PROCEEDINGS OF SCIENCE

Superconformal Symmetry and Index Theory

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Formulation and supersymmetry localization of superconformal indices for $\mathcal{N} = 2B$ superconformal quantum mechanics are reviewed by providing a generalization to fixed point submanifolds of resolved target space geometries, and future applications to gauged scaling quivers are discussed.

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1. Introduction

Strominger and Vafa [1] initiated a black hole microstate accounting programme calculating black hole entropy

$$S_{BH} = \frac{Ac^3}{4G\hbar}, \tag{1}$$

by a count of D-brane microstates in a regime where string coupling constant is very small. However, an explicit understanding of AdS_2/CFT_1 duality via [2]

$$e^{S_{BH}(\vec{q})} = \Omega(\vec{q}), \tag{2}$$

where $\Omega(\vec{q})$ denotes the degeneracy of ground states carrying charge $\vec{q}$ in the dual CFT_1, continues to remain as the least understood (and possibly enigmatic) corner of AdS/CFT duality.

To have a better control in this accounting programme, the microstates are typically constrained to be BPS, which are more robust under the variations of the string coupling constant and also of other parameters such as the asymptotic moduli. However, even this counting is not free from complications. In particular, one needs to make sure to exclude certain BPS states which can combine into long representations or decay into BPS constituents. For example, for the case of $su(1,1|1)$ superconformal algebra

$$[L_m, L_n] = (m-n)L_{m+n} \tag{3}$$

$$[L_0, \mathcal{G}_{\pm\frac{1}{2}}] = \mp\frac{1}{2}\mathcal{G}_{\pm\frac{1}{2}}, \qquad [L_{\mp 1}, \mathcal{G}_{\pm\frac{1}{2}}] = \pm\mathcal{G}_{\mp\frac{1}{2}}, \qquad [R, \mathcal{G}_{\pm\frac{1}{2}}] = \mathcal{G}_{\pm\frac{1}{2}} \tag{4}$$

$$\{\mathcal{G}_{\pm\frac{1}{2}}, \mathcal{G}^\dagger_{\pm\frac{1}{2}}\} = 2L_0 \pm R, \qquad \{\mathcal{G}_{\pm\frac{1}{2}}, \mathcal{G}^\dagger_{\mp\frac{1}{2}}\} = 2L_{\pm 1}, \qquad \{\mathcal{G}_\alpha, \mathcal{G}_\beta\} = 0, \tag{5}$$

the short spectrum is split into chiral and anti-chiral sectors :

$$[h]^{\text{chiral}}_{su(1,1|1)} = [(h,2h)]_{sl(2)\oplus u(1)} \oplus [(h+1/2, 2h-1)]_{sl(2)\oplus u(1)} \tag{6}$$

$$[h]^{\text{anti-chiral}}_{su(1,1|1)} = [(h,-2h)]_{sl(2)\oplus u(1)} \oplus [(h+1/2, -2h+1)]_{sl(2)\oplus u(1)}, \tag{7}$$

where we used the notation $[(sl(2), u(1))]$ for the respective quantum numbers of the bosonic subalgebra $sl(2)\oplus u(1)$, and (6,7) can be combined to obtain special long multiplets $[h]^{L_1}, [h]^{L_2}$:

$$[h]^{L_1} := [h]^{\text{chiral}}_{su(1,1|1)} \oplus [h+1/2]^{\text{chiral}}_{su(1,1|1)} \qquad [h]^{L_2} := [h]^{\text{anti-chiral}}_{su(1,1|1)} \oplus [h+1/2]^{\text{anti-chiral}}_{su(1,1|1)}, \tag{8}$$

whereas a generic long multiplet with $|r| < 2h$, is given by (where we drop the $sl(2)\oplus u(1)$ subscript for brevity)

$$[(h,r)]^{\text{chiral}}_{su(1,1|1)} = [(h,r)] \oplus [(h+1/2, r-1)] \oplus [(h, r+1)] \oplus [(h+1, r)]. \tag{9}$$

This algebra is the unique superalgebra of $\mathcal{N}=2$ superconformal quantum mechanics, which is the main subject of this note. In particular, the main interest is to be able to do the BPS counting in terms of the $su(1,1|1)$ unitary lowest weight irreducible representations (6,7) for a given such model, and even more optimistically to read the full BPS spectrum[1]. $su(1,1|1)$ superconformal indices [5]

$$\mathcal{I}_\pm(\zeta) = \mathrm{tr}\left((-1)^F e^{-\beta\mathcal{H}_\pm}\zeta^{\pm J}\right), \tag{10}$$

[1] A partial progress in this direction is achieved for $\mathcal{N}=(4,4)$ models through a detailed study of $osp(4^*|4)$ representation theory [3],[4].

serve precisely for this purpose, which we compute in Section 2 in a slightly more general setting than [5], namely by allowing for the presence of non-isolated fixed points, i.e for fixed point submanifolds.

Conformal invariance for one-dimensional sigma models requires that the target space has a conical geometry [6],[7], and hence in superconformal quantum mechanics[2] we have two complications being non-compactness and singularity of our target space cones, which are generally not considered in the applications of equivariant localization theorems. The first issue is resolved by considering the spectrum of L_0 instead of the original Hamiltonian H, which has an effect of introducing a harmonic potential given by the special conformal charge $K \sim r^2$, and hence acting as an IR-regulator. At the level of the superconformal algebra, this is realized through a similarity transformation such that the discrete spectrum of the dilatation operator maps to that of L_0. The second issue however is more serious and *always*[3] exists, since any cone is singular at least at a point $r = 0$, the tip of the cone, where the curvature does not vanish, but rather blows up. Hence, even the wave-functions corresponding to eigenstates of L_0 will be ill defined at this point, and so is a counting done by an index localizing to this singular point. We overcome this problem in two steps. The first step is to further modify our spectral problem by instead counting the wave-functions corresponding to the eigenstates of $L_0 \pm R$, which introduces an auxiliary magnetic background $\mathcal{A}^{\pm}$ to the original problem, and breaks the conformal invariance. Since both L_0 and R have a discrete spectrum (in terms of $su(1,1|1)$ irreps), this maps the counting problem to the *unitary* ground states of $L_0 \pm R$, as the BPS bound is a unitarity bound given by $2h \geq |r|$. The second and key point is that (10) is now computed as an equivariant Witten index [12] for a model which is *not* conformally invariant, and hence the fixed point locus is determined by an 'arbitrary' Reeb-like vector ρ^A (up to (21)), rather than the conformal one (23), and accordingly the refinement in (10) tracks through arbitrary smooth supersymmetry and global isometry preserving deformations of the metric. Concretely, given any SCQM (H) defined on a KT (Kähler with torsion) space X (with a fixed complex structure), with a singular metric G and the conformal Reeb vector fixed by the homothety, (10) computes the BPS spectrum of SQM ($\mathcal{H}_{\pm}$) defined on the same space X with a smooth metric[4] $\tilde{G}$ and a generically different Reeb vector determined by the holomorphic global isometry J, assuming that (X, G) possesses such smooth resolution. This interpretation is similar to the approach applied successfully for arbitrary quasi-regular Kähler cones by Martelli, Sparks, and Yau [15], whereas the relevance for superconformal quantum mechanics was observed in [16].

One physical motivation originates from a subset of solutions to $\mathcal{N} = 2$, $d = 4$ supergravity obtained from CY compactification of Type-II string theory, where one obtains two types of BPS solutions: single-centered, or multi-centered. The moduli space of the multi-centered ones is parameterized by Denef equation [17] :

$$|\mathcal{Z}_Q| \sin\left(\alpha_Q - \alpha\right)|_{r=\infty} = \sum_{p=1}^{n} \frac{\langle \Gamma_p, \Gamma_Q \rangle}{2|\vec{x}_p - \vec{x}_Q|}, \tag{11}$$

[2]See [8], [9] for reviews on superconformal mechanics, and [10],[11] for the original works.

[3]Except from the 'flat' $\mathbb{C}^k$ models, which are examined in full detail via various approaches in [5].

[4]The resolved metric $\tilde{G}$ must asymptotically agree with the singular metric G as the resolution parameter goes to zero. In the case of ADHM quiver mechanics, where the target space can be realized as a Hyper-Kähler quotient, this resolution parameter coincides with the FI parameter, and has the effect of lifting the singular isolated fixed point locus from the tip of the cone to regular points on the resolved space [4],[13],[14].

which at the same time gives a simple physical halo-like configuration for a classical BPS bound state. For such bound states with a non-vanishing intersection product one derives from (11) that the bound state radius becomes infinite at the wall of marginal stability, that is to say this class of BPS bound states typically exists at only one side of the wall, unlike the single centered black holes which can exist for any given value of asymptotic moduli. In particular, a BPS index constructed as a 'second helicity supertrace' [18], to count D6+D2-D0 halo states was studied in [19] :

$$\Omega_Q^{J_L} = \sum_{J_R} (-1)^{2J_R} N_Q^{J_L,J_R}, \tag{12}$$

where $N_Q^{J_L,J_R}$ is the dimension of the moduli space of $D2$ brane of charge Q and (j_L, j_R)-charge under $SU(2)_L \times SU(2)_R$ R-symmetry.

For a comprehensive understanding of BPS bound states, it is useful to consider another low energy effective description, which allows for an explicit count of generic microstates. This is provided by N=4 supersymmetric quiver quantum mechanics [20] where the wrapped D-branes appear as particles moving in Minkowski space. The interesting point is that one has both Higgs and Coulomb branches in this quiver mechanics. Concretely, starting from the supergravity description and taking $g_S \to 0$ limit one first arrives the Coluomb branch quivers consisting of multi-centered particles with a bound state radius that can be mapped to the one in the supergravity description, and lowering it further one obtains a more stringy Higgs-branch picture. It has been shown before that *generic* multi-centered BPS molecules of the Coulomb branch make only a sub-leading contribution to the black hole entropy [21], whereas there exists a subset of Higgs-branch solutions which form the exponential majority [22]. It remains however unclear what the physical interpretation of these pure-Higgs states is when the gravitational coupling constant is not vanishingly small.

This Coulomb branch effective quiver mechanics is originally described in terms of (3,4,1)[5] component form by [23], [20]

$$\begin{aligned} L &= -f_a D^a - U_a D^a + A_{ia}\dot{x}^{ia} + \partial_{ib} U_a \bar{\lambda}^a \sigma_i \lambda^b + \frac{1}{2} G_{ab} \left(\dot{x}^{ia}\dot{x}^{ib} + D^a D^b + i \left(\bar{\lambda}^a \dot{\lambda}^b - \dot{\lambda}^b \lambda^a \right) \right) \\ &- \frac{1}{2} \partial_{ic} G_{ab} \left(\bar{\lambda}^a \sigma_i \lambda^b D^c + \epsilon_{ijk} \bar{\lambda}^a \sigma_j \lambda^b \dot{x}^{kc} \right) - \frac{1}{8} \partial_{jc}\partial_{jd} G_{ab} \lambda^a \lambda^b \bar{\lambda}^c \bar{\lambda}^d, \end{aligned} \tag{13}$$

where the background scalar and gauge potentials are given as

$$U_a = \sum_{b, b\neq a} = \frac{\kappa_{ab}}{2r_{ab}} \qquad\qquad A_{ia} = -\sum_{b,b\neq a} \kappa_{ab} \frac{\epsilon_{ijk} n^j x^k_{ab}}{2r\left(x^l_{ab} n^l - r\right)}, \tag{14}$$

where κ_{ab}, μ_{ab}, f_a are respectively DSZ product, mass, FI parameter, and the metric can be written as[6]

$$G_{ab} = \delta_{ab} \left(\sum_{c,c\neq a} \frac{|\kappa_{ac}|}{4r^3_{ac}} \right) - \frac{|\kappa_{ab}|}{4r^3_{ab}} + \mu_{ab} H(x). \tag{15}$$

Any $\mathcal{N} = 4B$ supersymmetric mechanics described in terms of $(3, 4, 1)$ multiplets can be rewritten through the gauging of the auxiliary field [25],[26] in terms of the so called root multiplet

[5]The notation denotes number of (bosonic,fermionic,auxiliary) degrees of freedom.

[6]Explicit form of the function $H(x)$, which is not important for our purpose (since $\mu_{ab} = 0$ for scaling solutions), can be obtained from [24].

(4, 4, 0) [27], which then takes a general form [28] :

$$L = \frac{1}{2}G_{AB}D_t x^A D_t x^B + A_A D_t x^A - \frac{i}{2}F_{AB}\chi^A\chi^B + \frac{i}{2}G_{AB}\chi^A\hat{D}_t\chi^B - \frac{1}{12}\partial_{[A}C_{BCD]}\chi^A\chi^B\chi^C\chi^D, \tag{16}$$

and hence in particular the effective Coulomb branch description (13) can be brought into this form [24]. $\mathcal{N} = 4$ supersymmetry with $SU(2)_L \times SU(2)_R$ R-symmetry restricts the target space geometry of (16) to be Hyper-Kähler with Torsion (HKT).

At a generic point of the effective Coulomb branch, the first equation in (14) is equivalent to (11), which shows the correspondence between BPS bound states in $d = 4$ supergravity and $d = 1$ quiver descriptions [20]. There is a subset of these bound state solutions known as scaling solutions [20],[19] captured in a certain scaling limit of (13) [29],[24]. This has the net effect of putting mass and FI couplings to zero, while the action remains finite and develops a $D(2, 1; 0)$ superconformal symmetry. In this limit, the root formulation (16) takes the form of a gauged superconformal mechanics [28] with the target geometry restricted by a set of conformal constraints which can be interpreted as a deformed version of the well-known standard constraints [6],[30] for conformal invariance. We revisit this gauged superconformal mechanics in Section 3 particularly specializing to scaling quivers.

The root form of scaling quiver mechanics is especially useful for quantization since it gives a geometric description for the supersymmetric ground states. Moreover, it brings a possibility of an explicit count of these states as $D(2, 1; 0)$ irreducible representations via corresponding superconformal indices [31] :

$$\mathrm{tr}\left((-1)^{2J_L^3} y^{L_0 \pm J_L} z^{L_0 \pm J_R}\right), \tag{17}$$

which we hope to be able to compute in a future work, in particular for scaling quivers. For a similar, yet simpler $\mathcal{N} = 2B$ superconformal quantum mechanics, this is recently achieved [5] for the indices (10) under the assumption of a resolved target, and with a fixed point locus consisting of a single isolated point. While we keep the first assumption, in this note we generalize that computation to the presence of a fixed point submanifold, determined by the holomorphic Killing vector field ρ^A of the resolved space[7].

2. $\mathcal{N} = 2B$ superconformal quantum mechanics

On a complex topological space $\mathcal{M}$, we consider $\mathcal{N} = (0, 2)$ supersymmetric non-linear sigma models with N-many chiral multiplets. Each such multiplet contains two bosonic, two fermionic real degrees of freedom and no auxiliary degree of freedom. These non-linear sigma models can generically be defined on a family of different Hermitian metrics $\{G_{AB}\}$; $A = 1, \cdots, 2N$. In this case, since we are considering (0, 2) models but not (1, 1) models, we also have a complex structure J, and $\mathcal{N} = 2$ supersymmetry restricts the geometry (G, J) with the constraints :

$$G_{AC}J^C{}_B + G_{BC}J^C{}_A = 0 \qquad J^A{}_C J^C{}_B = -\delta^A_B \qquad \mathcal{N}(J)^A{}_{BC} = 0 \qquad \hat{\nabla}_{(A}J_{B)C} = 0. \tag{18}$$

[7]Indeed the superconformal index of the analog type-A models for various examples (remarkably for the case of generic toric Calabi-Yau 3-folds [16]) can be computed as a regular Dolbeault cohomology on the resolved space, and the fixed point data of the resolved space is sufficient.

In addition to $\mathcal{N} = 2$ supersymmetry we also demand $u(1)_R$ R-symmetry transforming the coordinates as

$$\delta_r x^A = -2r\rho^A \qquad \delta_r \chi^A = -2r\left(\partial_B \rho^A + \frac{1}{2}J^A{}_B\right)\chi^B, \tag{19}$$

for some vector field ρ^A. This restricts the last constraint in (18) slightly further :

$$\hat{\nabla}_A J_{BC} = 0, \tag{20}$$

and requires the extra constraints :

$$L_\rho G_{AB} = 0 \qquad L_\rho J^A{}_B = 0. \tag{21}$$

These models become also conformally invariant if there exists a holomorphic closed homothety ξ such that the refined geometry (G, J, ξ) further satisfies the constraints

$$L_\xi G_{AB} = -G_{AB} \qquad L_\xi J^A{}_B = 0 \qquad \xi_A = -\frac{1}{2}\partial_A K, \tag{22}$$

which are restrictive enough to fix the vector field ρ as

$$\rho^A_{\text{conformal}} = -J^A{}_B \xi^B. \tag{23}$$

Hence, *only* for $\mathcal{N} = 2$ non-linear sigma models which are additionally conformally invariant, (21) becomes a consequence of the conformal symmetry conditions (22,23). Such metrics G satisfying (22) are always[8] singular at least at a single point $\{0\}$, since the condition

$$\nabla_A \xi^B = -\frac{1}{2}\delta^A_B, \tag{24}$$

implied by (22), itself implies [7] that the conformal metric always takes the form of a cone metric

$$G = dr^2 + r^2 g_{ij}(\{x\})dx^i dx^j, \tag{25}$$

where $\{0\}$ corresponds to the tip of the cone.

This non-linear sigma model for N chiral multiplets can be obtained easily from the $\mathcal{N} = 1$ superspace action [32] as

$$\mathcal{L} = \frac{1}{2}G_{AB}\dot{x}^A\dot{x}^B + A_A\dot{x}^A - \frac{i}{2}F_{AB}\chi^A\chi^B + \frac{i}{2}G_{AB}\chi^A\left(\dot{\chi}^B + \hat{\Gamma}^B{}_{CD}\dot{x}^C\chi^D\right) - \frac{1}{12}\partial_{[A}C_{BCD]}\chi^A\chi^B\chi^C\chi^D, \tag{26}$$

where in general one can pick $\hat{\Gamma}$ as to be slightly more general[9] than the Bismut one, but we work with this special choice, i.e.

$$\hat{\Gamma}^A{}_{BC} = \Gamma^A{}_{BC} + \frac{1}{2}C^A{}_{BC}, \tag{27}$$

where the torsion $C^A{}_{BC}$ is accordingly fixed due to (20) by the complex structure $J^A{}_B$ as

$$C_{ABC} = J_A{}^D J_B{}^E J_C{}^F \left(\nabla_D J_{EF} + \nabla_E J_{FD} + \nabla_F J_{DE}\right), \tag{28}$$

[8] Except for the flat geometries.

[9] In that case then the extra part $\mathcal{B}_{ABC}$ should necessarily contain a symmetric part because Bismut is the unique totally-antisymmetric one. In case one chooses to work with this slightly more general torsion tensor than the Bismut one, then there is an additional constraint $J^B{}_{[E}\partial_A C_{BCD]} = 0$, which becomes automatic in the Bismut case due to (28).

which should further satisfy

$$L_\rho C_{ABC} = 0, \tag{29}$$

for the invariance of (26) under R-symmetry. Moreover, the target space geometry becomes Kähler if

$$\rho^A C_{ABC} = 0. \tag{30}$$

The conformal invariance further requires

$$L_\xi C_{ABC} = -C_{ABC} \qquad\qquad \xi^A C_{ABC} = 0, \tag{31}$$

which together with (23) implies (29). Now via (20,24) and the second constraint in (31), we obtain a nice identity

$$\rho^C C_{CAB} = J_{AB} + 2\nabla_A \rho_B. \tag{32}$$

Finally, existence of the background gauge-field A_A, with the corresponding field strength $F = dA$, enhances the previous set of R-symmetry constraints to

$$F_{AC} J^C{}_B + F_{CB} J^C{}_A = 0 \qquad\qquad i_\rho F = 0, \tag{33}$$

whereas conformal symmetry requires

$$i_\xi F = 0, \tag{34}$$

which together with (23) implies the second condition in (33).

The corresponding superconformal symmetry algebra realized by (26) is $su(1,1|1)$ [6], [30], which we prefer to express in a basis such that the complex supercharges $\mathcal{G}_{\pm\frac{1}{2}}$ satisfy[10]

$$\{\mathcal{G}_{\pm\frac{1}{2}}, \mathcal{G}^\dagger_{\pm\frac{1}{2}}\} = 2L_0 \pm R \qquad\qquad [R, \mathcal{G}_{\pm\frac{1}{2}}] = \mathcal{G}_{\pm\frac{1}{2}}, \tag{35}$$

where $L_0 = \frac{1}{2}(H + K)$, and H, K, R are respectively the Hamiltonian, special conformal charge, and the R-charge. These fermionic charges $\mathcal{G}_{\pm\frac{1}{2}}$ are constructed by combining supercharges and conformal supercharges of the parent superconformal model (26), which we now review. From [28] we recall that the supercharges Q^α and conformal supercharges S^α are given by

$$Q^1 = -\chi^A J_A{}^B \Pi_B + \frac{i}{2} J_{[A}{}^D C_{BC]D} \chi^A \chi^B \chi^C, \qquad\qquad S^1 = 2\chi^A J_{AB} \xi^B \tag{36}$$

$$Q^2 = \chi^A \Pi_A - \frac{i}{6} C_{ABC} \chi^A \chi^B \chi^C, \qquad\qquad S^2 = -2\chi^A \xi_A, \tag{37}$$

with

$$\Pi_A = \tilde{p}_A - A_A - \frac{i}{2}\left(\omega_{ABC} - \frac{1}{2} C_{ABC}\right)\chi^B \chi^C. \tag{38}$$

The generators $\mathcal{G}_{\pm\frac{1}{2}}$ are then defined as

$$\mathcal{G}_{\pm\frac{1}{2}} = \mathcal{Q} \mp i\mathcal{S}, \qquad \mathcal{Q} = \frac{1}{2}(Q^1 + iQ^2) \qquad \mathcal{S} = \frac{1}{2}(S^1 + iS^2), \tag{39}$$

[10] More properly, we define a similarity transformation which maps the dilatation generator to $L_0 = \frac{1}{2\omega}\left(H + \omega^2 K\right)$, so that the supercharges $\mathcal{G}_{\pm\frac{1}{2}}$ satisfy $\{\mathcal{G}_{\pm\frac{1}{2}}, \mathcal{G}^\dagger_{\pm\frac{1}{2}}\} = \omega\,(2L_0 \pm R)$. We then choose to work in the units such that $\omega = 1$.

which explicitly give

$$\mathcal{G}_{\pm\frac{1}{2}} = \frac{1}{2}\left(\delta_A^B + iJ_A^B\right)\chi^A\left(\Pi_B - \mathcal{A}_B^{\pm}\right) + \frac{1}{12}\left(\delta_A^D + 3iJ_{[A}{}^D\right)C_{BC]D}\chi^A\chi^B\chi^C, \qquad \mathcal{A}_A^{\pm} = \mp 2\rho_A. \tag{40}$$

Remaining superconformal charges of (26) are given as [6] (in the conventions of [28][11])

$$H = \frac{1}{2}G_{AB}\Pi^A\Pi^B + \frac{i}{2}F_{AB}\chi^A\chi^B + \frac{1}{12}\partial_{[A}C_{BCD]}\chi^A\chi^B\chi^C\chi^D \tag{41}$$

$$D = \xi^A\Pi_A \tag{42}$$

$$K = 2\xi^A\xi_A \tag{43}$$

$$R = 2\rho^A\Pi_A - 2i\nabla_A\rho_B\chi^A\chi^B. \tag{44}$$

On the spectrum of the models (26), there exists a well defined fermion number F which satisfies [5]

$$[F, \mathcal{G}_{\pm\frac{1}{2}}] = F, \qquad F = \frac{i}{2}J_{AB}\chi^A\chi^B + \frac{N}{2}, \tag{45}$$

and hence

$$[J, \mathcal{G}_{\pm\frac{1}{2}}] = 0 \qquad J := R - F + c, \tag{46}$$

extending the superconformal algebra to $su(1,1|1) \oplus u(1)_J$. This also provides the classical expression for the J-charge

$$\begin{aligned} J &= -2\rho^A\Pi_A + 2i\nabla_A\rho_B\chi^A\chi^B \\ &= -2\rho^A\left(\tilde{p}_A - A_A - \frac{i}{2}\omega_{ABC}\chi^B\chi^C\right) + i\nabla_A\rho_B\chi^A\chi^B - \frac{i}{2}\rho^C C_{CAB}\chi^A\chi^B, \end{aligned} \tag{47}$$

where we used the identity (32). The action of this isometry $u(1)_J$ on the coordinates is given by (cfr. (19))

$$\delta_\epsilon x^A = \epsilon\rho^A \qquad \delta_\epsilon\chi^A = \epsilon\partial_B\rho^A\chi^B. \tag{48}$$

So, a character-valued index [33] with respect to this global isometry is introduced [5] as

$$\mathcal{I}_\lambda^{\pm} = Tr\left[(-1)^F e^{-\beta(H+K\pm R)}\zeta^{\pm J}\right] = Tr\left[(-1)^F e^{-\beta\mathcal{H}_\lambda^{\pm}}\right], \tag{49}$$

where we defined

$$\mathcal{H}_\lambda^{\pm} = H + K \pm R \mp i\lambda J \qquad \zeta := e^{\beta(i\lambda)}, \; i\lambda \in \mathbb{R}. \tag{50}$$

We compute this index in the standard manner via supersymmetry localization à la Álvarez-Gaumé [34] for the corresponding real supercharge (which corresponds to torsionful Dirac operator [5])

$$D := \mathcal{G}_{\pm\frac{1}{2}} - \mathcal{G}^{\dagger}_{\pm\frac{1}{2}}, \tag{51}$$

and obtain that it is given by the formula

$$\begin{aligned} \mathcal{I}_\pm^\lambda &= i^N \int_{\mathcal{M}_0} \prod_{m=1}^{\dim\mathcal{M}_0} dx_0^m d\eta_0^m \frac{\exp\left(\frac{i}{2}\omega_{AB}^\lambda\eta_0^A\eta_0^B\right)}{\det{}'_{\mathrm{PBC}}\left(-\delta_{AB}\partial_\tau - i\hat{R}_{AB}^{\pm,\lambda}\right)^{1/2}} = \int_{\mathcal{M}_0} ch(\omega^\lambda)\wedge\hat{A}(\hat{R}^{\pm,\lambda}); \\ \hat{R}_{AB}^{\pm,\lambda} &= \hat{R}_{ABCD}\eta_0^C\eta_0^D \mp 2i\lambda\partial_A\rho_B \qquad \omega_{AB}^\lambda = \tilde{F}_{AB}^{\pm} + i\frac{\lambda}{2}\mathcal{F}_{AB}^{\pm}, \end{aligned} \tag{52}$$

[11] with $R = -2R_{\text{there}}$

where

$$\mathcal{F}^{\pm} = d\mathcal{A}^{\pm} = \mp 4\nabla_{[A}\rho_{B]} \tag{53}$$

denotes the auxiliary potential. This result for the index is equivalent to Niemi-Tirkkonen equivariant localization theorem [35],[36] as expected (since D corresponds to Dirac operator). To evaluate this formula for singular conical geometries of various $\mathcal{N} = 2B$ superconformal quantum mechanical non-linear sigma models, one has to find the fixed point locus $\mathcal{M}_0$ on the corresponding resolved space, which is given by[12]

$$D\chi = 0 \quad \Rightarrow \quad \rho^A = 0, \tag{54}$$

where the vector field ρ^A is determined by the conditions (21). Moreover, since the inverse Legendre transform $\mathcal{L}_\lambda$ of the 'Hamiltonian' $\mathcal{H}_\lambda$ is not conformally invariant, ρ^A is in general different than the conformal one (23), since on the resolved space there exists no homothety.

In the remaining parts of this section, we give more details for the supersymmetry localization computation of (49), starting with the unrefined case $\zeta = 1$, for general $\mathcal{N} = 2B$ superconformal mechanics on resolved targets.

2.1 Superconformal index as an index for auxiliary quantum mechanics

The path integral for the index

$$\mathcal{I}_{\pm} = Tr\left[(-1)^F e^{-\beta(H+K\pm R)}\right] = \int [D\chi][Dx]e^{-\beta\int_0^\beta d\tau \mathcal{L}^E_{\pm}} \tag{55}$$

involves the Lagrangian $\mathcal{L}^E_{\pm}$ which corresponds (after Wick rotation) to the inverse Legendre transform of $\{\mathcal{G}_{\pm}, \mathcal{G}^{\dagger}_{\pm}\}$, and is related to the vanilla model (26) by a simple shift of the background gauge potential, i.e.[13]

$$\mathcal{L}_{\pm} = \mathcal{L}[A \to \tilde{A}_{\pm} = A + \mathcal{A}_{\pm}], \tag{56}$$

which is invariant under $\mathcal{N} = 2$ supersymmetries :

$$\delta_{\mathcal{G}^{\dagger}_{\pm 1/2}} x^A = -\frac{i\epsilon}{2}(J^A{}_B - i\delta^A_B)\chi^B \tag{57}$$

$$\delta_{\mathcal{G}^{\dagger}_{\pm 1/2}} \chi^A = -\frac{\epsilon}{2}\left(J^A{}_B + i\delta^A_B\right)\dot{x}^B + \frac{i\epsilon}{2}\partial_C J^A{}_B \chi^C \chi^B \tag{58}$$

$$\delta_{(\mathcal{G}_{\pm 1/2} - \mathcal{G}^{\dagger}_{\pm 1/2})} x^A = \epsilon\chi^A \qquad \delta_{(\mathcal{G}_{\pm 1/2} - \mathcal{G}^{\dagger}_{\pm 1/2})} \chi^A = i\epsilon\dot{x}^A, \tag{59}$$

as well as under the $u(1)_J$ transformation (48).

We note that when $\mathcal{L}_{\pm} \to \mathcal{L}$, and the torsion is put to zero, the index is manifestly equivalent to the index of Dirac-operator [37],[38], first computed by Atiyah and Singer [39],[40], and then by Álvarez-Gaumé [34],[41] and Friedan and Windey [42] via supersymmetry path integral, except that in our case we have a noncompact target. For simplicity we will assume that the Kähler form Ω satisfies the condition

$$\partial\bar{\partial}\Omega = 0, \tag{60}$$

[12] In general we also have time-dependent classical vacua (instanton) configurations given by $\dot{x}^A = \mp 2i\lambda\rho^A$, which we do not consider.

[13] Note that ξ^2 term cancels with K in the derivation of (56).

so that the four-fermion term in (26) drops out and thus we are able to apply standard supersymmetry localization [38]. Such target geometries are called strong Kähler with torsion (SKT), but our results straightforwardly generalize to weak Kähler with torsion (wKT) geometries where the derivative of the torsion tensor is non-vanishing. However, this brings a simplification rather than a complication because in the wKT cases, the whole contribution from the torsion can be made vanishing through a continuous deformation while preserving the $\mathcal{N} = 2$ supersymmetry of (26) in computing the index [38].

Let us now summarize the computation of (55) through explicit supersymmetry localization for SKT geometries. First, we note that the supersymmetry generator (51) acts on $\mathcal{L}_\pm$ as given in (59), and hence we observe that a supersymmetry-exact generalization of (26) is given by

$$\mathcal{L}_\pm(\kappa) = \kappa\delta_{(\mathcal{G}_{\pm 1/2}-\mathcal{G}^\dagger_{\pm 1/2})}\left(-\frac{i}{2}G_{AB}\dot{x}^A\chi^B - \frac{1}{12}C_{ABC}\chi^A\chi^B\chi^C\right) + \tilde{A}^\pm_A\dot{x}^A - \frac{i}{2}\tilde{F}^\pm_{AB}\chi^A\chi^B, \tag{61}$$

with an arbitrary parameter κ. Note that (61) reduces to (56) for $\kappa = 1$. Now, expanding (61) over the fluctuations

$$x = x_0 + \frac{\xi}{\sqrt{\kappa}} \qquad\qquad \chi = \eta_0 + \frac{\eta}{\sqrt{\kappa}}, \tag{62}$$

and considering the limit $\kappa \longrightarrow \infty$, the Euclidean path integral (55) evaluates to

$$\mathcal{I} = i^N \int_{\mathcal{M}} \prod_{K=1}^{\dim\mathcal{M}} dx_0^K\, d\eta_0^K \frac{e^{\frac{i}{2}\tilde{F}^\pm_{AB}\eta_0^A\eta_0^B}}{\det'_{\mathrm{PBC}}\left(-\delta_{AB}\partial_\tau - i\hat{R}_{ABCD}\eta_0^C\eta_0^D\right)^{1/2}} = \int_{\mathcal{M}} ch(\tilde{F}^\pm)\wedge\hat{A}(\hat{R}), \tag{63}$$

where $\hat{R}$ is the torsionful Riemann tensor. We obtain the corresponding index for the wKT geometry simply by replacing $\hat{R}$ with the standard torsionless Riemann tensor [38]. We also note that it is the 'net' field $F + \mathcal{F}^\pm$ rather than only the background field F that appears inside the Chern character. Up to these differences, (63) coincides with the standard result of Atiyah-Singer index. However, in this case, since $\mathcal{M}$ is noncompact the result (63) is divergent.

Finally let us also note that to compute the index (55), we merely used the $\mathcal{N} = 2$ supersymmetry invariance of the auxiliary Lagrangian $\mathcal{L}^E_\pm$ appearing in the index path integral. In fact, unlike the original superconformal model (26), $\mathcal{L}_\pm$ is not invariant under $u(1)_R$ and conformal transformations due to the appearance of the extra potential $\mathcal{A}_\pm$. In other words, Witten indices constructed from $\mathcal{H}_\pm$ do not make a good use of this freedom.

2.2 Supersymmetric Localization of the Refined Index

We now return to the refined superconformal index (49), which can be interpreted as the equivariant Witten index for the refined Hamiltonian (50), with the $u(1)_J$ isometry generated by a holomorphic Killing vector field ρ^A. Crucially, the refined Lagrangian $L^\pm_\lambda$ appearing in the index path integral is not conformally invariant. Hence, only $\mathcal{N} = 2$ supersymmetry and $u(1)_J$ symmetry are preserved in the index path integral, whereas the stringent condition (23) on ρ^A does not hold.

The Lagrangian corresponding to the refined Hamiltonian

$$H^\pm_\lambda = H + K \pm R \mp i\lambda J, \;\; i\lambda \in \mathbb{R}, \tag{64}$$

can be easily obtained via inverse Legendre transform as

$$L^{\pm}_{\lambda} = \frac{1}{2}G_{AB}\dot{x}^A\dot{x}^B + \tilde{A}^{\pm}_A\dot{x}^A + \frac{i}{2}G_{AB}\chi^A\hat{\nabla}_t\chi^B - \frac{i}{2}\tilde{F}^{\pm}_{AB}\chi^A\chi^B \mp \lambda\left(\nabla_A\rho_B + \frac{1}{2}\rho^C C_{CAB}\right)\chi^A\chi^B$$
$$\mp 2i\lambda\rho_A\dot{x}^A - 2\lambda^2\rho_A\rho^A - \frac{1}{12}\partial_{[A}C_{BCD]}\chi^A\chi^B\chi^C\chi^D, \tag{65}$$

and is invariant under *refined* $\mathcal{N} = 2$ supersymmetries :

$$\delta_{\mathcal{G}^{\lambda}_{\pm\frac{1}{2}}} x^A = -\frac{i\epsilon}{2}\left(J^A{}_B + i\delta^A_B\right)\chi^B \tag{66}$$

$$\delta_{\mathcal{G}^{\lambda}_{\pm\frac{1}{2}}} \chi^A = -\frac{\epsilon}{2}\left(J^A{}_B - i\delta^A_B\right)\left(\dot{x}^B \mp 2i\lambda\rho^B\right) + \frac{i\epsilon}{2}\partial_C J^A{}_B\chi^C\chi^B, \tag{67}$$

$$\delta_{\mathcal{G}^{\lambda\dagger}_{\pm 1/2}} x^A = -\frac{i\epsilon}{2}(J^A{}_B - i\delta^A_B)\chi^B \tag{68}$$

$$\delta_{\mathcal{G}^{\lambda\dagger}_{\pm 1/2}} \chi^A = -\frac{\epsilon}{2}\left(J^A{}_B + i\delta^A_B\right)\left(\dot{x}^B \mp 2i\lambda\rho^B\right) + \frac{i\epsilon}{2}\partial_C J^A{}_B\chi^C\chi^B. \tag{69}$$

$$\delta_{(\mathcal{G}^{\lambda}_{\pm 1/2} - \mathcal{G}^{\lambda\dagger}_{\pm 1/2})} x^A = \epsilon\chi^A, \qquad \delta_{(\mathcal{G}^{\lambda}_{\pm 1/2} - \mathcal{G}^{\lambda\dagger}_{\pm 1/2})} \chi^A = i\epsilon\left(\dot{x}^A \mp 2i\lambda\rho^A\right). \tag{70}$$

Comparing these with the unrefined analogs (57-59), we see that the supersymmetry transformations of fermions receive a fugacity-dependent extra contribution determined by the vector ρ^A. This leads to localization of the index path integral

$$\mathcal{I}^{\pm}_{\lambda} = \int [Dx][D\chi]e^{-\int_0^\beta d\tau L^{\pm,E}_{\lambda}}, \tag{71}$$

to a subspace determined by the fixed point locus $\mathcal{M}_0$ of ρ^A :

$$\mathcal{M}_0 = \{x_0|\rho(x_0) = 0\}, \tag{72}$$

instead of an integration over the full target space $\mathcal{M}$, and hence eventually brings a finite result unlike the unrefined index.

As similar to before, we find a useful supersymmetry-exact generalization of (65) :

$$L^{\pm}_{\lambda,\kappa} = \kappa\delta_\lambda\left(-\frac{i}{2}G_{AB}\dot{x}^A\chi^B - \frac{1}{12}C_{ABC}\chi^A\chi^B\chi^C\right) - \frac{i}{2}\tilde{F}^{\pm}_{AB}\chi^A\chi^B \mp \lambda\nabla_A\rho_B\chi^A\chi^B - 2\lambda^2\rho_A\rho^A, \tag{73}$$

where δ_λ is used as a shorthand for (70), and we did not write $O(\dot{x}^A)$ terms which do not contribute to path integral index as $\kappa \to \infty$.

We are now ready to use the localization principle for the index path integral (71) computed with the Euclidean continuation of (73), which after expanding over (62) up to quadratic order in fluctuations, and considering the limit $\kappa \to \infty$, gives (52)[14]. For the cases where $\mathcal{M}_0$ consists of only isolated fixed points $\{x_0\}$, it reduces to a simple form

$$\mathcal{I}^{\pm}_{\lambda} = i^N \sum_{\{x_0\}} \underset{\text{PBC}}{\det}{}'\left(-\delta_{AB}\partial_\tau \mp 2\lambda\partial_A\rho_B(x_0)\right)^{-1/2}, \tag{74}$$

[14]which is valid for a SKT geometry. Similarly as before, one obtains the corresponding result for the wKT geometry simply by replacing the Riemann tensor with the torsionless one.

as obtained in [5]. Hence, unlike the unrefined index which generically gives an infinite result for noncompact spaces, the refined index gives a finite result (at a fixed charge) since the domain of integration is now restricted to be $\mathcal{M}_0$, but not over the full target space $\mathcal{M}$.

Finally, we note that

$$\mp \lambda\delta_\lambda \left(\rho_A\chi^A\right) = \mp i\lambda\rho_A\dot{x}^A \mp \lambda\nabla_A\rho_B\chi^A\chi^B - 2\lambda^2\rho_A\rho^A, \tag{75}$$

and thus we can deform $\mathcal{L}^\lambda_\pm$ by a δ_λ-exact term as

$$\mathcal{L}^\lambda_\pm \to \mathcal{L}^\lambda_\pm \mp \lambda\tau\delta_\lambda(\rho_A\chi^A), \tag{76}$$

so that (73) generalizes

$$\mathcal{L}^{\lambda_\tau,\kappa}_\pm = \kappa\delta_{\lambda_\tau}\left(G_{AB}\dot{x}^A\chi^B - \frac{i}{12}C_{ABC}\chi^A\chi^B\chi^C\right) + \tilde{A}^\pm_A\dot{x}^A - \frac{i}{2}\tilde{F}^\pm_{AB}\chi^A\chi^B \mp \lambda_\tau\nabla_A\rho_B\chi^A\chi^B - 2\lambda_\tau^2\rho_A\rho^A, \tag{77}$$

where we introduced the shorthand

$$\lambda_\tau := \tau\lambda, \qquad \tau \in \mathbb{R}. \tag{78}$$

So, we see that the deformation (76) has the net effect of introducing an arbitrary relabeling factor τ in front of the fugacity λ in the main result (52). Similarly to the unrefined case, (52) is valid for SKT targets, and the generalization to the wKT cases is simply obtained by replacing the torsionful equivariant Riemann curvature with the torsionless one.

To conclude, the superconformal index defined as an equivariant Witten index on the resolved target space can be obtained from the general localization formula (52), once the fixed point locus of the holomorphic Killing vector ρ on this resolved space is determined.

3. Gauged Quivers

We now move on to the $\mathcal{N} = 4$ case in order to make a connection with the quiver model of our interest mentioned in the Introduction. Due to $\mathcal{N} = 4B$ supersymmetry we now have two additional complex structures, which we express in the covariant form

$$J^\rho = (J^i, \mathbb{1}) \qquad \bar{J}^\rho = (-J^i, \mathbb{1}), \qquad i = 1, 2, 3. \tag{79}$$

This leads to HyperKähler with torsion target spaces for the corresponding sigma models, that can be parametrized by the coordinates

$$x^A := x^{\mu a} \qquad A \equiv \mu a = 1, \cdots, 4n, \qquad \mu = (i, 4),\ a = 1, \cdots, n. \tag{80}$$

We now assume that there exists a set of global isometries described by n-many commuting[15] Killing vectors $\{k_a\}$ acting on the bosonic and fermionic coordinates as[16]

$$\delta_\lambda x^A = \lambda^b k^A_b \qquad \delta_\lambda\chi^A = \lambda^b\partial_C k^A_b\chi^C. \tag{81}$$

[15]Generalization to non-abelian case is given in [28].

[16]Comparing (81) with (48) reveals that one can consider $\mathcal{N} = 2$ gauged models where the refinement generator J is gauged [43], and thereby building an explicit connection with [15] in superconformal mechanics.

Gauging this global isometry by promoting $\lambda^a \longrightarrow \lambda^a(t)$, and correspondingly introducing a world-line valued gauge fields $a^a(t)$, (26) becomes [28]

$$L = \frac{1}{2}G_{AB}D_t x^A D_t x^B + A_A \dot{x}^A + a^a v_a - \frac{i}{2}F_{AB}\chi^A\chi^B + \frac{i}{2}G_{AB}\chi^A \hat{D}_t\chi^B - \frac{1}{12}\partial_{[A}C_{BCD]}\chi^A\chi^B\chi^C\chi^D, \tag{82}$$

where the gauge covariant derivatives are

$$D_t x^A = \dot{x}^A - a^b k_b^A \qquad \hat{D}_t\chi^A = \dot{\chi}^A + \hat{\Gamma}^A{}_{BC}\dot{x}^B\chi^C + a^b\left(\nabla^A k_{bC} + \frac{1}{2}C^A{}_{CD}k_b^D\right)\chi^C, \tag{83}$$

and $v_a(x)$ are arbitrary target space valued potentials satisfying the constraint

$$i_{k_a}F = dv_a. \tag{84}$$

We now specialize to a specific choice of a target space data :

$$G_{\mu a \nu b} = \delta_{\mu\nu}G_{ab}, \qquad G_{ab} = \delta_{ab}\left(\sum_{c,c\neq a}\frac{|\kappa_{ac}|}{4r_{ac}^3}\right) - \frac{|\kappa_{ab}|}{4r_{ab}^3} + \mu_{ab}H(x) \tag{85}$$

$$C_{\mu a\nu b\rho c} = \epsilon_{\lambda\mu\nu\rho}\partial_{\lambda a}G_{bc} \tag{86}$$

$$(J^i)^{\mu a}{}_{\nu b} = (j_+^i)_{\mu\nu}\delta_b^a, \qquad (j_\pm^i)_{\mu\nu} = \mp\left(\delta_{\mu i}\delta_{\nu 4} - \delta_{\mu 4}\delta_{\nu i}\right) - \epsilon_{i\mu\nu 4} \tag{87}$$

$$A_A = A_{\mu a} = (A_{ia}, -v_a) = (A_{ia}, -f_a - U_a), \tag{88}$$

which is evidently invariant under the isometry

$$k_a^A = \delta_4^A\partial_a. \tag{89}$$

Second, we note that by fixing the gauge such that x^{4a} is constant, (82) becomes the $(3,4,1)$ effective Coulomb quiver mechanics (13) :

$$L_{(4,4,0)}(D_t x^{4a} = -a^a := D^a) = L_{(3,4,1)}. \tag{90}$$

When $\mu_{ab} = 0, f_a = 0$, it was shown [29],[24],[28] that (82) is invariant under $D(2,1;0)$ action. A detailed analysis of the algebra closure and the derivation of geometric constraints for a general target space data, as well as the corresponding restrictions for the specific choice (85-88), can be found in [28]. In the $(4,4,0)$ language, this has the interpretation that the conformal symmetry is realized after a reduction from the HKT space to a subspace which is determined by the $U(1)^n$ gauge symmetry (81) :

$$M_a = -v_a \approx 0 \Leftrightarrow U_a \approx 0. \tag{91}$$

Gauged formalism allows us instead to work on the covering HKT space, and try to compute the corresponding $D(2,1;0)$ index (17) with respect to this gauged model. From the computation in the previous section, we can now see the simple reason why this would probably be a powerful tool for computing the index : because in the gauged description of scaling quiver mechanics (82) the analog ρ-vector field takes the form [28] :

$$\rho_i = -\epsilon^{ijk}x^{ja}\partial_{ka} - x^{ia}\partial_{4a}, \tag{92}$$

where the shift due to gauged isometry (81) given by the second term lifts the fixed point locus from the singular tip of the cone of the unresolved conformal target space as $x^{4a} \neq 0$. Indeed, in the gauged model ρ is given by

$$\rho = -J \cdot \xi_{\perp}, \tag{93}$$

where $\xi_{\perp}$ is not a homothety but rather related to that with a shift given by the Killing vector (89) [28]. This suggests [43] that the gauged superconformal mechanical sigma models might be useful to obtain a more precise description of the fixed point locus $\mathcal{M}_0$ on resolved targets, which is a key ingredient of the localization formula (52).

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