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Yano F structures and extended Supersymmetry in a BiLP

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We look at a sigma model based on the $(4, 4)$ twisted chiral multiplet [3]. It admits two geometric descriptions: The ususal biquaternionic geometry on the tangent space and a new geometry involving two Yano F-structures on a doubled tangent space. This analysis sheds light on the recent discussion of semichiral models in [1] and on an upcoming discussion of complex linears.

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1. Introduction

In a recent article[1] [1] on $(2,2)$ symplectic sigma models, it is shown that additional supersymmetries imply the existence of two integrable Yano F structures.

The analysis deals with a general symplectic model and would benefit from an explicit example. In this note we illustrate the general procedure in the transparent case of the $(4,4)$ twisted chiral multiplet [3]. Unlike in the symplectic model there are no auxiliary fields to be integrated out so the relation between $(2,2)$ and $(1,1)$ formulations is very direct. Nevertheless the two geometries found in the symplectic model are found here too.

2. Background

2.1 $(1,1)$ Geometry of sigma models

Additional symmetries of a $(1,1)$ supersymmetric non linear sigma model in two dimensions are associated with additional complex structures J^i. The transformations follow the pattern

$$\delta^i\varphi = \epsilon_i^{\pm} J^i \mathbb{D}_{\pm}\varphi \ . \tag{1}$$

When the extra symmetry is $(4,4)$, the corresponding complex structures form an $SU(2)$ algebra and are covariantly constant with respect to the Levi Civita connection (hyperkähler) or the Levi-Civita connection plus torsion (bi hermitian). The metric is hermitian with respect to all complex structures.

2.2 $(2,2)$ Geometry of pre-sigma models

The $(2,2)$ algebra is

$$\{\mathbb{D}_{\pm}, \bar{\mathbb{D}}_{\pm}\} = 2i\partial_{\substack{+\\=}} \ . \tag{2}$$

The symplectic model discussed in [1] is based on left ℓ and right $\mathfrak{r}$ semichiral superfields

$$\bar{\mathbb{D}}_{+}\ell = 0 \ , \quad \bar{\mathbb{D}}_{-}\mathfrak{r} = 0 \ . \tag{3}$$

These $(2,2)$ fields contain auxiliary spinorial $(1,1)$ superfields which have to be integrated out before the sigma model structure can be seen. So the $(2,2)$ formulation is not a sigma model and we don't expect it to carry the usual geometry. Instead it was found in [1] that an ansatz

$$\delta\varphi = U^{(\alpha)}\bar{\epsilon}^{\alpha}\bar{\mathbb{D}}_{\alpha}\varphi + V^{(\alpha)}\epsilon^{\alpha}\mathbb{D}_{\alpha}\varphi \ , \quad \alpha \in (+,-) \tag{4}$$

leads to the existence of two Yano F-structures $\mathcal{F}_{(\alpha)}$ on the doubled tangent bundle.

[1] Partly drawing on [2].

2.3 Strategy

The procedure in [1] can be summarised as follows:

1. In $(2,2)$ superspace, consider a sigma model based on $(2,2)$ superfields φ and make the most general ansatz for additional susy transformations in terms of φ, preserving all their chirality conditions.

2. Use closure of the susy algebra and invariance of the action to constrain the transformations.

3. Identify the geometric objects that result from step 2. E.g. the Yano F-stuctures for the symplectic model in [1].

4. Reduce the resulting transformations and structures to $(1,1)$ superspace where the geometry corresponding to extended susy is wellknown.

Below we check these steps for the twisted chiral $(4,4)$ multiplet [3] where all structure is known.

3. The $N = 4$ twisted chiral multiplet

In $(2,2)$ superspace the $N = 4$ twisted chiral multiplet is given in terms of chiral superfields ϕ and twisted chiral superfields χ

$$\begin{aligned}&\bar{\mathbb{D}}_{\pm}\phi = 0\\&\bar{\mathbb{D}}_{+}\chi = \mathbb{D}_{-}\chi = 0\ .\end{aligned} \tag{5}$$

A model based on these has an action

$$S = \int d^2x d^2\theta d^2\bar{\theta} K(\phi, \bar{\phi}, \chi, \bar{\chi})\ . \tag{6}$$

Reduction to $(1,1)$ superspace reveals that the target space geometry has two commuting complex structures, a local product structure and a metric that is hermitean wrt both complex structures: A Bihermitean Local Product (BiLP) geometry [3], [4]. This model has $(4,4)$ off-shell supersymmetry when the $(2,2)$ fields sit in the $(4,4)$ twisted chiral multiplet, [3] provided that

$$K,_{\phi\bar{\phi}} + K,_{\chi\bar{\chi}} = 0 \tag{7}$$

and that the complex structures are covariantly constant wrt connections with torsion, as shown in [3].

A description in terms of transformations of the $(2,2)$ fields instead leads to Yano F structures, which we now discuss.

3.1 Transformations

To simplify the description we consider the case of one chiral ϕ and one twisted chiral multiplet χ transforming under the extra susy as [3]

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}^{+}\bar{\mathbb{D}}_{+}\bar{\chi} + \bar{\epsilon}^{-}\bar{\mathbb{D}}_{-}\chi \\
\delta\bar{\phi} &= \epsilon^{+}\mathbb{D}_{+}\chi + \epsilon^{-}\mathbb{D}_{-}\bar{\chi} \\
\delta\chi &= -\bar{\epsilon}^{+}\bar{\mathbb{D}}_{+}\bar{\phi} - \epsilon^{-}\mathbb{D}_{-}\phi \\
\delta\bar{\chi} &= -\epsilon^{+}\mathbb{D}_{+}\phi - \bar{\epsilon}^{-}\bar{\mathbb{D}}_{-}\bar{\phi}\ .
\end{aligned} \tag{8}$$

Defining

$$\varphi := \left(\begin{array}{c} \phi \\ \bar{\phi} \\ \chi \\ \bar{\chi} \end{array}\right)\ , \tag{9}$$

we write (8) as

$$\delta\varphi = U^{(\pm)}\bar{\epsilon}^{\pm}\bar{\mathbb{D}}_{\pm}\varphi + V^{(\pm)}\epsilon^{\pm}\mathbb{D}_{\pm}\varphi \tag{10}$$

where $V = \bar{U}$ suitably rearranged, and

$$U^{(+)} = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\ ,\quad V^{(+)} = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right)\ , \tag{11}$$

and similarly for $U^{(-)}$ and $V^{(-)}$. We find that $U^{(+)}V^{(+)} = -diag(1,0,1,0)$, $V^{(+)}U^{(+)} = -diag(0,1,0,1)$ and that $U^{(-)}V^{(-)} = -diag(1,0,0,1)$ and $V^{(-)}U^{(-)} = -diag(0,1,1,0)$.

Closure of the extended susy algebra leads to the Nijenhuis tensors for U^{α} and V^{α} vanishing.

3.2 Geometry

The above results allows us to define two Yano F-structures $\mathcal{F}_{(\pm)}$ [5] [6] acting on the doubled[2] tangent space $T \oplus T$:

$$\mathcal{F}_{(\alpha)} := \left(\begin{array}{cc} 0 & U^{(\alpha)} \\ V^{(\alpha)} & 0 \end{array}\right)\ ,\quad \mathcal{F}^{3}_{(\alpha)} + \mathcal{F}_{(\alpha)} = 0\ . \tag{12}$$

An F structure defines projection operators

$$l := -\mathcal{F}^{2} = -\left(\begin{array}{cc} UV & 0 \\ 0 & VU \end{array}\right) \text{and } m := \mathbb{1} + \mathcal{F}^{2} = \left(\begin{array}{cc} \mathbb{1} + UV & 0 \\ 0 & \mathbb{1} + VU \end{array}\right) \tag{13}$$

[2]Whitney sum

that obey

$$l+m=\mathbb{1},\quad l^2=l,\quad m^2=m\,,\quad lm=ml=0$$
$$\mathcal{F}l=l\mathcal{F}=\mathcal{F},\quad m\mathcal{F}=\mathcal{F}m=0\,. \tag{14}$$

The subspace corresponding to the projector $m_{(+)}$ is $(\phi,\chi)\oplus(\bar\phi,\bar\chi)$ while the complement $(\bar\phi,\bar\chi)\oplus(\phi,\chi)$ corresponds to the projector $l_{(+)}$; $m_{(-)}$ projects onto $(\phi,\bar\chi)\oplus(\bar\phi,\chi)$ with the complement $(\bar\phi,\chi)\oplus(\phi,\bar\chi)$.

In the special case when all entries in the U and V matrices are derivatives of a function,[3] the condition from *invariance of the action* (6) for the $\bar\epsilon^\alpha$ transformations in (10) reads:

$$0=\left(K_{,i}\,U^{(\alpha)i}{}_{[j}\right)_{k]}=K_{,i[j}\,U^{(\alpha)i}{}_{k]}\,, \tag{15}$$

The conditions for invariance under ϵ^α transformations follow by complex conjugation, replacing $U\to V$. Using the explicit form of U and V in (11), we indeed recover the conditions (7).
The requirements on U and V mentioned in Sec. 3.1 are sufficient to show that the F-structures are integrable. The details follow in parallel to the discussion in [1].

3.3 The $(1,1)$ description recovered.

At the $(1,1)$ level extended susy is governed by additional complex structures. For $(4,4)$ susy there are three extra complex structures for plus and three for minus directions $\mathcal{J}^{\mathfrak{A}}_{(\alpha)}$. They form a bi-quaternion structure:

$$\mathcal{J}^{\mathfrak{A}}_{(\alpha)}\mathcal{J}^{\mathfrak{B}}_{(\alpha)}=-\delta^{\mathfrak{A}\mathfrak{B}}+\epsilon^{\mathfrak{A}\mathfrak{B}\mathfrak{C}}\mathcal{J}^{\mathfrak{C}}_{(\alpha)}\,. \tag{16}$$

We now relate the $(2,2)$ superspace results to the bi-quaternion structure in $(1,1)$ superspace.

Reducing to $(1,1)$ entails the following split of $(2,2)$ derivatives $\mathbb{D}$ into $(1,1)$ derivatives D and explicit susy transformation generators Q

$$\mathbb{D}_\alpha=\tfrac{1}{2}\left(D_\alpha-iQ_\alpha\right)\,,\quad \bar{\mathbb{D}}_\alpha=\tfrac{1}{2}\left(D_\alpha+iQ_\alpha\right)\,, \tag{17}$$

so that the (twisted) chirality constraints become

$$Q_\pm\phi^A=J^A_{\ B}D_\pm\phi^B\,,\quad Q_\pm\chi^{A'}=\pm J^{A'}_{\ B'}D_\pm\chi^{B'} \tag{18}$$

where J has the canonical form $diag(i\mathbb{1},-i\mathbb{1})$. When acting on φ we then have

$$\mathcal{J}^{(3)}_{(\pm)}=\begin{pmatrix} i&0&0&0\\ 0&-i&0&0\\ 0&0&\pm i&0\\ 0&0&0&\mp i\end{pmatrix}, \tag{19}$$

where we have labelled the complex structure 3 to indicate that it is one of the $\mathcal{J}^{\mathfrak{A}}_{(\pm)}$s. Using (19) we conclude that

$$\mathbb{D}_\pm\varphi=\tfrac{1}{2}\left(D_\pm-iQ_\pm\right)\varphi=\pi_{(\pm)}D_\pm\varphi \tag{20}$$
$$\bar{\mathbb{D}}_\pm\varphi=\tfrac{1}{2}\left(D_\pm+iQ_\pm\right)\varphi=\bar\pi_{(\pm)}D_\pm\varphi\,, \tag{21}$$

[3] Certainly true in the present $(4,4)$ twisted chiral case.

where

$$\pi_{(\pm)} := \tfrac{1}{2}\left(\mathbb{1} - i\mathcal{J}^{(3)}_{(\pm)}\right) . \tag{22}$$

To compare the transformations (10) to the transformations of the $(1,1)$ components, we need to calculate

$$\begin{aligned} U^{(+)}\bar{\epsilon}^{+}\bar{\mathbb{D}}_{+}\varphi &= \bar{\epsilon}^{+}U^{(+)}\bar{\pi}_{(+)}D_{+}\varphi \\ &= \bar{\epsilon}^{+}D_{+}\begin{pmatrix} \bar{\chi} \\ 0 \\ -\bar{\phi} \\ 0 \end{pmatrix} , \end{aligned} \tag{23}$$

all evaluated at the $(1,1)$ level.

Additional supersymmetries in the $(1,1)$ formulation lead to an additional two complex structures $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$ that can be read off from the transformations (8) (which look the same in $(1,1)$). On general grounds we expect the transformations to be a linear combinations of $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$ as described in [1]. Their form is[4]

$$\delta^{\pm}\varphi + \bar{\delta}^{\pm}\varphi = \tfrac{1}{2}\left(\left(\mathcal{J}^{(1)}_{(\pm)} + i\mathcal{J}^{(2)}_{(\pm)}\right)\epsilon^{\pm}D_{\pm}\varphi + \left(\mathcal{J}^{(1)}_{(\pm)} - i\mathcal{J}^{(2)}_{(\pm)}\right)\bar{\epsilon}^{\pm}D_{\pm}\varphi\right) . \tag{24}$$

The plus complex structures we get from (8) are

$$\mathcal{J}^{(1)}_{(+)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} , \quad \mathcal{J}^{(2)}_{(+)} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} , \tag{25}$$

so that

$$\tfrac{1}{2}\left(\mathcal{J}^{(1)}_{(\pm)} - i\mathcal{J}^{2}_{(\pm)}\right) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{26}$$

Using these we read off the $\bar{\epsilon}^{+}$ transformation from the last term in (24):

$$\bar{\delta}_{(+)}\begin{pmatrix} \phi \\ \bar{\phi} \\ \chi \\ \bar{\chi} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\bar{\epsilon}^{+}D_{+}\begin{pmatrix} 0 \\ \bar{\phi} \\ 0 \\ \bar{\chi} \end{pmatrix} = \bar{\epsilon}^{+}D_{+}\begin{pmatrix} \bar{\chi} \\ 0 \\ -\bar{\phi} \\ 0 \end{pmatrix} \tag{27}$$

Which indeed reproduces the $\bar{\epsilon}^{+}$ transformations in (8). The $\bar{\epsilon}^{-}$ and $\epsilon^{\pm}$ transformations are similarly reproduced.

[4]It again checks with the transformations given in [3].

4. Conclusions

We have illuminated a method for finding extra non manifest supersymmetries directly in $(2,2)$ superspace for $2d$ sigma models describing sectors of generalised Kähler geometry. We have done that using the twisted chiral multiplet of [3] where all steps can be explicitly verified. This corroborates the new results for semi chiral multiplets in [1] as well as results on complex linear fields presently being investigated.

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