim_{V\to\infty}\;{}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\; e^{V\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\ln\left(1-\sin^2\theta\,|V_{\mathbf{k}}|^2\right)^2} = 0\,, \tag{28}$$

i.e. flavor and massive fields belong to unitarily inequivalent representations of the anticommutation relations.

The previous discussions suggest that flavor states $|\nu^r_{\mathbf{k},\sigma}\rangle$ can be built as one particle states of the flavor Fock space

$$|\nu^r_{\mathbf{k},\sigma}\rangle \;\equiv\; \alpha^{r\dagger}_{\mathbf{k},\sigma}|0\rangle_{e,\mu}\,, \tag{29}$$

and similarly for the antineutrino ($|\beta^r_{\mathbf{k},\sigma}\rangle \equiv \beta^{r\dagger}_{\mathbf{k},\sigma}|0\rangle_{e,\mu}$). One can prove that these states are exact eigenstates of the charge operators at the reference (production/detection) time, i.e.

$$Q_{\nu_\sigma}(0)|\nu^r_{\mathbf{k},\sigma}\rangle \;=\; |\nu^r_{\mathbf{k},\sigma}\rangle\,. \tag{30}$$

In this approach the flavor oscillation probability is computed by taking the expectation value of the time-dependent flavor charges with respect to a reference time flavor state [28]

$$\mathcal{Q}_{\sigma\to\rho}(t) \;=\; \langle Q_{\nu_\rho}(t)\rangle_\sigma\,, \tag{31}$$

where $\langle\cdots\rangle_\sigma \equiv \langle\nu^r_{\mathbf{k},\sigma}|\cdots|\nu^r_{\mathbf{k},\sigma}\rangle$, which gives

$$\mathcal{Q}_{\sigma\to\rho}(t) = \sin^2(2\theta)\left[|U_{\mathbf{k}}|^2\sin^2\left(\frac{\Omega^-_{\mathbf{k}}}{2}t\right) + |V_{\mathbf{k}}|^2\sin^2\left(\frac{\Omega^+_{\mathbf{k}}}{2}t\right)\right], \quad \sigma\neq\rho\,, \tag{32}$$

$$\mathcal{Q}_{\sigma\to\sigma}(t) = 1 \,-\, \mathcal{Q}_{\sigma\to\rho}(t)\,, \quad \sigma\neq\rho\,, \tag{33}$$

with $\Omega^\pm_{\mathbf{k}} \;\equiv\; \omega_{\mathbf{k},1}\pm\omega_{\mathbf{k},2}$. Note the presence of the term proportional to $|V_{\mathbf{k}}|^2$ in the oscillation probability Eq. (32), which introduces rapid oscillations not found in the conventional quantum mechanics formula. As previously mentioned, $|V_{\mathbf{k}}|^2 \to 0$ in the relativistic limit $|\mathbf{k}| \gg m_j$, $j = 1, 2$, and $\Omega^-_{\mathbf{k}} \;\approx\; \frac{\delta m^2}{4|\mathbf{k}|} \;=\; \frac{\pi}{L^{osc}}$, where we introduced the *oscillation length* L_{osc}. Consequently, the oscillation formula reduces to the standard result

$$\mathcal{Q}_{\sigma\to\rho}(L) \;\approx\; \sin^2(2\theta)\sin\left(\frac{\pi L}{L_{osc}}\right)\,, \tag{34}$$

where we took $t \approx L$, i.e. the distance travelled by the neutrino. It has been demonstrated, particularly in the case of scalar field mixing, that the oscillation formula (32) represents the time component of a Lorentz-covariant formula, although the presence of the flavor vacuum breaks Lorentz invariance [41].

We have already seen that the lepton charges $Q_{\nu_\sigma}(t)$ do not commute with the neutrino Hamiltonian H

$$\left[Q_{\nu_\sigma}(t)\,,\,H\right] \;=\; i\,\frac{\mathrm{d}Q_{\nu_\sigma}(t)}{\mathrm{d}t} \;\neq\; 0\,. \tag{35}$$

This fact suggested to compute TEUR in the Mandelstam–Tamm form [42], which is a *flavor–energy* uncertainty relation [29]

$$\Delta E\,\Delta Q_{\nu_\sigma} \;\geq\; \frac{1}{2}\left|\frac{\mathrm{d}\mathcal{Q}_{\sigma\to\sigma}(t)}{\mathrm{d}t}\right|\,. \tag{36}$$

where Δ indicates the standard deviations on the neutrino state $\Delta O \equiv \sqrt{\langle O^2_{\nu_\sigma}(t)\rangle_\sigma \,-\, \langle O_{\nu_\sigma}(t)\rangle^2_\sigma}$. Explicitly

$$\Delta E\,\sqrt{\mathcal{Q}_{\sigma\to\sigma}(t)\,(1-\mathcal{Q}_{\sigma\to\sigma}(t))} \;\geq\; \frac{1}{2}\left|\frac{\mathrm{d}\mathcal{Q}_{\sigma\to\sigma}(t)}{\mathrm{d}t}\right|\,. \tag{37}$$

Because the square root on the l.h.s. has the maximum value $\frac{1}{2}$, it follows

$$\Delta E \;\geq\; \left|\frac{\mathrm{d}Q_{\sigma\to\sigma}(t)}{\mathrm{d}t}\right| . \tag{38}$$

Integrating both members and using the triangular inequality, we get

$$\Delta E\, T \;\geq\; Q_{\sigma\to\rho}(T)\,, \qquad \sigma\neq\rho. \tag{39}$$

Evaluating the inequality (39) at the leading order in the relativistic limit and for $T \approx L = L^{osc}/2$, one finds

$$\Delta E \;\geq\; \frac{2\sin^2(2\theta)}{L^{osc}} . \tag{40}$$

Conditions like (40) are known in literature and are usually interpreted in the following way: if neutrino energies or masses are measured with great accuracy, it becomes possible to infer which massive neutrino was produced in the weak interaction. Consequently, in such a scenario, oscillations would not occur [43]. This line of reasoning relies on the notion that flavor neutrinos are essentially a superposition of the“physical” massive neutrinos. In other words, the flavor eigenstates $|\nu_\sigma\rangle$ of neutrinos are just linear combinations of the mass eigenstates $|\nu_j\rangle$. Therefore, if one could precisely determine the masses or energies of neutrinos, it could be distinguished which mass eigenstate was produced, and consequently, no oscillations would be observed. However, the above analysis suggests a different interpretation of the inequality (40). In fact, Eq. (28) implies

$$\lim_{V\to\infty}\langle \nu^r_{\mathbf{k},i}|\nu^r_{\mathbf{k},\sigma}\rangle \;=\; 0\,, \qquad i=1,2\,, \tag{41}$$

i.e. neutrino flavor eigenstates, which are are the exact eigenstates of the lepton charges, and which are produced in weak decays, *cannot* be generally written as a linear superposition of single-particle massive neutrino states. Therefore, if we accept the physical basis is the flavor one, the inequality (40) imposes a fundamental lower bound on neutrino energy uncertainty.

4. Neutrino oscillations in the interaction picture

As seen in the previous Section, TEUR deeply encodes flavor oscillations. This fact reveals the importance of treating neutrino oscillations in finite-time QFT. In fact, energy uncertainty which is required by TEUR is only observed at finite time (see Eq.(39)). This considerations suggested to compute the transition amplitudes among different flavors, performing a perturbative calculation of the time evolution operator [33]

$$U(t_i,t_f) \;=\; \mathcal{T}\exp\left[i\int_{t_i}^{t_f}\mathrm{d}^4x\;:\mathcal{L}_{int}(x):\right] \;=\; \mathcal{T}\exp\left[-i\int_{t_i}^{t_f}\mathrm{d}^4x\;:\mathcal{H}_{int}(x):\right], \tag{42}$$

where $\mathcal{L}_{int}\equiv\mathcal{L}^{g=0}_{int} = -m_{e\mu}\left(\overline{\nu}_e\nu_\mu+\overline{\nu}_\mu\nu_e\right)$, $\mathcal{H}_{int}(x) = -\mathcal{L}_{int}(x)$ is the interaction Hamiltonian density and $\mathcal{T}$ is the chronological product. In the following we stop at the second order in $m_{e\mu}$:

$$U(t_i,t_f) \;=\; 1-i\int_{t_i}^{t_f}\mathrm{d}t_1\,H_{int}(t_1)+(-i)^2\int_{t_i}^{t_f}\mathrm{d}t_1\,H_{int}(t_1)\int_{t_i}^{t_1}\mathrm{d}t_2\,H_{int}(t_2)+\ldots \tag{43}$$

where $H_{int} = \int d^3\mathbf{x}\, \mathcal{H}_{int}(x)$ is the interaction Hamiltonian.

Although this approach is apparently unrelated to the flavor Fock space construction we have reviewed in the previous section, we will show that it leads to the same oscillation formula (32), within the approximation employed. Let us emphasize the importance of examining the time evolution operator rather than the S-matrix. This choice stems from the fact that the phenomenon of flavor oscillations can only be adequately described at finite time intervals, as deduced from the TEUR. Consequently, it follows that flavor neutrino states are not well-defined as asymptotically stable states. As elucidated by the examples below, the limits $t_i \to -\infty$ and $t_f \to +\infty$ preclude the flavor-changing processes, while simultaneously ensuring strict energy conservation, as required by TEUR. This bears resemblance to the treatment of unstable particles [34, 35, 44–47] (see also [48, 49], where the significance of finite-time Quantum Field Theory in the analysis of decay processes has been underscored). Indeed, both the decay of unstable particles [50] and neutrino oscillations can be understood in terms of TEUR.

In the interaction picture ν_σ $(\sigma = e, \mu)$, defined by the Lagrangian Eq.(1), can be expanded as free fields, whose evolution is only due to $\mathcal{L}_0$:

$$\nu_\sigma(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},\sigma}(t)\, \alpha^r_{\mathbf{k},\sigma} + v^r_{-\mathbf{k},\sigma}(t)\, \beta^{r\dagger}_{-\mathbf{k},\sigma} \right] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{44}$$

with $u^r_{\mathbf{k},\sigma}(t) = e^{-i\omega_{\mathbf{k},\sigma} t}\, u^r_{\mathbf{k},\sigma}$, $v^r_{\mathbf{k},\sigma}(t) = e^{i\omega_{\mathbf{k},\sigma} t}\, v^r_{\mathbf{k},\sigma}$, $\omega_{\mathbf{k},\sigma} = \sqrt{|\mathbf{k}|^2 + m^2_\sigma}$. Annihilation operators satisfy

$$\alpha^r_{\mathbf{k},\sigma}|0\rangle = 0 = \beta^r_{\mathbf{k},\sigma}|0\rangle\ . \tag{45}$$

Let us stress that the perturbative vacuum $|0\rangle$ does not coincide with the flavor vacuum $|0\rangle_{e,\mu}$ or with mass vacuum $|0\rangle_{1,2}$ above introduced. The anticommutation relations are

$$\{\alpha^r_{\mathbf{k},\rho}, \alpha^{s\dagger}_{\mathbf{q},\sigma}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{\rho\sigma} \quad , \quad \{\beta^r_{\mathbf{k},\rho}, \beta^{s\dagger}_{\mathbf{q},\sigma}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{\rho\sigma}, \tag{46}$$

and the spinors are normalized so that

$$u^{r\dagger}_{\mathbf{k},\rho} u^s_{\mathbf{k},\rho} = v^{r\dagger}_{\mathbf{k},\rho} v^s_{\mathbf{k},\rho} = \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},\rho} v^s_{-\mathbf{k},\rho} = 0\ . \tag{47}$$

Then, the interacting part of the Hamiltonian reads:

$$\begin{aligned} H_{int}(t) \;=\; & m_{e\mu} \sum_{s,s'=1,2} \sum_{\mathbf{p}} \Big[\beta^s_{\mathbf{p},\mu}\beta^{s\dagger}_{\mathbf{p},e}\delta_{ss'} W^*_{\mathbf{p}}(t) + \alpha^{r\dagger}_{\mathbf{p},\mu}\alpha^r_{\mathbf{p},e}\delta_{ss'} W_{\mathbf{p}}(t) \\ & + \beta^s_{-\mathbf{p},\mu}\alpha^{s'}_{e,\mathbf{p}} \left(Y^{ss'}_{\mathbf{p}}(t)\right)^* + \alpha^{s\dagger}_{\mathbf{p},\mu}\beta^{s'\dagger}_{-\mathbf{p},e} Y^{ss'}_{\mathbf{p}}(t) + e \leftrightarrow \mu \Big], \end{aligned} \tag{48}$$

where we introduced the notation

$$W_{\mathbf{p}}(t) = \overline{u}^s_{\mathbf{p},\mu} u^s_{\mathbf{p},e} e^{i\left(\omega_{\mathbf{k},\mu} - \omega_{\mathbf{k},e}\right)t} = W_{\mathbf{p}}\, e^{i\left(\omega_{\mathbf{p},\mu} - \omega_{\mathbf{p},e}\right)t} \tag{49}$$

$$Y^{ss'}_{\mathbf{p}}(t) = \overline{u}^s_{\mathbf{p},\mu} v^{s'}_{-\mathbf{p},e} e^{i\left(\omega_{\mathbf{k},\mu} + \omega_{\mathbf{k},e}\right)t} = Y^{ss'}_{\mathbf{p}}\, e^{i\left(\omega_{\mathbf{p},\mu} + \omega_{\mathbf{p},e}\right)t} \tag{50}$$

Explicitly

$$W_{\mathbf{p}} = \sqrt{\frac{(\omega_{\mathbf{p},e}+m_e)\,(\omega_{\mathbf{p},\mu}+m_\mu)}{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(1-\frac{|\mathbf{p}|^2}{(\omega_{\mathbf{p},e}+m_e)(\omega_{\mathbf{p},\mu}+m_\mu)}\right), \tag{51}$$

$$Y^{22}_{\mathbf{p}} = -Y^{11}_{\mathbf{p}} = \frac{p_3}{\sqrt{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(\sqrt{\frac{\omega_{\mathbf{p},\mu}+m_\mu}{\omega_{\mathbf{p},e}+m_e}}+\sqrt{\frac{\omega_{\mathbf{p},e}+m_e}{\omega_{\mathbf{p},\mu}+m_\mu}}\right), \tag{52}$$

$$Y^{12}_{\mathbf{p}} = \left(Y^{21}_{\mathbf{p}}\right)^* = -\frac{p_1-ip_2}{\sqrt{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(\sqrt{\frac{\omega_{\mathbf{p},\mu}+m_\mu}{\omega_{\mathbf{p},e}+m_e}}+\sqrt{\frac{\omega_{\mathbf{p},e}+m_e}{\omega_{\mathbf{p},\mu}+m_\mu}}\right). \tag{53}$$

A first non-trivial process we study is

$$|\nu^r_{\mathbf{p},e}\rangle \;\rightarrow\; |\nu^s_{\mathbf{k},\mu}\rangle\,, \qquad |\nu^r_{\mathbf{p},\sigma}\rangle \equiv \alpha^{r\dagger}_{\mathbf{p},\sigma}|0\rangle\,. \tag{54}$$

Its first-order amplitude can be written as

$$\begin{aligned}
\mathcal{A}^{rs}_{e\to\mu}(\mathbf{p},\mathbf{k},;t_i,t_f) &\approx -im_{e\mu}\delta_{rs}\delta_{\mathbf{k},\mathbf{p}}W_{\mathbf{p}}\int_{t_i}^{t_f}\mathrm{d}t\, e^{i\left(\omega_{\mathbf{k},\mu}-\omega_{\mathbf{p},e}\right)t}\\
&= m_{e\mu}\,\delta_{rs}\delta_{\mathbf{k},\mathbf{p}}\left(e^{i\left(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e}\right)t_f}-e^{i\left(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e}\right)t_i}\right)\frac{W_{\mathbf{p}}}{\omega_{\mathbf{k},e}-\omega_{\mathbf{k},\mu}}\\
&= \delta_{rs}\delta_{\mathbf{k},\mathbf{p}}\,\tilde{\mathcal{A}}_{e\to\mu}(\mathbf{k};t_i,t_f)\,,
\end{aligned} \tag{55}$$

where

$$\tilde{\mathcal{A}}_{e\to\mu}(\mathbf{p};t_i,t_f) = \frac{m_{e\mu}\,W_{\mathbf{p}}}{\omega_{\mathbf{p},e}-\omega_{\mathbf{p},\mu}}\left(e^{i\left(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e}\right)t_f}-e^{i\left(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e}\right)t_i}\right). \tag{56}$$

The oscillation probability is computed including a sum over the final density of states

$$\begin{aligned}
\mathcal{P}_{e\to\mu}(\mathbf{p};\Delta t) &= \sum_{\mathbf{k},s}|\mathcal{A}^{rs}_{e\to\mu}(\mathbf{p},\mathbf{k};t_i,t_f)|^2 = |\tilde{\mathcal{A}}_{e\to\mu}(\mathbf{p},t_i,t_f)|^2\\
&= W^2_{\mathbf{p}}\frac{2m^2_{e\mu}}{\left(\omega_{\mathbf{p},e}-\omega_{\mathbf{p},\mu}\right)^2}\left[1-\cos\left[\left(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e}\right)\Delta t\right]\right], \qquad \Delta t\equiv t_f-t_i\,.
\end{aligned} \tag{57}$$

A second non-trivial process to consider is the decay

$$|\nu^r_{\mathbf{p},e}\rangle \;\rightarrow\; |\nu^{s_1}_{\mathbf{k}_1,e}\rangle|\nu^{s_2}_{\mathbf{k}_2,\mu}\rangle|\overline{\nu}^{s_3}_{\mathbf{k}_3,e}\rangle\,. \tag{58}$$

Its amplitude reads

$$\begin{aligned}
&\mathcal{A}^{rs_1s_2s_3}_{e\to e\overline{e}\mu}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;t_i,t_f) \approx -i\,m_{e\mu}\,Y^{s_3s_2}_{\mathbf{k}_2}\,\delta_{\mathbf{k}_1,\mathbf{p}}\delta_{\mathbf{k}_2,-\mathbf{k}_3}\,\delta_{rs_1}\int_{t_i}^{t_f}\mathrm{d}t\,e^{-i\left(\omega_{\mathbf{k}_2,\mu}+\omega_{\mathbf{k}_2,e}\right)t}\\
&= -m_{e\mu}\,\delta_{rs_1}\,\delta_{\mathbf{k}_1,\mathbf{p}}\delta_{\mathbf{k}_2,-\mathbf{k}_3}\left(e^{-i\left(\omega_{\mathbf{k}_2,\mu}+\omega_{\mathbf{k}_2,e}\right)t_f}-e^{-i\left(\omega_{\mathbf{k}_2,\mu}+\omega_{\mathbf{k}_2,e}\right)t_i}\right)\frac{Y^{s_2s_3}_{\mathbf{k}_2}}{\omega_{\mathbf{k}_2,e}+\omega_{\mathbf{k}_3,\mu}}\\
&= \delta_{\mathbf{k}_1,\mathbf{p}}\delta_{\mathbf{k}_2,-\mathbf{k}_3}\,\delta_{rs_1}\,\tilde{\mathcal{A}}^{s_2s_3}_{e\to e\overline{\mu}\mu}(\mathbf{k}_2;t_i,t_f)\,,
\end{aligned} \tag{59}$$

where

$$\tilde{\mathcal{A}}^{s_2 s_3}_{e\to e\bar{e}\mu}(\mathbf{k};t_i,t_f) \;=\; -\frac{m_{e\mu}\,Y^{s_2 s_3}_{\mathbf{k}}}{\omega_{\mathbf{k},e}+\omega_{\mathbf{k},\mu}}\,\left(e^{-i\left(\omega_{\mathbf{k},\mu}+\omega_{\mathbf{k},e}\right)t_f}-e^{-i\left(\omega_{\mathbf{k},\mu}+\omega_{\mathbf{k},e}\right)t_i}\right)\,. \tag{60}$$

We thus find the probability as

$$\mathcal{P}_{e\to e\bar{e}\mu}(\mathbf{p};\Delta t) \;=\; \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\sum_{s_1,s_2,s_3}|\mathcal{A}^{r s_1 s_2 s_3}_{e\to e\bar{e}\mu}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;t_i,t_f)|^2 \;=\; \sum_{\mathbf{k}}\sum_{s_2,s_3}|\tilde{\mathcal{A}}^{s_2 s_3}_{e\to e\bar{e}\mu}(\mathbf{k};t_i,t_f)|^2\,. \tag{61}$$

In the large-V limit we get

$$\mathcal{P}_{e\to e\bar{e}\mu}(\mathbf{p};\Delta t) \;=\; V\sum_{s_2,s_3}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\frac{\left(Y^{s_2 s_3}_{\mathbf{k}}\right)^2}{\left(\omega_{\mathbf{k},e}+\omega_{\mathbf{k},\mu}\right)^2}\,\sin^2\left(\frac{\left(\omega_{\mathbf{k},\mu}+\omega_{\mathbf{k},e}\right)\Delta t}{2}\right)\,. \tag{62}$$

This is an infrared divergent expression due to a vacuum diagram which must be subtracted from the final result.

Finally, we consider the process

$$|\nu^r_{\mathbf{p},e}\rangle \;\to\; |\nu^{s_1}_{\mathbf{k}_1,e}\rangle|\nu^{s_2}_{\mathbf{k}_2,e}\rangle|\bar{\nu}^{s_3}_{\mathbf{k}_3,\mu}\rangle\,, \qquad \mathbf{k}_1\neq\mathbf{k}_2 \;\vee\; s_1\neq s_2\,. \tag{63}$$

Its amplitude explicitly reads

$$\begin{aligned}\mathcal{A}^{r s_1 s_2 s_3}_{e\to ee\bar{\mu}}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;t_i,t_f) \;&=\; \delta_{\mathbf{k}_1,\mathbf{p}}\delta_{\mathbf{k}_2,-\mathbf{k}_3}\,\delta_{r s_1}\,\tilde{\mathcal{A}}^{s_2 s_3}_{e\to ee\bar{\mu}}(\mathbf{k}_2;t_i,t_f)\\ &-\; \delta_{\mathbf{k}_2,\mathbf{p}}\delta_{\mathbf{k}_1,-\mathbf{k}_3}\,\delta_{r s_2}\,\tilde{\mathcal{A}}^{s_1 s_3}_{e\to ee\bar{\mu}}(\mathbf{k}_1;t_i,t_f)\,.\end{aligned} \tag{64}$$

where $\tilde{\mathcal{A}}^{s_2 s_3}_{e\to ee\bar{\mu}}(\mathbf{k};t_i,t_f)=\tilde{\mathcal{A}}^{s_2 s_3}_{e\to e\bar{e}\mu}(\mathbf{k};t_i,t_f)$. Let us observe that this expression goes to zero when $\mathbf{k}_1=\mathbf{k}_2$ and $s_1=s_2$, as it should be due to the Pauli principle. We thus compute the probability as

$$\begin{aligned}\mathcal{P}_{e\to ee\bar{\mu}}(\mathbf{p};\Delta t) \;&=\; \frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\sum_{s_1,s_2,s_3}|\mathcal{A}^{r s_1 s_2 s_3}_{e\to ee\bar{\mu}}(\mathbf{p},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3;t_i,t_f)|^2\\ &=\sum_{\mathbf{k},s_2,s_3}|\tilde{\mathcal{A}}^{s_2 s_3}_{e\to ee\bar{\mu}}(\mathbf{k};t_i,t_f)|^2-\sum_{s_3}|\tilde{\mathcal{A}}^{r s_3}_{e\to ee\bar{\mu}}(\mathbf{p};t_i,t_f)|^2\,.\end{aligned} \tag{65}$$

Because of the Pauli principle, the vacuum cannot carry the contribution with $\boldsymbol{k}=\boldsymbol{p}$.Thus, we must isolate the contribution with $\mathbf{k}=\mathbf{p}$

$$\begin{aligned}\mathcal{P}_{e\to ee\bar{\mu}}(\mathbf{p};\Delta t) \;&=\sum_{\mathbf{k}\neq\mathbf{p},s_2,s_3}|\tilde{\mathcal{A}}^{s_2 s_3}_{e\to ee\bar{\mu}}(\mathbf{k};t_i,t_f)|^2+\sum_{s_2,s_3}|\tilde{\mathcal{A}}^{s_2 s_3}_{e\to ee\bar{\mu}}(\mathbf{p};t_i,t_f)|^2-\sum_{s_3}|\tilde{\mathcal{A}}^{r s_3}_{e\to ee\bar{\mu}}(\mathbf{p};t_i,t_f)|^2\\ &=\sum_{\mathbf{k}\neq\mathbf{p},s_2,s_3}|\tilde{\mathcal{A}}^{s_2 s_3}_{e\to ee\bar{\mu}}(\mathbf{k};t_i,t_f)|^2+\sum_{s_3}|\tilde{\mathcal{A}}^{r s_3}_{e\to ee\bar{\mu}}(\mathbf{p};t_i,t_f)|^2\,.\end{aligned} \tag{66}$$

In the large-V limit

$$\mathcal{P}_{e\to ee\bar{\mu}}(\mathbf{p};\Delta t) \;=\; V\sum_{s_2,s_3}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,|\tilde{\mathcal{A}}^{s_2 s_3}_{e\to ee\bar{\mu}}(\mathbf{k};t_i,t_f)|^2+\sum_{s_3}|\tilde{\mathcal{A}}^{r s_3}_{e\to ee\bar{\mu}}(\mathbf{p};t_i,t_f)|^2\,. \tag{67}$$

The first piece diverges and must be subtracted, while the second piece gives a finite contribution. Explicitly

$$\mathcal{P}_{e\to ee\overline{\mu}}(\mathbf{p};\Delta t) \;=\; \frac{4m_{e\mu}^2 Y_{\mathbf{p}}^2}{\left(\omega_{\mathbf{p},e}+\omega_{\mathbf{p},\mu}\right)^2}\sin^2\left(\frac{\left(\omega_{\mathbf{p},\mu}+\omega_{\mathbf{p},e}\right)\Delta t}{2}\right), \tag{68}$$

where

$$Y_{\mathbf{p}}^2 \;=\; \sum_{\mathbf{s}}\left(Y_{\mathbf{p}}^{rs}\right)^* Y_{\mathbf{p}}^{rs}, \tag{69}$$

and

$$Y_{\mathbf{p}} \;=\; \frac{|\mathbf{p}|}{\sqrt{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(\sqrt{\frac{\omega_{\mathbf{p},\mu}+m_\mu}{\omega_{\mathbf{p},e}+m_e}}+\sqrt{\frac{\omega_{\mathbf{p},e}+m_e}{\omega_{\mathbf{p},\mu}+m_\mu}}\right). \tag{70}$$

Therefore, the total decay probability of ν_e is

$$\mathcal{P}_D^e(\mathbf{p};\Delta t) \;=\; 4m_{e\mu}^2\left[\frac{W_{\mathbf{p}}^2}{\left(\omega_{\mathbf{p},e}-\omega_{\mathbf{p},\mu}\right)^2}\sin^2\left(\frac{\left(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e}\right)\Delta t}{2}\right)\right.$$
$$\left.+\frac{Y_{\mathbf{p}}^2}{\left(\omega_{\mathbf{p},e}+\omega_{\mathbf{p},\mu}\right)^2}\sin^2\left(\frac{\left(\omega_{\mathbf{p},\mu}+\omega_{\mathbf{p},e}\right)\Delta t}{2}\right)\right]. \tag{71}$$

It is noteworthy that transitions between neutrino flavors, or flavor decays, are generally not permitted when considering the time evolution from t_i to t_f as $t_i\to-\infty$ and $t_f\to+\infty$, unless the masses of the involved neutrinos are equal (i.e., $m_e = m_\mu$). Such transitions are typically forbidden due to the conservation of energy. In such scenarios, the three-dimensional delta functions appearing in the above transition amplitudes would be replaced by delta functions which ensures the conservation of four-momentum. In other words, uncertainty in energy is fundamental to the occurrence of neutrino flavor oscillations, as it allows for the transition between different flavor eigenstates. These considerations agree with the ones derived by TEUR (39).

We now recognize that, at the leading order in $m_{e\mu}$,

$$|U_{\mathbf{p}}| \;=\; W_{\mathbf{p}}\frac{m_\mu-m_e}{\omega_{\mathbf{p},e}-\omega_{\mathbf{p},\mu}} \;=\; \sqrt{\frac{\left(\omega_{\mathbf{p},e}+m_e\right)\left(\omega_{\mathbf{p},\mu}+m_\mu\right)}{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(1+\frac{|\mathbf{p}|^2}{\left(\omega_{\mathbf{p},e}+m_e\right)\left(\omega_{\mathbf{p},\mu}+m_\mu\right)}\right), \tag{72}$$

$$|V_{\mathbf{p}}| \;=\; Y_{\mathbf{p}}\frac{m_\mu-m_e}{\omega_{\mathbf{p},e}+\omega_{\mathbf{p},\mu}} \;=\; \sqrt{\frac{\left(\omega_{\mathbf{p},e}+m_e\right)\left(\omega_{\mathbf{p},\mu}+m_\mu\right)}{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(\frac{|\mathbf{p}|}{\omega_{\mathbf{p},e}+m_e}-\frac{|\mathbf{p}|}{\omega_{\mathbf{p},\mu}+m_\mu}\right). \tag{73}$$

Then, we write the probability as

$$\mathcal{P}_D^e(\mathbf{p};\Delta t) \;=\; \sin^2 2\theta\left[|U_{\mathbf{p}}|^2\sin^2\left(\frac{\left(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e}\right)\Delta t}{2}\right)+|V_{\mathbf{p}}|^2\sin^2\left(\frac{\left(\omega_{\mathbf{p},\mu}+\omega_{\mathbf{p},e}\right)\Delta t}{2}\right)\right]. \tag{74}$$

with $\theta = m_{e\mu}/(m_\mu - m_e) \approx \sin\theta$. In the approximation we used, this coincides with the oscillation probability (32). This fact is relevant because in computing it we did not use flavor vacuum or Fock space construction and it thus represents an independent derivation of the QFT oscillation formula Eq.(32) [33].

The survival probability is the one associated to the process

$$|\nu_{\mathbf{p},e}^r\rangle \;\to\; |\nu_{\mathbf{k},e}^s\rangle\,. \tag{75}$$

The final result is

$$\mathcal{P}^{e}_{S}(\mathbf{p};\Delta t) \;=\; 1-\sin^{2}2\theta\left[|U_{\mathbf{p}}|^{2}\sin^{2}\left(\frac{\left(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e}\right)\Delta t}{2}\right)+|V_{\mathbf{p}}|^{2}\sin^{2}\left(\frac{\left(\omega_{\mathbf{p},\mu}+\omega_{\mathbf{p},e}\right)\Delta t}{2}\right)\right], \qquad (76)$$

so that

$$\mathcal{P}^{e}_{D}(\mathbf{p};\Delta t)+\mathcal{P}^{e}_{S}(\mathbf{p};\Delta t) \;=\; 1\,, \qquad (77)$$

as expected.

5. Conclusions

We delved into the significance of finite time and energy uncertainties within the framework of QFT for describing neutrino oscillations. To accomplish this, we revisited TEUR in the flavor Fock-space approach and the interaction picture approach to flavor oscillations.

TEUR, in the Mandelstam-Tamm form, relates energy and flavor charge uncertainties. This relation, coupled with the distinction between mass and flavor neutrino Fock spaces, imposes a lower bound on the energy uncertainty for neutrino with a definite flavor.

Similar insights emerge from a perturbative analysis of flavor transitions, where the Dirac picture is utilized to compute the probability of finite time flavor transitions. Remarkably, both the flavor Fock-space and the interaction picture approaches yield identical oscillation probabilities, given the approximation employed in the perturbative calculation. It is important to stress that time-evolution operator must be used instead of S-matrix, which would lead to an exact four-momentum conservation leading to a trivial result, in agreement with TEUR.

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