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Neutrino Mixing Sum Rules and Littlest Seesaw Models

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In this talk we discuss *solar* neutrino mixing sum rules, arising from charged lepton corrections to Tri-bimaximal (TB), Bi-maximal (BM), Golden Ratios (GRs) and Hexagonal (HEX) neutrino mixing. We also discuss *atmospheric* neutrino mixing sum rules, arising from preserving one of the columns of the above types of mixing. We consider the sum rule predictions for θ_{12} and the cosine of the Dirac CP phase δ, in terms of the other mixing angles, using up-to-date global fits of the neutrino oscillation data. We also consider Littlest Seesaw models, based on constrained sequential dominance CSD(n) with $n = 2.5$, $n = 3$ and $n \approx 3.45$ (the latter favoured by modular symmetry), using a new analysis to accurately predict the observables θ_{12}, θ_{23} and δ.

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1. Introduction

Neutrino physics has made remarkable progress since the discovery of neutrino mass and mixing in 1998 [1], with the reactor angle, unknown before 2012, accurately measured by Daya Bay [2]. The oscillation parameters are determined from global fits to be in the three sigma ranges given in Table 1 [3] where the meaning of the angles is given in Table 2.

	NuFIT 5.2
θ_{12} [°]	$31.3 \to 35.7$
$\sin^2\theta_{12}$	$0.27 \to 0.34$
θ_{13} [°]	$8.2 \to 8.9$
$\sin^2\theta_{13}$	$0.020 \to 0.024$
θ_{23} [°]	$40 \to 52$
$\sin^2\theta_{23}$	$0.405 \to 0.62$
δ [°]	$0 \to 44$ & $108 \to 360$
Δm^2_{21} [10^{-5}eV^2]	$6.2 \to 8.3$
Δm^2_{31} [10^{-3}eV^2]	$2.4 \to 2.6$

Table 1: The nu-fit 5.2 results with approximate three sigma ranges without SK atmospheric data for the normal ordered (NO) case, favoured by current data [3].

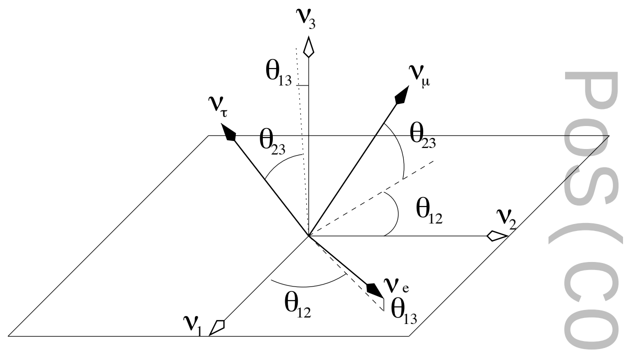

Table 2: Neutrino mixing angles may be represented as Euler angles relating the states in the charged lepton mass basis $(\nu_e, \nu_\mu, \nu_\tau)$ to the mass eigenstate basis states (ν_1, ν_2, ν_3).

The measurement of the reactor angle had a major impact on models of neutrino mass and mixing as reviewed in [4, 5, 6, 7, 62] (for earlier reviews see e.g. [9, 10, 11]).

2. Quark vs Lepton Mixing

The PMNS mixing matrix is (in the PDG parametrisation):

$$U_{li} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \tag{2.1}$$

where $l = e, \mu, \tau$ labels the charged lepton mass eigenstates and $i = 1,2,3$ labels the three neutrino mass eigenstates, and $s_{13} = \sin\theta_{13}$, etc. A similar parameterisation applies to the CKM matrix, but with (very) different angles for quarks and leptons. In the case of Majorana neutrinos, the PMNS matrix also involves the Majorana phase matrix: $P_M = \mathrm{diag}(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}})$ which post-multiplies the above matrix.

It is interesting to compare quark mixing, which is small,

$$s^q_{12} = \lambda, \;\; s^q_{23} \sim \lambda^2, \;\; s^q_{13} \sim \lambda^3 \tag{2.2}$$

where the Wolfenstein parameter is $\lambda = 0.226 \pm 0.001$, to lepton mixing, which is large,[1]

$$s_{13} \sim \lambda/\sqrt{2}, \;\; s_{23} \sim 1/\sqrt{2}, \;\; s_{12} \sim 1/\sqrt{3}. \tag{2.3}$$

[1] As in section 1 lepton parameters are denoted without a superscript l.

The smallest lepton mixing angle θ_{13} (the reactor angle), is of order the largest quark mixing angle $\theta^q_{12} = \theta_C = 13.0^\circ$ (the Cabibbo angle, where $\sin\theta_C = \lambda$). There have been attempts to relate quark and lepton mixing angles such as postulating a reactor angle $\theta_{13} = \theta_C/\sqrt{2}$ [12, 13, 14] and the CP violating lepton phase $\delta \sim -\pi/2$ (c.f. the well measured CP violating quark phase $\delta^q \sim (\pi/2)/\sqrt{2}$).

3. Simple patterns of lepton mixing

Before the measurement of θ_{13}, various simple ansatzes for the PMNS matrix were proposed involving a zero reactor angle and bimaximal atmospheric mixing, $s_{13} = 0$ and $s_{23} = c_{23} = 1/\sqrt{2}$, leading to a PMNS matrix of the form,

$$U_0 = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{3.1}$$

where the solar angle θ_{12} may take various values and the zero subscript reminds us that this form has $\theta_{13} = 0$ (and $\theta_{23} = 45^\circ$).

For tri-bimaximal (TB) mixing [15, 16, 17], one assumes a solar angle $s_{12} = 1/\sqrt{3}$, $c_{12} = \sqrt{2/3}$ ($\theta_{12} = 35.26^\circ$) in Eq. (3.1),

$$U_{\rm TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3.2}$$

For bimaximal (BM) mixing (see e.g. [18, 19, 20, 21] and references therein), one has maximal solar angle $s_{12} = c_{12} = 1/\sqrt{2}$ ($\theta_{12} = 45^\circ$) into Eq. (3.1),

$$U_{\rm BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3.3}$$

For golden ratio (GRa) mixing [22, 23, 24, 25, 26], the solar angle is given by $\tan\theta_{12} = 1/\phi$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio which implies $\theta_{12} = 31.7^\circ$. There are two alternative versions where $\cos\theta_{12} = \phi/2$ and $\theta_{12} = 36^\circ$ [27] which we refer to as GRb mixing, and GRc where $\cos\theta_{12} = \phi/\sqrt{3}$ and $\theta_{12} \approx 20.9^\circ$. Finally another possibility is the hexagonal mixing (HEX) with solar angle $\theta_{12} = \pi/6$ [28, 29].

All these proposals are typically enforced by finite discrete symmetries such as A_4, S_4, S_5 (for a discussion see e.g. [4]). For example S_4 can lead to TB mixing as depicted in Fig. 1.

After the reactor angle was measured, which excluded all these ansatze, there were various proposals to maintain the notion of predictivity of the leptonic mixing parameters. Indeed the measurement of the reactor angle opens up the possibility to predict the CP phase δ, which is not directly measured so far and remains poorly determined even indirectly. Two approaches have been developed, in which some finite symmetry (typically a subgroup of A_4, S_4, S_5) can enforce a particular structure of the PMNS matrix consistent with a non-zero reactor angle, leading to *solar* and *atmospheric* sum rules, as depicted in Fig. 2, and discussed further in the following sections.

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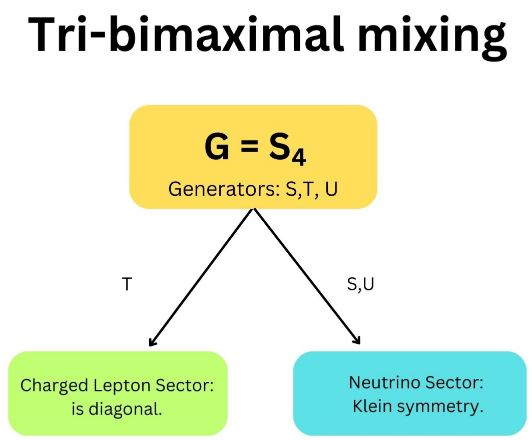

Figure 1: A schematic diagram that illustrate the way that the two subgroups $Z_2^U \times Z_2^S$ and Z_3^T of a finite group work in the charged lepton and neutrino sectors in order to enforce a particular pattern of PMNS mixing. In this example, the group S_4 leads to TB mixing.

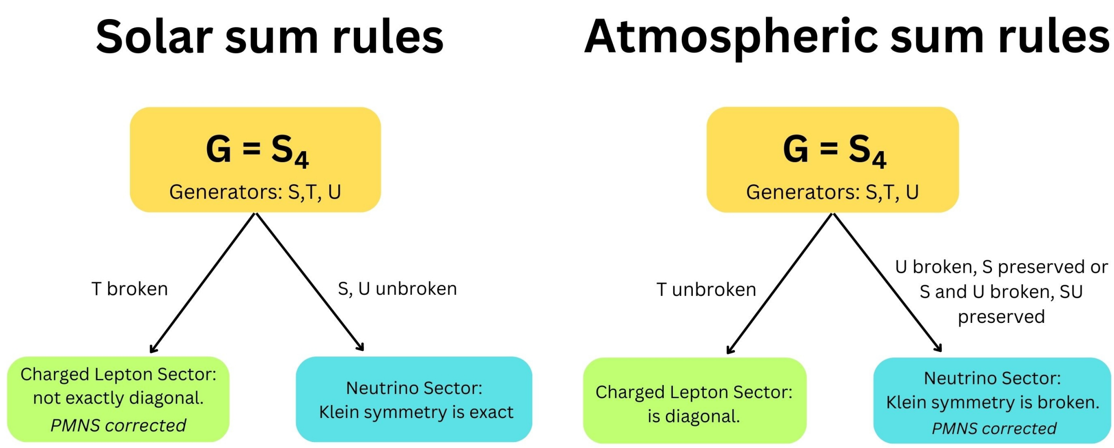

Figure 2: In order to generate a non-zero (13) PMNS element, one or more of the generators S,T,U must be broken. In the left panel we depict T breaking leading to charged lepton mixing corrections and possible *solar* sum rules. In the right panel, U is broken, while either S or SU is preserved leading to neutrino mixing corrections and *atmospheric* sum rules.

4. Solar sum rules

The first approach, which leads to *solar* sum rules, is to assume that the above patterns of mixing still apply to the neutrino sector, but receive charged lepton mixing corrections due to the PMNS matrix being the product of two unitary matrices, which in our convention is written as $U_{eL}U^{\dagger}_{\nu_L}$, where $U^{\dagger}_{\nu_L}$ is assumed to take the BM, TB or GR form, while U_{eL} differs from the unit matrix. If U_{eL} involves negligible 13 charged lepton mixing, then it is possible to generate a non-zero 13 PMNS mixing angle, while leading to correlations amongst the physical PMNS parameters, known as *solar* mixing sum rules [30, 31, 32, 33]. This scenario may be enforced by a subgroup of A_4, S_4, S_5 which enforces the U_ν structure [4] while allowing charged lepton corrections.

From the above discussion we see that the physical PMNS matrix in Eq.2.1 is given by $U_{\rm PMNS} = U^e U^\nu$, where we write $U^e = U_{eL}$ and $U^\nu = U^{\dagger}_{\nu_L}$. Now suppose that $U^\nu = U_0$ is the matrix in Eq.3.1, while U^e corresponds to small but unknown charged lepton corrections. This was first discussed in [30, 31, 32, 33] for the case of TB neutrino mixing where the following sum rule involving the lepton mixing parameters, including crucially the CP phase δ, was first derived [30, 31, 32, 33] :

$$\theta_{12} \approx 35.26^o + \theta_{13}\cos\delta. \tag{4.1}$$

For trimaximal mixing $\theta_{12} \approx 35.26^o$ (where $35.26^o = \sin^{-1}\frac{1}{\sqrt{3}}$) this sum rule predicts $\cos\delta \approx 0$ consistent with $\delta \approx 90^o$ or 270^o, with the former being disfavoured by the global fits.

To derive this sum rule, let us consider the case of the charged lepton mixing corrections involving only (1,2) mixing, so that the PMNS matrix is given by [33],

$$U_{\rm PMNS} = \begin{pmatrix} c^e_{12} & -s^e_{12}e^{-i\delta^e_{12}} & 0 \\ s^e_{12}e^{i\delta^e_{12}} & c^e_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c^\nu_{12} & s^\nu_{12} & 0 \\ -\frac{s^\nu_{12}}{\sqrt{2}} & \frac{c^\nu_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{s^\nu_{12}}{\sqrt{2}} & -\frac{c^\nu_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cdots & \cdots & -\frac{s^e_{12}}{\sqrt{2}}e^{-i\delta^e_{12}} \\ \cdots & \cdots & \frac{c^e_{12}}{\sqrt{2}} \\ \frac{s^\nu_{12}}{\sqrt{2}} & -\frac{c^\nu_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{4.2}$$

Comparing Eq. 4.2 to the PMNS parametrisation in Eq.2.1, we identify the exact sum rule relations [33], in terms of the elements $|U_{e3}|, |U_{\tau 1}|, |U_{\tau 2}|, |U_{\tau 3}|$ identified above. The first element $|U_{e3}| = \frac{s^e_{12}}{\sqrt{2}}$ implies a reactor angle $\theta_{13} \approx 9^\circ$ if $\theta_e \approx \theta_C$ (see e.g. the models in [12]). The second and third elements, $|U_{\tau 1}|, |U_{\tau 2}|$ after eliminating θ_{23}, yield a new relation between the PMNS parameters, θ_{12}, θ_{13} and δ. Expanding to first order, such charged lepton mixing corrections to TB neutrino mixing gives the approximate solar sum rule relations in Eq.4.1 [30].

The above derivation assumes only θ^e_{12} charged lepton corrections. However it is possible to derive an accurate sum rule which is valid for both θ^e_{12} and θ^e_{23} charged lepton corrections (while keeping $\theta^e_{13} = 0$). Indeed, using a similar matrix multiplication method to that employed above leads to the exact results [34]

$$U_{\tau 1} = s^\nu_{12}(s^\nu_{23}c^e_{23} - c^\nu_{23}s^e_{23}e^{i\delta^e_{23}}), \quad U_{\tau 2} = -c^\nu_{12}(s^\nu_{23}c^e_{23} - c^\nu_{23}s^e_{23}e^{i\delta^e_{23}}), \quad \frac{|U_{\tau 1}|}{|U_{\tau 2}|} = \frac{s^\nu_{12}}{c^\nu_{12}} = t^\nu_{12}. \tag{4.3}$$

This relation is easy to understand if we consider only one charged lepton angle θ^e_{12} to be non-zero, then the third row of the PMS matrix in Eq. (4.2) is unchanged, so the elements $U_{\tau i}$ are uncorrected. However, the last relation in Eq. 4.3 clearly holds even if both θ^e_{12} and θ^e_{23} are non-zero due to a cancellation in the ratio $\frac{U_{\tau 1}}{U_{\tau 2}}$. However it fails if $\theta^e_{13} \neq 0$ [35].

The last relation in Eq. 4.3 can be translated into a prediction for $\cos\delta$ as [34][2]

$$\cos\delta = \frac{\tan\theta_{23}\sin\theta^2_{12} + \sin\theta^2_{13}\cos\theta^2_{12}/\tan\theta_{23} - (\sin\theta^\nu_{12})^2\left(\tan\theta_{23} + \sin\theta^2_{13}/\tan\theta_{23}\right)}{\sin 2\theta_{12}\sin\theta_{13}}, \tag{4.4}$$

where only the parameter $\sin\theta^\nu_{12}$ is model dependent and we have respectively $\sin\theta^\nu_{12} = 1/\sqrt{3}$, $\sin\theta^\nu_{12} = 1/\sqrt{2}$, $\tan\theta^\nu_{12} = 1/\varphi$ and $\theta^\nu_{12} = \pi/5$, $\cos\theta^\nu_{12} = \varphi/\sqrt{3}$ and $\theta^\nu_{12} = \pi/6$ for mixing based on TBM, BM, GRa, GRb, GRc and HEX where $\varphi = (1+\sqrt{5})/2$.

To leading order in θ_{13}, Eq.4.4 for the case of TB neutrino mixing returns the sum rule in Eq.4.1. There has been much activity in exploring the phenomenology of various such *solar mixing sum rules* (see e.g. [34, 37]). On the other hand, for a GUT example with $\theta^e_{12} \sim \theta_C/3$ and $\theta^e_{13} \sim \theta_C$ which violates the *solar mixing sum rules* see [38].

[2]For an alternative derivation of an equivalent sum rule see [36].

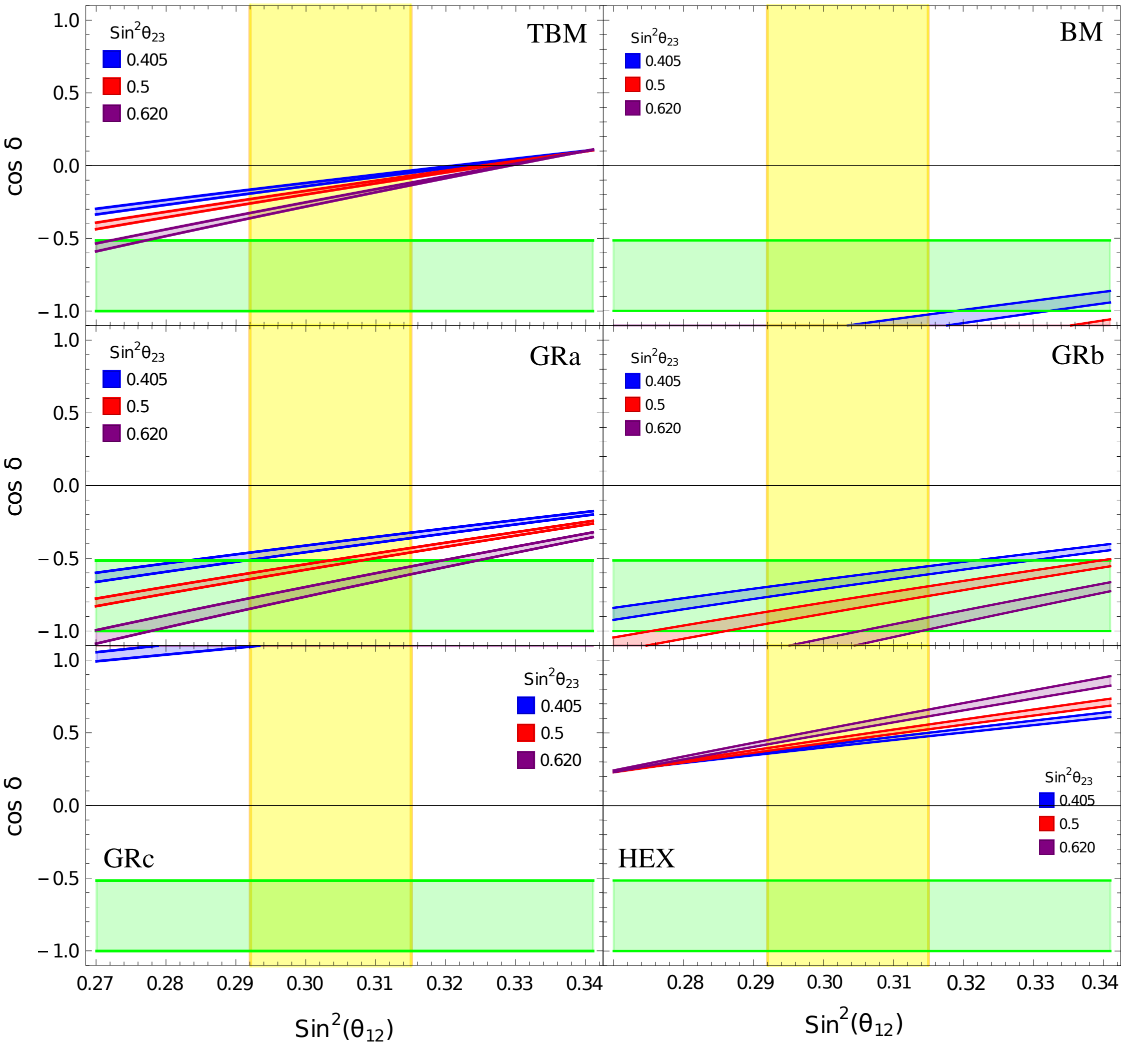

Figure 3: Summary of exact *solar* sum rule predictions derived from Eq. (4.4) for different types of neutrino mixing. For example, in the top left hand panel, we present the sum rule prediction for TBM for $\cos\delta$, as a function of $\sin^2\theta_{12}$, for the different choices of $\sin\theta_{23}$ given in the legend. The width of the diagonal bands are given by the 3σ range of $\sin\theta_{13}$. The horizontal (green) and vertical (yellow) bands are the 1σ ranges for $\cos\delta$ and $\sin^2\theta_{12}$, respectively, while the 3σ range for $\sin^2\theta_{12}$ in Table 1 defines the entire range shown in the plot. Similar plots for BM, GRa, GRb, GRc and HEX are presented respectively on the top right, center right, center left, bottom left, bottom right panels.

In Figure 3 we present the exact sum rules prediction arising from charged lepton corrections as in Eq. (4.4) for TBM, BM, GRa, GRb, GRc and HEX and the constraints from the fit of the neutrino oscillation data [39]. We require $\cos\delta$ to fall in the physical range $-1 < \cos\delta < 1$ and we present it in the y-axis [3]. In all panels the x-axis is $\sin^2\theta_{12}$ and the different colour bands are

[3]Note that the 3σ excluded region $\delta = 45° - 107°$ does not mean that $\cos\delta = 0.71 \to -0.29$ is excluded since the entire range $\delta = 180° - 360°$ is allowed and hence the entire range of $\cos\delta$ is allowed. This highlights a weakness of the sum rules which only predict $\cos\delta$ and not δ itself.

sampled in the allowed $\sin^2\theta_{23}$ region. The width of the band is given by allowing $\sin^2\theta_{13}$ to vary in its 3σ range. The yellow and green bands are the 1σ range respectively of $\sin^2\theta_{12}$ and $\cos\delta$.

For TBM mixing (top-left panel), where $\sin\theta^{\nu}_{12} = 1/\sqrt{3}$ in the neutrino sector, the charged lepton corrections lead to consistent results in all parameter space, with the prediction for $\cos\delta$ being consistent with the leading order prediction in Eq. 4.1 $\cos\delta \approx 0$ for trimaximal mixing $\sin^2\theta_{12} \approx 0.33$. For BM mixing (top-right panel), where $\theta^{\nu}_{12} = 45°$, the sum rule predicts $\cos\delta$ almost outside the physical range and so is close to being excluded at 3σ and only low values of $\sin^2\theta_{12}$ and high values of $\sin^2\theta_{12}$ are still viable. Similarly for GRc mixing (bottom-left panel), with $\cos\theta^{\nu}_{12} = \varphi/3$, the viable parameter space is very tight, only for maximal values of $\sin\theta_{13}$ and minimal values of $\sin\theta_{12}$ and $\sin\theta_{23}$ we can obtain physical results for the CP phase. The yellow and green bands 1σ ranges favour GRa and GRb mixing in the centre panels. For both these models we see that the prediction of $\cos\delta$ are in the negative plane. For GRa (center-left panel), with $\tan\theta^{\nu}_{12} = 1/\varphi$, the whole parameter space leads to physical prediction of $\cos\delta$. For GRb (center-right panel), with $\theta^{\nu}_{12} = \pi/5$ mixing, larger values $\sin\theta_{23}$ are excluded for small values of $\sin^2\theta_{12}$. We finally notice that TBM and HEX are the only models predicting positive values of $\cos\delta$ and HEX (bottom-right panel), with $\theta^{\nu}_{12} = \pi/6$ in particular the only predicting values of $\cos\delta \gtrsim 0.2$.

In summary, of the mixing patterns studied, GRa and GRb are favoured by the current 1σ ranges, while BM and GRc are strongly disfavoured and only consistent with the far corners of the parameter space with a prediction of $|\cos\delta| \approx 1$. The other mixings TBM and HEX are also allowed.

5. Atmospheric sum rules

We now turn to *atmospheric* sum rules, where it is assumed that the physical PMNS mixing matrix takes the BM, TB or GR form but only in its first or second column, while the third column necessarily departs from these structures due to the non-zero 13 angle. Such patterns again lead to correlations amongst the physical PMNS parameters, known as *atmospheric* mixing sum rules. This scenario may be enforced by a subgroup of A_4, S_4, S_5 which enforces the one column U^{ν} structure [4] while forbidding charged lepton corrections.

For example, let us consider again $G = S_4$ and the TB mixing in Eq. (3.2). If we break S and U but preserve SU the first column of the TB matrix is preserved and we have the so-called TM1 mixing pattern [40, 41]

$$U_{\rm TM1} \approx \begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \end{pmatrix}, \tag{5.1}$$

For TM1 where the first column of TB matrix is conserved we have

$$|U_{e1}| = \sqrt{\frac{2}{3}}, \quad |U_{\mu 1}| = |U_{\tau 1}| = \frac{1}{\sqrt{6}}, \tag{5.2}$$

and given the parametrisation in Equation (2.1) we have

$$|U_{e1}| = |c_{12}c_{13}|, \quad |U_{\mu 1}| = |s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta}|, \quad |U_{\tau 1}| = |s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta}|. \tag{5.3}$$

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By comparing these last two equations we obtain two *atmospheric* sum rules for TM1 mixing, for example, $c_{12}c_{13} = \sqrt{\frac{2}{3}}$ and $|s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta}| = \frac{1}{\sqrt{6}}$. The first equation predicts θ_{12} in terms of the accurately measured θ_{13}. The second equation can be expanded to yield a prediction for $\cos\delta$ in terms of the other parameters. The corresponding equation for $|U_{\tau 1}|$ yields equivalent results.

The above *atmospheric* sum rules give powerful constraints on the mixing parameters which may or may not be consistent with present data, and can be tested by future neutrino data. For example, the first *atmospheric* sum rule for TM1 can be expressed as

$$c^2_{12} = \frac{2}{3\,c^2_{13}}, \quad s^2_{12} = \frac{(1-3s^2_{13})}{3\,(1-s^2_{13})} \tag{5.4}$$

which predicts $sin^2\theta_{12}$ in terms of the accurately measured $\sin^2\theta_{13}$, as shown in Fig. 4, where it is easy to understand why it is $\lesssim \frac{1}{3}$. The second *atmospheric* sum rule for TM1 [40] yields, after eliminating θ_{12},

$$\cos\delta = -\frac{\cot 2\theta_{23}(1-5s^2_{13})}{2\sqrt{2}s_{13}\sqrt{1-3s^2_{13}}}. \tag{5.5}$$

If instead S is unbroken the second column is preserved and we have the second mixing pattern TM2 [40, 42]

$$U_{\text{TM2}} \approx \begin{pmatrix} - & \sqrt{\frac{1}{3}} & - \\ - & \sqrt{\frac{1}{3}} & - \\ - & -\sqrt{\frac{1}{3}} & - \end{pmatrix}. \tag{5.6}$$

For TM2 where the second column of TB matrix is conserved we have

$$|U_{e2}| = |U_{\mu 2}| = |U_{\tau 2}| = \frac{1}{\sqrt{3}}, \tag{5.7}$$

and given the parametrisation in Equation (2.1) we have

$$|U_{e2}| = |s_{12}c_{13}|, \quad |U_{\mu 2}| = |c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta}|, \quad |U_{\tau 2}| = |-c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta}|. \tag{5.8}$$

By comparing these last two equations we obtain two *atmospheric* sum rules for TM2 mixing, for example, $s_{12}c_{13} = \sqrt{\frac{1}{3}}$ and $|c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta}| = \frac{1}{\sqrt{3}}$.

As before, the above *atmospheric* sum rules give powerful constraints on the mixing parameters which may or may not be consistent with present data, and can be tested by future neutrino data. For example, the first *atmospheric* sum rule for TM2 can be expressed as

$$s^2_{12} = \frac{1}{3\,c^2_{13}} = \frac{1}{3\,(1-s^2_{13})}, \tag{5.9}$$

which predicts $sin^2\theta_{12}$ in terms of the accurately measured $\sin^2\theta_{13}$, as shown in Fig. 4, where it is easy to understand why it is $\gtrsim \frac{1}{3}$. The second *atmospheric* sum rule for TM2 [40] yields, after eliminating θ_{12},

$$\cos\delta = \frac{2c_{13}\cot 2\theta_{23}\cot 2\theta_{13}}{\sqrt{2-3s^2_{13}}}. \tag{5.10}$$

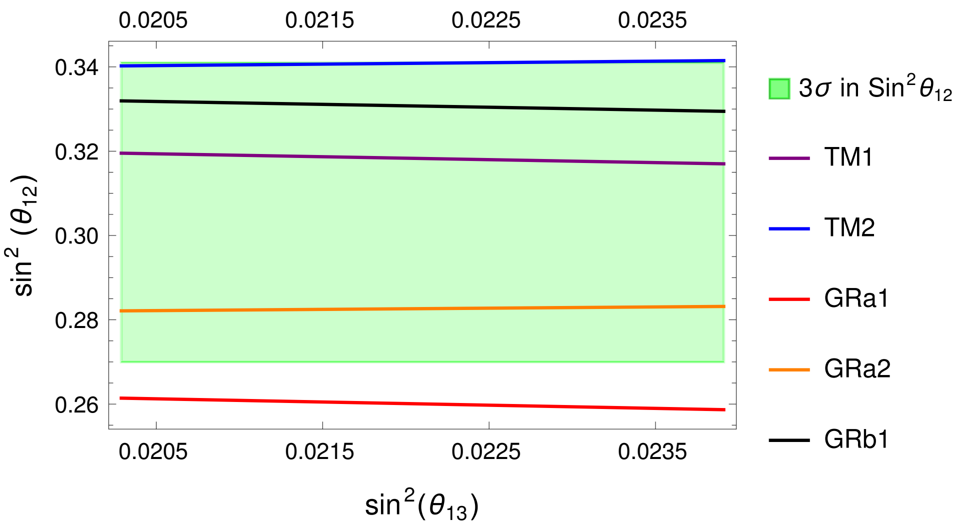

Figure 4: Summary of exact *atmospheric* sum rule predictions which predict the solar angle for different types of lepton mixing corresponding to a preserved column of the PMNS matrix, with only a mild dependence on the reactor angle. The pink, blue, red, orange and black curves are respectively the predictions for the surviving TM1, TM2, GRa1, GRa2 and GRb1 mixing patterns (with GRa1 just outside and TM2 just inside the 3σ allowed region in green). Other possibilities not plotted are further outside the allowed region.

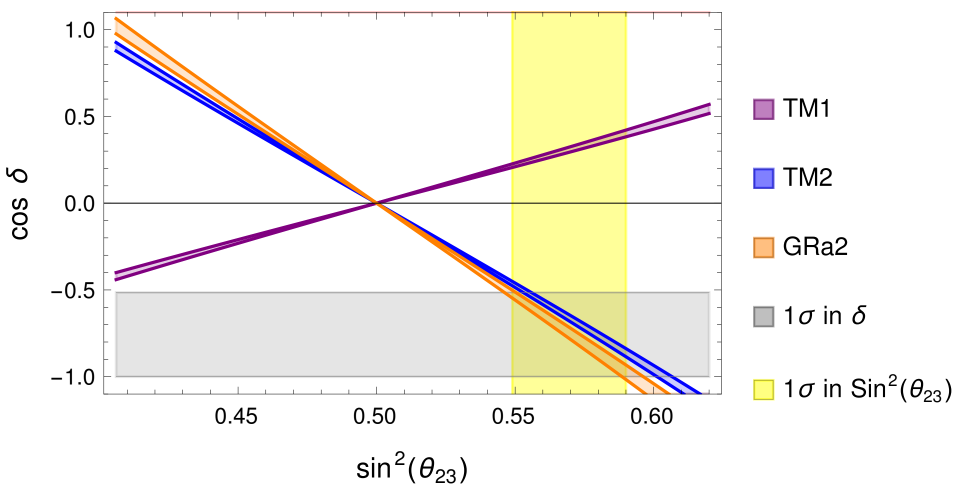

Figure 5: Summary of exact *atmospheric* sum rule predictions which predict $\cos\delta$ in terms of the other mixing angles for different types of lepton mixing corresponding to a preserved column of the PMNS matrix. We present with the blue band the exact sum rule prediction for TM2 for $\cos\delta$ letting $\sin^2\theta_{13}$ vary in its 3σ range. In orange and purple we present the exact the sum rule predictions for GRa2 and TM1. Only the viable mixing patterns are plotted. The yellow and grey regions are respectively the 1σ range of $\sin\theta_{23}$ and $\cos\delta$, while the plot covers the whole 3σ range. These predictions can be further tested at future neutrino experiments [43].

For the other mixing patterns BM, GRa, GRb, GRc, HEX the discussion is similar where we call BM1 and BM2 the *atmospheric* sum rules respectively derived by preserving the first and second column of BM, and similarly for the other mixing patterns. However BM1, BM2, GRa1, GRb2, GRc1, GRc2, HEX1 and HEX2 are excluded by the solar mixing prediction. Only four possibilities survive the solar mixing prediction, namely TM1, TM2, GRa2 and GRb1, and these are plotted in Fig. 4.

In Figure 5 we show the exact *atmospheric* sum rules for $\cos\delta$ for the models that are still allowed from Figure 4. However GRb1 mixing does not appear in the plot because it predicts unphysical values of $\cos\delta$.

In summary, only TM1, TM2 and GRa2 survive the *atmospheric* sum rules (with TM2 on the edge of solar angle exclusion as discussed above).

6. The Littlest Seesaw

The sequential dominance (SD) [44, 45] of right-handed neutrinos (RHNs) proposes that the mass spectrum of heavy Majorana neutrinos is strongly hierarchical, where the RHN with mass $M_{\rm atm}$ is responsible for the atmospheric neutrino mass, that with mass $M_{\rm sol}$ gives the solar neutrino mass, and a third largely decoupled RHN gives a suppressed lightest neutrino mass. It leads to an effective two right-handed neutrino (2RHN) model [46, 47] with a natural explanation for the physical neutrino mass hierarchy, with normal ordering $m_3 > m_2$ where the lightest neutrino is approximately massless, $m_1 \approx 0$ (being exactly massless in the 2RHN limit).

A very predictive minimal seesaw model with two right-handed neutrinos and one texture zero is the so-called constrained sequential dominance (CSD) model [30, 48, 49, 50, 51, 52, 53, 54, 55, 56]. The CSD(n) scheme assumes the flavour basis (i.e. diagonal RHN and charged lepton mass matrices) and a Dirac neutrino mass matrix of the form, [4]

$$m^D = \begin{pmatrix} 0 & a \\ e & na \\ e & (n-2)a \end{pmatrix}, \quad \begin{pmatrix} 0 & a \\ e & (n-2)a \\ e & na \end{pmatrix} \tag{6.1}$$

where e, a are complex mass matrix elements, n is a real number and the first option is denoted "normal", the second one being "flipped". For example the CSD(3) (also called Littlest Seesaw model) [49, 50, 51, 52, 53], CSD(4) models [54, 55] and CSD(2.5) [57] can give rise to phenomenologically viable predictions for lepton mixing parameters and the two neutrino mass squared differences Δm^2_{21} and Δm^2_{31}, corresponding to special constrained cases of lepton mixing which preserve the first column of the TB mixing matrix, namely TM1 and hence satisfy *atmospheric* mixing sum rules. As was observed, modular symmetry remarkably suggests CSD($1+\sqrt{6}$) $\approx$ CSD(3.45) [58, 59, 60, 61, 62]. We define the Littlest Seesaw (LS) as CSD(~ 3) since $n \sim 3$ is the preferred phenomenological choice.

[4]In the flavour basis, the form of m^D can be multiplied by phases acting on the left-handed electroweak lepton doublets L_e, L_μ, L_τ (compensated by equal phases acting on the right-handed charged leptons) without changing the physics. For example, the first, second or third rows of m^D can be independently multiplied by -1, as can the first or second columns due to RHN rephasings of -1 (but not more general phases for RHN Majorana masses).

For the "normal" case of CSD(n), after the seesaw mechanism the light effective neutrino Majorana mass matrix in the diagonal charged lepton basis is given by

$$m^{\nu} = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_b e^{i\eta} \begin{pmatrix} 1 & n & n-2 \\ n & n^2 & n(n-2) \\ n-2 & n(n-2) & (n-2)^2 \end{pmatrix}, \tag{6.2}$$

where the two real mass parameters are

$$m_a = \frac{|e|^2}{M_{\text{atm}}} \qquad m_b = \frac{|a|^2}{M_{\text{sol}}}, \tag{6.3}$$

and the only relevant phase is $\eta = 2\arg(a/e)$. The "flipped" case gives a mass matrix similar to Eq. 6.2 but with the second and third rows and columns interchanged.

Interestingly, the mass matrix in Eq.6.2 and the flipped version both predict TM1 mixing [5], and hence CSD(n) already satisfies the two sum rules in Eqs. 5.4, 5.5 for any choice of n and the parameters m_b/m_a and η. However since there are only three real input parameters responsible for the entire PMNS matrix and the neutrino mass ratios, it is more predictive, as we now discuss.

In the diagonal charged lepton mass basis which we are using, the PMNS mixing matrix is fully specified by the choice of n and the parameters m_b/m_a and η. Indeed it is possible to derive exact analytic results for all the neutrino masses and the PMNS mixing parameters [50] in terms of these parameters, for "normal" case. The expressions are a little complicated, but the key observation is that $\sin\theta_{13}$, $\tan\theta_{23}$, $\sin\delta$ and $\Delta m^2_{21}/\Delta m^2_{31} = m_2^2/m_3^2$ are predicted, in addition to the TM1 sum rule predictions for $\sin\theta_{12}$ and $\cos\delta$, all in terms of the three real parameters n, m_b/m_a and η. Note that CSD(n) predicts not just $\cos\delta$ as in the sum rules, but also $\sin\delta$, and hence δ is fully predicted without ambiguity. However the predictions for the "flipped" (f) case are related to those of the "normal" (n) case by $\theta^f_{23} = 90^\circ - \theta^n_{23}$ and $\delta^f = \delta^n + 180^\circ$ which flips the octant and reflects the phase.

It is debatable whether n should be regarded as a continuous free parameter since it may be fixed via models based on S_4 symmetry [50, 51]. From this point of view the only free parameters are m_b/m_a and η which together fix everything in the neutrino sector. Indeed the best measured parameters are $\sin\theta_{13}$ and $\Delta m^2_{21}/\Delta m^2_{31} = m_2^2/m_3^2$ and these two parameters may be used to accurately fix the two inputs m_b/m_a and η for any given choice of n, as shown in Fig. 6 for three values of $2 < n < 4$ since the contours would not intersect for values of n outside this range. Thus the sweet spot is $n \sim 3$ and this case is called the "Littlest Seesaw" (LS). For each value of n there are thus sharp predictions for θ_{23} and δ which go beyond the TM1 sum rule predictions for θ_{12} and $\cos\delta$. This is illustrated In Table 3 where we focus on $n = 1+\sqrt{6} \approx 3.45$, which can be realised with modular symmetry [58, 59, 60, 61, 62]. For both "normal" and "flipped" cases there are two possible 3σ ranges of atmospheric angle, one in the first octant and one in the second octant, depending on η. Renormalisation group corrections to the LS results have been shown to be generally quite small [63, 64, 65], since the neutrino masses are hierarchical and normal ordered with $m_1 = 0$.

[5]This is because the first column of the TBM mixing matrix in Eq. 5.1 is an eigenvector of the neutrino mass matrix in Eq. 6.2 with zero eigenvalue corresponding to the lightest physical neutrino mass being zero.

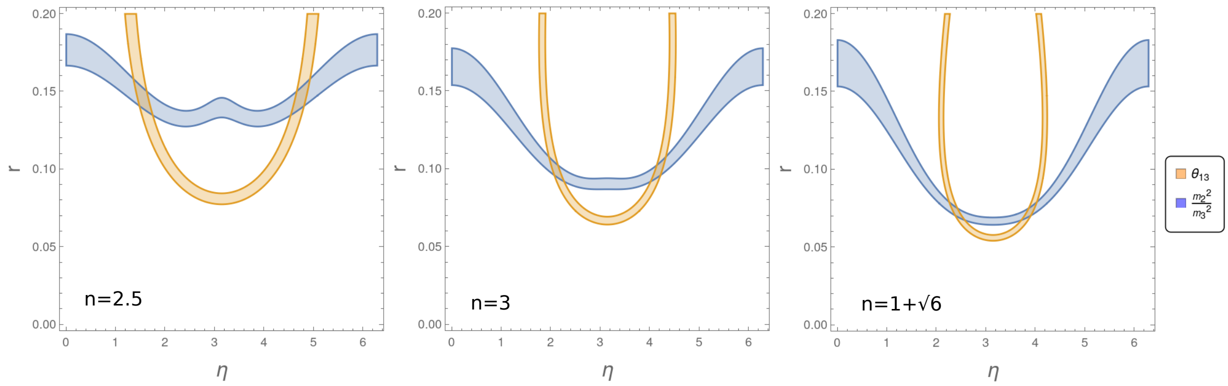

Figure 6: The results for the LS models with $n \approx 3$. The input parameters η and $r = m_b/m_a$ are constrained to a good degree of accuracy by only two experimental observables, namely θ_{13} and the mass ratio m_2^2/m_3^2. The 3σ allowed region for θ_{13} and the mass ratio are respectively the blue and orange band. The area of intersection is the allowed parameter space for η and r. From the left to the right we assume, $n = 2.5$, 3 and $1+\sqrt{6} \approx 3.45$.

	$n = 1+\sqrt{6}$	$\eta = 2.42 \pm 0.16$	$\eta = 3.87 \pm 0.16$
	θ_{12} [°]	$34.36^{+0.18}_{-0.21}$	$34.36^{+0.18}_{-0.21}$
normal	θ_{23} [°]	$41.4^{+2.6}_{-2.6}$	$41.5^{+2.6}_{-2.6}$
normal	δ [°]	$253.8^{+11.7}_{-13.8}$	$105.7^{+13.7}_{-11.6}$
flipped	θ_{23} [°]	$48.6^{+2.6}_{-2.6}$	$48.5^{+2.6}_{-2.6}$
flipped	δ [°]	$74.8^{+11.7}_{-13.8}$	$285.8^{+13.7}_{-11.6}$

Table 3: The LS predictions for $n = 1+\sqrt{6} \approx 3.45$ where the two most accurately measured observables, θ_{13} and the mass squared ratio m_2^2/m_3^2, are used to accurately determine the two input parameters $r = m_b/m_a = 0.072 \pm 0.004$ for two η ranges as shown above, corresponding to the right panel of Fig. 6. This then leads to highly constrained predictions for the less accurately determined observables θ_{12}, θ_{23} and δ, which may be compared to the current experimental ranges as shown in Table 1. All results are given to 3σ accuracy.

7. Conclusion

In conclusion we have discussed *solar* neutrino mixing sum rules, arising from charged lepton corrections to Tri-bimaximal (TB), Bi-maximal (BM), Golden Ratios (GRs) and Hexagonal (HEX) neutrino mixing, and saw that GRa and GRb are favoured, TBM and HEX are allowed, but BM and GRc are strongly disfavoured. We also discussed *atmospheric* neutrino mixing sum rules, arising from preserving one of the columns of the above types of mixing, and saw that only TM1, TM2 and GRa2 survive, with TM2 on the edge of solar angle exclusion. We also studied CSD(n) with $n \sim 3$ (also known as the Littlest Seesaw (LS)) as an ultraviolet completion of the successful TM1. It is possible to realise CSD(n) and fix n from theory, for example modular symmetry yields $n \approx 3.45$. Then θ_{13} and $\Delta m_{21}^2/\Delta m_{31}^2$ may be used to fix the two input parameters, resulting in predictions for θ_{23}, in addition to the TM1 *atmospheric* neutrino mixing sum rule predictions for θ_{12} and $\cos\delta$.

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