PROCEEDINGS OF SCIENCE

Static and radiative cylindrically symmetric spacetimes

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Cylindrical symmetry allows for potentially significant astrophysical applications. Even the general relativistic vacuum is quite rich, including a static ground state, the Levi–Civita spacetime and multiple types of Einstein–Rosen exact wave solutions. We summarize our recent study on the interpretation of the static background, then proceed to discuss the backreaction of the cylindrical waves at second order, in the WKB approximation and under the Brill–Hartle averaging scheme, identifying the lasting effect of a passing wave on the background as an emerging exact radiative spacetime. We also briefly discuss extensions of the Einstein–Rosen waves in generalized Brans–Dicke theories, where the scalar field is sourcing these waves.

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1. Introduction

Being as complicated as a system of 10 second order nonlinear coupled partial differential equations in 10 dependent (metric components) and 4 independent (coordinates) variables, the Einstein equations need simplifying assumptions to be solved, even for the most simple matter sources, in particular vacuum. After spherical symmetry, the next simplest would be cylindrical. Studying cylindrically symmetric spacetimes also qualifies a precursor of investigations under axial symmetry, which is omnipresent in nature, as it faithfully characterizes rotating astrophysical objects.

Cylindrical symmetry has been useful in discussing extended sources, including cosmic strings [1], [2]. It is also a suitable local approximation for cosmic filaments of galaxies and dark matter extending across hundreds of million of light years [3], [4]. Beside, energetic jets of supermassive black holes are modeled either as having conical or cylindrical symmetry [5], [6], [7].

Cylindrical symmetry was the prerequisite to derive the existence of first exact (nonperturbative) gravitational wave solutions by Einstein and Rosen [8], which included for both standing wave and approximate progressive wave behaviors. This class also contains solitonic waves [9] and impulsive waves [10], [11].

Finally, these cylindrically symmetric waves were the testbed of canonical quantization, as the earliest example of the midisuperspace approach, by Kuchař [12]. The approach provided a much richer structure than previous midisuperspace quantizations of the Friedmann universe by DeWitt or of the mixmaster universe by Misner. The topic has been accurately revisited by Torre and Varadarajan and an alternative approach developed by Ashtekar and Pierri. All these are presented in detail in the review [13]. Cylindrical symmetry shows up as a meaningful compromise between the simplicity induced by too many degrees of freedom frozen by symmetry assumptions in the minisuperspace approach and the full complexity of the gravitational degrees of freedom, hence it provides an ideal testbed for comparing quantization approaches.

The cylindrically symmetric vacuum appears as much richer than the spherically symmetric one. The latter, constrained by the Birkhoff–Jebsen unicity theorem, allows only for either the static Schwarzschild or the homogeneous Kantowski–Sachs spacetime. The cylindrically symmetric vacuum however is quite rich, with the plethora of Einsten–Rosen type gravitational waves but also a static ground state, the Levi-Civita spacetime [14].

2. Interpretation of the Levi-Civita spacetime

The Levi-Civita spacetime has a single parameter, however its interpretation cannot be farther from the mass parameter of the Schwarzschild spacetime. Although is often interpreted as a mass density on the axis of the cylinder, this interpretation stands only for its small values. Indeed, the vanishing of the parameter leads to a flat spacetime. Nevertheless, so it does another nonvanishing but finite value of the parameter. Moreover, when the parameter diverges either on the positive or negative range, flat spacetime emerges again.

In Ref. [15] the Komar mass density was introduced for the Levi-Civita spacetime. Although the Komar mass requires to integrate over a closed surface encompassing the source, a condition the cylinder axis obviously violates, it was possible to employ the procedure to compactify it, calculate a Komar mass density, then decompactify. The resulting Komar mass density, when employed as new metric parameter, lifted the original quadruple fold degeneracy of the flat limit. Minkowski spacetime arises only in the limit of vanishing Komar mass density and in its infinite limit. For small values, the Komar mass density can still be interpreted as a mass density on the symmetry axis. At its infinite value limit however the Levi-Civita spacetime is Rindler, with topology $S^1 \times \mathbb{R}$, as can be most easily seen from its Kasner form.

All 4 curvature invariants of the vacuum Levi-Civita spacetime can be expressed in terms of the Komar mass density and the proper radial distance from the axis. These dependencies show that the metric is singular on the axis. Moreover, radial null geodesics satisfy both the Tipler and the Królak singularity conditions, hence the singularity is strong [15]. The singularity is also naked, as there is no horizon.

The explanation for the double flat limit lies in the difference of Newtonian and Einsteinian gravity concepts. A stationary observer at fixed proper distance from the symmetry axis senses a Newtonian gravitational acceleration increasing monotonically with the Komar mass density. By contrast, Einsteinian gravity, defined through the mutual tidal acceleration of nearby geodesics towards each other, increases up to the value 0.5 of the Komar mass density, after which it starts to decrease and asymptote to zero with increasing Komar mass density. What happens here can be visualized through the following process. The increasing Komar mass density deepens the Newtonian gravity well until the field lines become almost parallel, leading to disappearing Einsteinian gravity and the emerging Rindler limit with $S^1 \times \mathbb{R}$ topology.

3. Backreaction of cylindrically symmetric gravitational waves

The Einstein–Rosen waves are much richer cylindrically symmetric, exact vacuum solutions of the Einstein equations. When it comes to gravitational waves treated perturbatively, in the usual approach a linearization about the flat metric is employed, resulting in a wave equation (in the transverse-traceless gauge) for the perturbation, which is then interpreted as a wave. On a curved backgroud however the very concept of wave needs to be defined. This is possible whenever the wavelength is much smaller than the curvature radius of the background (geometric optics or high-frequency approximation). Such a condition is always fulfilled on flat background, but on curved background it needs to be imposed. Then there are two small parameters in the formalism: one is describing the weakness of the perturbation (similar to and generalizing a post-Newtonian parameter), while a second assures the validity of the geometrical optics approximation. Isaacson gave arguments based on which he identified these two small parameters. Compared to the flat case, the existence of the second parameter results in order shifts in the expansions, some of the derivatives carrying inverse powers of it [16]. This is how in the leading order the Einstein equation emerges with an effective source term, representing a quadratic contribution in the gravitational wave, the Isaacson energy-momentum tensor. The interpretation of it is that a gravitational wave sweeping over the background causes a lasting effect.

In the next order a wave equation still appears, obviously with covariant derivatives and also some correction terms including the Riemann and Ricci curvatures. This is the Lichnerowitz wave equation. When the Wentzel–Kramers–Brillouin (WKB) approximation is imposed (slowly changing amplitude and fast changing phase), the Lichnerowitz wave equation degenerates into the covariant wave equation, on a curved background.

Isaacson has also proven that gravitational waves with spherically symmetric Brill–Hartle average have a lasting effect on the flat background, changing it into a Vaidya spacetime [17]. The null dust source of it popped out from the quadratic backreaction of the gravitational waves in the WKB approximation and upon integration over spacetime regions containing a significant number of wavelengths.

We have repeated the procedure, demonstrating similar behaviour under cylindrical symmetry. The null dust source emerging as a Brill–Hartle average over WKB waves in this case generates the radiating Levi–Civita spacetime identified by Rao [18]. The details of this work will be shown elsewhere [19].

4. Einstein–Rosen waves in generalized Brans–Dicke theories

Scalar-tensor gravity theories also admit Einstein–Rosen type gravitational waves. In the simplest Brans–Dicke theory Akyar and Delice have shown that the Einstein–Rosen waves are quite similar to the general relativistic ones, describing linearly polarized gravitational waves [20]. In a work in preparation [21] we will show our recent finding, that if one allows the Brans–Dicke coupling constant to be scalar field dependent, the same wave equation under cylindrical symmetry can be derived for one of the metric functions, as in the Brans–Dicke theory, however with a slight, but important modification, as it contains a new, scalar field dependent source term. In this case the Einstein–Rosen wave is sourced by the scalar field.

5. Summary

Cylindrical symmetry is worth studying, allowing for conceptually interesting and astrophysically relevant applications in strings, galactic dark matter filaments, AGN and quasar jets. The cylindrically symmetric vacuum is not unique. Its static ground state, the Levi-Civita spacetime can be conveniently described in terms of the Komar mass density of the axis, which represents a strong singularity. Einsteinian and Newtonian gravity concepts behave quite differently with increasing Komar mass density, the Newtonian part containing eventually only an acceleration-type contribution (a homogeneous gravitational field), which although diverges, leaves no Einsteinian (tidal) gravity in the limit.

Several types of Einstein–Rosen waves are also cylindrically symmetric vacuum solutions. We have shown that the backreaction of weak cylindrical waves through the Isaacson procedure and WKB approach gives the radiating Levi-Civita (Rao) spacetime, sourced by the incoherent superposition of electromagnetic or gravitational waves with random phases and polarisations (null dust). This is in full analogy with the emergence of Vaidya spacetime through backreaction in the spherically symmetric case. Further details will be given elsewhere [19].

Einstein–Rosen waves were also known in Brans–Dicke scalar-tensor theories. Generalizing them, we identified an inhomogeneous Einstein–Rosen wave equation for a class of generalised Brans–Dicke theories. This generalized Einstein–Rosen wave is sourced by the scalar. Further details will be given elsewhere [21].

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