PROCEEDINGS OF SCIENCE

Renormalization group and quantum error correction

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We show that quantum error correction is realized by the renormalization group in scalar field theories. We construct a q-level system in the IR region by using coherent states. We encode it in the UV region by acting on the states in the q-level system the inverse of the unitary operator that gives the renormalization group flow of the ground state. We find that the condition for quantum error correction is satisfied for operators that create coherent states. We confirm this to the first order in the perturbation theory. This result suggests a general relationship between the renormalization group and quantum error correction and should give insights into their role in the AdS/CFT correspondence.

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1. Introduction

Emergence of space-time is seen in various contexts such as the large-N reduction, the matrix models for noncritical strings and superstrings and the AdS/CFT correspondence (the gauge/gravity correspondence)[1]. It seems natural in quantum gravity, since the space-time itself fluctuates there. Here we focus on the AdS/CFT correspondence.

In the AdS/CFT correspondence, the bulk geometry emerges from the degrees of freedom of field theories on the boundary. While the mechanism of emergence of space-time in the AdS/CFT correspondence has not been revealed completely, there are some insights into it. First, the structure of the renormalization group is seen in the AdS/CFT correspondence. The metric of $(d+2)$-dimensional AdS in the Poincare coordinates is given by

$$ds^2 = \frac{dz^2 + dx^\mu dx^\mu}{z^2} \ , \tag{1.1}$$

where μ run 0 to d. z corresponds to the coordinate in the bulk direction. The CFT lives on the (regularized) boundary specified by $z = \epsilon$, where ϵ is the UV cutoff. The metric (1.1) is invariant under $z \to \rho z$, $x^\mu \to \rho x^\mu$. This implies that z is interpreted as the scale of the renormalization group and that small and large z correspond to the UV and IR regions, respectively. Second, it has been recognized that quantum information plays crucial roles in the AdS/CFT correspondence. We note here that it has been argued that the structure of quantum error correction is needed for the bulk operators to be consistently described by the boundary operators[2]. As seen above, the bulk and the boundary correspond to the IR and UV regions in field theories on the boundary, respectively. This suggests that in field theories quantum error correction can be associated with the renormalization group.

It was shown in [3, 4] that quantum error correction is realized by the renormalization group in so-called magic cMERA[5], which is a free scalar field theory whose action has a particular scale dependence, by using coherent states. In this paper, motivated by this work, we realize quantum error correction by the renormalization group in scalar field theories including interactions[6][1]. For this, we need the scale dependence of wave functionals, since quantum error correction is examined in a Hilbert space. Thus, we begin with reviewing the exact renormalization group (ERG) equation for wave functions in scalar field theories derived in [7], which determines the scale dependence of wave functionals.

This paper is organized as follows. In section 2, we review the ERG equation for the wave functional. In section 3, we present a general procedure of encoding a q-level system by the renormalization group. In section 4, we examine the renormalization group flow in scalar field theories, in particular that of the creation and annihilation operators. In section 5, we construct a q-level system by using the coherent states and show that quantum error correction is realized by encoding the q-level system in the IR region into that in the UV region. Section 6 is devoted to conclusion and discussion. In appendices, the Knill-Laflamme condition and the Polchinski equation are briefly reviewed.

[1]The roles of IR and UV in our case seem to be exchanged compared to those in [4].

2. ERG equation for wave functionals

In this section, we review the ERG equation for the wave functionals[7].

2.1 ERG equation for wave functionals in the scalar field theory

In the AdS/CFT correspondence, the bulk direction corresponds to the scale of the renormalization group. Furthermore, since the classical bulk geometry corresponds to strongly-coupled gauge theory, non-perturbative treatment is essential.

The exact renormalization group(ERG) is a non-perturbative method in which scale dependence is described by a functional differential equation. We derived an ERG equation describing the scale dependence of the wave functionals in scalar field theories. This equation is based on the Polchinski equation[8], summarized in appendix B.

Throughout this paper, we consider scalar field theories in $d + 1$ dimensions with UV cutoff and use a shorthand notation:

$$\int_p \equiv \int \frac{d^d p}{(2\pi)^d} \,, \quad \tilde{\delta}(p) = (2\pi)^d \delta^d(p) \,, \tag{2.1}$$

where p stands for d-dimensional spatial momentum. We denote the effective momentum cutoff by Λ.

The path-integral representation of the ground-state wave functional $\Psi[\varphi]$ is given by

$$\Psi_\Lambda[\varphi] = \int_{\phi(0,p)=\varphi(p)} \mathcal{D}\phi e^{-\int_{-\infty}^{0} d\tau L_\Lambda[\phi]} \,, \tag{2.2}$$

where L_Λ is the effective Lagrangian. We impose the boundary condition for the field $\phi(\tau, p)$ as

$$\phi(0, p) = \varphi(p) \,. \tag{2.3}$$

L_Λ is assumed to be real so that $\Psi_\Lambda[\varphi]$ is also real.

The scale dependence of the wave functional $\Psi_\Lambda[\varphi]$ is described by the ERG equation[7]

$$\begin{aligned} -\Lambda\frac{\partial}{\partial\Lambda}\Psi_\Lambda = &-\frac{1}{2}\int_p \dot{C}_\Lambda(0,p)\left\{\frac{\delta^2\Psi_\Lambda}{\delta\varphi(p)\delta\varphi(-p)} + \frac{1}{\Psi_\Lambda}\frac{\delta\Psi_\Lambda}{\delta\varphi(p)}\frac{\delta\Psi_\Lambda}{\delta\varphi(-p)}\right\} \\ &- \int_p \frac{\dot{C}_\Lambda(0,p)}{C_\Lambda(0,p)}\varphi(p)\frac{\delta\Psi_\Lambda}{\delta\varphi(p)} - \frac{V}{2}\Psi_\Lambda\int_p \frac{\dot{C}_\Lambda(0,p)}{C_\Lambda(0,p)} \,, \end{aligned} \tag{2.4}$$

where $\dot{C}_\Lambda \equiv -\Lambda\partial_\Lambda C_\Lambda$ is an ERG integration kernel(see appendix B). The details of the derivation is in [7].

2.2 The free and perturbative solutions

Here we introduce the free and perturbative solutions of the ERG equation(2.4).

We assume that $C_\Lambda(\tau, p)$ is factorized as

$$C_\Lambda(\tau, p) = f(\tau, p) g_\Lambda(p) \,, \tag{2.5}$$

where f is independent of Λ. We use the following C_Λ:

$$C_\Lambda(p) = \frac{K(p^2/\Lambda^2)}{p_0^2 + p^2 + m^2}\ , \quad C_\Lambda(0,p) = \frac{K(p^2/\Lambda^2)}{2\omega_p}\ , \tag{2.6}$$

where the property shown in (2.5) is satisfied and $\omega_p = \sqrt{p^2+m^2}$. $K(x)$ is assumed to have the following properties: $K(0) = 1$, $K(x) \sim 1$ for $x < 1$, and $K(x)$ damps rapidly for $x > 1$. In this case, $\dot{C}_\Lambda(0,p)$ is given by

$$\dot{C}_\Lambda(0,p) = \frac{\dot{K}(p^2/\Lambda^2)}{2\omega_p}\ , \tag{2.7}$$

where $\dot{K}(p^2/\Lambda^2) = -\Lambda\partial_\Lambda K(p^2/\Lambda^2)$.

We consider the Lagrangian L_Λ which consists of the free part $L_{0,\Lambda}$ and the interaction part $L_{\mathrm{int},\Lambda}$: $L_\Lambda = L_{0,\Lambda} + \alpha L_{\mathrm{int},\Lambda}$, with

$$L_{0,\Lambda} = \int_p \frac{1}{2}K_p^{-1}\left[\partial_\tau\phi(\tau,p)\partial_\tau\phi(\tau,-p) + \omega_p^2\phi(\tau,p)\phi(\tau,-p)\right]\ , \tag{2.8}$$

$$\begin{aligned} L_{\mathrm{int},\Lambda} = {} & \frac{\delta m^2}{2}\int_p \phi(\tau,p)\phi(\tau,-p) \\ & + \frac{\lambda}{4!}\int_{p_1\ldots p_4} \phi(\tau,p_1)\phi(\tau,p_2)\phi(\tau,p_3)\phi(\tau,p_4)\tilde{\delta}(p_1+p_2+p_3+p_4)\ , \end{aligned} \tag{2.9}$$

where we have introduced an expansion parameter α and a shorthand notation

$$K_p = K(p^2/\Lambda^2)\ . \tag{2.10}$$

The flow equation for δm^2 is

$$\frac{\dot{\delta m^2}}{2} = -\frac{\lambda}{4!}\int_p \frac{6\dot{K}_p}{2\omega_p}\ . \tag{2.11}$$

The free solution of the ERG equation(2.4) is given by

$$\Psi_\Lambda^{(0)}[\varphi] = N_0 \exp\left[-\int_p \frac{1}{2}K_p^{-1}\omega_p\varphi(p)\varphi(-p)\right]\ , \tag{2.12}$$

where N_0 is the normalization constant which is fixed by the condition $1 = \int \mathcal{D}\varphi|\Psi_0[\varphi]|^2$ as

$$N_0 = \exp\left[\frac{V}{4}\int_p \log\left(2K_p^{-1}\omega_p\right)\right]\ . \tag{2.13}$$

The perturbative solution in the first order takes the following form:

$$\Psi_\Lambda^{(1)} = A_\Lambda \Psi_\Lambda^{(0)} \tag{2.14}$$

with

$$\begin{aligned} A_\Lambda = & -\frac{\delta m^2}{2}\int_p \varphi(p)\varphi(-p)\frac{1}{2\omega_p} \\ & -\frac{\lambda}{4!}\int_{p_1p_2} \varphi(p_1)\varphi(-p_1)\ \frac{3K_2}{2\omega_1(\omega_1+\omega_2)} \\ & -\frac{\lambda}{4!}\int_{p_1\cdots p_4} \varphi_1\cdots\varphi_4\ \frac{\tilde{\delta}(p_1+p_2+p_3+p_4)}{\omega_1+\omega_2+\omega_3+\omega_4} + C\ , \end{aligned} \tag{2.15}$$

where

$$C = \left\{ \frac{\delta m^2}{2} + \frac{\lambda}{4!} \int_p \frac{6K_p}{2\omega_p} \right\} \int_k \frac{K_k V}{4\omega_k^2} - \frac{\lambda}{4!} \int_{p_1,p_2} \frac{3K_1 K_2 V}{\omega_1 \omega_2 (\omega_1 + \omega_2)} . \tag{2.16}$$

Note that A_Λ is an anti-Hermitian operator. In the following sections, we use the free and perturbative solutions (2.12) and (2.14) to realize quantum error correction in the scalar field theory.

3. Encoding

In the AdS/CFT correspondence, the bulk and boundary operators should have the property of quantum error correction(QEC). If the solution of the ERG equation(2.4) describes the bulk reconstruction(spacetime emergence), the Knill-Laflamme condition should hold in a code subspace.

In the following, by making a rescaling such as

$$p \to \Lambda p , \quad \varphi(p) \to \Lambda^{-\frac{d+1}{2}} \varphi(p) , \tag{3.1}$$

$$K_p = K(p^2/\Lambda^2) \to K_p = K(p^2), \quad \omega_p = \sqrt{p^2 + m^2} \to \Lambda\omega_p = \Lambda\sqrt{p^2 + m^2/\Lambda^2} , \tag{3.2}$$

$$\delta m^2 \to \Lambda^2 \delta m^2, \quad \lambda \to \Lambda^{3-d} \lambda , \tag{3.3}$$

we make all quantities dimensionless. The functional Ψ_Λ is rewritten in terms of the rescaled quantities.

We construct q-level states $|r\rangle$ $(r = 0, 1, \ldots, q-1)$, which correspond to span$(\{|i\rangle\})$, by using coherent states in the IR region. We describe the renormalization group flow of the ground state $|\Psi\rangle_\Lambda$ by a unitary operator U as

$$|\Psi\rangle_\Lambda = U(\Lambda, \Lambda_{\mathrm{UV}}) \, |\Psi\rangle_{\Lambda_{\mathrm{UV}}} \ , \tag{3.4}$$

where Λ is the effective cutoff and Λ_{UV} is the UV cutoff.

We assume that $U(\Lambda, \Lambda_{\mathrm{UV}})$ can be represented as

$$U(\Lambda, \Lambda_{\mathrm{UV}}) = T \exp\left[\int_\Lambda^{\Lambda_{\mathrm{UV}}} \frac{d\Lambda'}{\Lambda'} X_{\Lambda'} \right] . \tag{3.5}$$

Here T is an ordering operator defined by

$$T(X_\Lambda X_{\Lambda'}) = \begin{cases} X_\Lambda X_{\Lambda'} & \text{for } \Lambda < \Lambda' \\ X_{\Lambda'} X_\Lambda & \text{for } \Lambda > \Lambda' \end{cases} . \tag{3.6}$$

As in the context of continuum MERA[9, 10], it is natural to call an anti-Hermitian operator X_Λ the disentangler because it removes entanglement and reduces degrees of freedom along with the renormalization group flow.

By acting $-\Lambda\partial_\Lambda$ on both sides of (3.4) and using (3.5), we obtain

$$-\Lambda\partial_\Lambda \, |\Psi\rangle_\Lambda = X_\Lambda \, |\Psi\rangle_\Lambda \ . \tag{3.7}$$

This is the flow equation for the ground state. If we obtain the scale dependence of the ground state in another way, we can calculate X_Λ by using this equation.

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By acting on the q-level states the inverse of the unitary operator U, we encode them into states in the UV region. Namely, we identify $U^\dagger$ with W in (A.1) and have

$$|\tilde{r}\rangle = U^\dagger(\Lambda, \Lambda_{\rm UV})|r\rangle \ . \tag{3.8}$$

We find that the condition (A.2) is satisfied in an approximate sense in the IR region and exactly in the IR limit for a class of operators which correspond to E_a in (A.2). We verify this to the first order in the perturbation theory, using the free and perturbative solutions (2.12) and (2.14) of the ERG equation(2.4).

4. Renormalization group flow of the ground state

In this section, we derive the renormalization group flow of the ground state in the perturbation theory for evaluating the Knill-Laflamme condition in the next section.

4.1 The effective Hamiltonian for scalar field theory

We define the creation and annihilation operators at the scale Λ by[2]

$$[a_{\Lambda,p}, a^\dagger_{\Lambda,p'}] = \tilde{\delta}(p-p') \ , \quad [a_{\Lambda,p}, a_{\Lambda,p'}] = 0 \ , \quad [a^\dagger_{\Lambda,p}, a^\dagger_{\Lambda,p'}] = 0 \ , \tag{4.1}$$

$$a_{\Lambda,p}|\Psi\rangle_\Lambda = 0 \ . \tag{4.2}$$

Then, the renormalization group flow of the creation and annihilation operators is defined by

$$\begin{aligned} a_{\Lambda,p} &= U(\Lambda, \Lambda_{\rm UV}) a_{\Lambda_{\rm UV},p} U(\Lambda, \Lambda_{\rm UV})^\dagger \ , \\ a^\dagger_{\Lambda,p} &= U(\Lambda, \Lambda_{\rm UV}) a^\dagger_{\Lambda_{\rm UV},p} U(\Lambda, \Lambda_{\rm UV})^\dagger \ . \end{aligned} \tag{4.3}$$

Or equivalently,

$$\begin{aligned} -\Lambda\partial_\Lambda a_{\Lambda,p} &= [X_\Lambda, a_{\Lambda,p}] \ , \\ -\Lambda\partial_\Lambda a^\dagger_{\Lambda,p} &= [X_\Lambda, a^\dagger_{\Lambda,p}] \ . \end{aligned} \tag{4.4}$$

(4.1) and (4.2) are preserved under (4.3) and (4.4). For later convenience, we also introduce linear combinations of the creation and annihilation operators as

$$a^+_{\Lambda,p} = a_{\Lambda,p} + a^\dagger_{\Lambda,-p} \ , \tag{4.5}$$

$$a^-_{\Lambda,p} = a_{\Lambda,p} - a^\dagger_{\Lambda,-p} \ . \tag{4.6}$$

The flow equations for the above operators follow from (4.4):

$$-\Lambda\partial_\Lambda a^\pm_\Lambda = \left[X_\Lambda, a^\pm_{\Lambda,p}\right] \ . \tag{4.7}$$

[2]It seems nontrivial whether there exist the creation and annihilation operators that satisfy (4.1) and (4.2). In the following, we show this is indeed the case to the first order in the perturbation theory.

In this paper, we consider a perturbation theory in which we expand $|\Psi\rangle_\Lambda$, X_Λ, $a_{\Lambda,p}$ and $a^\pm_{\Lambda,p}$ in terms of α as follows:

$$\begin{aligned}
|\Psi\rangle_\Lambda &= |\Psi^{(0)}\rangle_\Lambda + \alpha|\Psi^{(1)}\rangle_\Lambda + \alpha^2|\Psi^{(2)}\rangle_\Lambda + \cdots , \\
X_\Lambda &= X^{(0)}_\Lambda + \alpha X^{(1)}_\Lambda + \alpha^2 X^{(2)}_\Lambda + \cdots , \\
a_{\Lambda,p} &= a^{(0)}_{\Lambda,p} + \alpha a^{(1)}_{\Lambda,p} + \alpha^2 a^{(2)}_{\Lambda,p} + \cdots , \\
a^\pm_{\Lambda,p} &= a^{\pm(0)}_{\Lambda,p} + \alpha a^{\pm(1)}_{\Lambda,p} + \alpha^2 a^{\pm(2)}_{\Lambda,p} + \cdots .
\end{aligned} \tag{4.8}$$

In the remaining part of this section, we use the φ-representation.

4.2 Free field theory

In this subsection, we examine the free field theory, namely the zeroth order in α. The creation and annihilation operators for the free Hamiltonian $H_{0,\Lambda}$ are given by

$$\begin{aligned}
a^{(0)}_{\Lambda,p} &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\omega_{\Lambda,p}}{K_p}}\varphi(p) + \sqrt{\frac{K_p}{\omega_{\Lambda,p}}}\frac{\delta}{\delta\varphi(-p)}\right), \\
a^{(0)\dagger}_{\Lambda,p} &= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\omega_{\Lambda,p}}{K_p}}\varphi(-p) - \sqrt{\frac{K_p}{\omega_{\Lambda,p}}}\frac{\delta}{\delta\varphi(p)}\right).
\end{aligned} \tag{4.9}$$

Then, the free Hamiltonian is rewritten as

$$H_{0,\Lambda} = \int_p \omega_{\Lambda,p} a^{(0)\,\dagger}_{\Lambda,p} a^{(0)}_{\Lambda,p} + \frac{V}{2}\int_p \omega_{\Lambda,p} , \tag{4.10}$$

where V is the volume of space.

By acting $-\Lambda\partial_\Lambda$ on the free solution(2.12) of the ERG equation(2.4), we obtain

$$-\Lambda\partial_\Lambda \Psi^{(0)}_\Lambda = -\frac{1}{4}\int_p \frac{\dot{\omega}_{\Lambda,p}}{\omega_{\Lambda,p}} a^{(0)\dagger}_{\Lambda,-p} a^{(0)\dagger}_{\Lambda,p} \Psi^{(0)}_\Lambda . \tag{4.11}$$

Taking the anti-Hermiticity into account, we read off $X^{(0)}_\Lambda$ as

$$X^{(0)}_\Lambda = -\frac{1}{4}\int_p \frac{\dot{\omega}_{\Lambda,p}}{\omega_{\Lambda,p}}\left(a^{(0)\dagger}_{\Lambda,-p} a^{(0)\dagger}_{\Lambda,p} - a^{(0)}_{\Lambda,p} a^{(0)}_{\Lambda,-p}\right), \tag{4.12}$$

where we use $a^{(0)}_{\Lambda,p}\Psi^{(0)}_\Lambda = 0$.

Finally, we derive the scaling of the creation and annihilation operators in the free field theory. While we can easily read off from (4.9), we derive it by using the disentangler as a preparation for the analysis of the interacting theory. By using (4.12), we calculate the zeroth order of (4.7) in α as

$$\begin{aligned}
-\Lambda\partial_\Lambda a^{\pm(0)}_{\Lambda,p} &= \left[X^{(0)}_\Lambda, a^{\pm(0)}_{\Lambda,p}\right] \\
&= \pm\frac{1}{2}\frac{\dot{\omega}_{\Lambda,p}}{\omega_{\Lambda,p}} a^{\pm(0)}_{\Lambda,p} ,
\end{aligned} \tag{4.13}$$

from which, we obtain the scaling for $a^{\pm(0)}$ as

$$a^{+(0)}_{\Lambda,p} = \sqrt{\frac{\omega_{\Lambda,p}}{\omega_{\text{UV},p}}} a^{+(0)}_{\text{UV},p} , \tag{4.14}$$

$$a^{-(0)}_{\Lambda,p} = \sqrt{\frac{\omega_{\text{UV},p}}{\omega_{\Lambda,p}}} a^{-(0)}_{\text{UV},p} , \tag{4.15}$$

where $\omega_{\text{UV},p}$ and $a^{\pm(0)}_{\text{UV},p}$ stand for $\omega_{\Lambda,p}$ and $a^{\pm(0)}_{\Lambda,p}$ with $\Lambda = \Lambda_{\text{UV}}$, respectively.

4.3 The first order in the perturbation theory

In this subsection, we examine the first order in the perturbation theory. By expanding (3.7), we obtain

$$-\Lambda\partial_\Lambda \Psi^{(0)}_\Lambda = X^{(0)}_\Lambda \Psi^{(0)}_\Lambda , \tag{4.16}$$

$$-\Lambda\partial_\Lambda \Psi^{(1)}_\Lambda = X^{(0)}_\Lambda \Psi^{(1)}_\Lambda + X^{(1)}_\Lambda \Psi^{(0)}_\Lambda . \tag{4.17}$$

Substituting perturbative solution (2.14) and (4.16) into (4.17) leads to

$$-\Lambda\partial_\Lambda A_\Lambda = X^{(1)}_\Lambda + [X^{(0)}_\Lambda, A_\Lambda] . \tag{4.18}$$

One can determine $X^{(1)}_\Lambda$ if A_Λ is known.

From (4.4), we obtain

$$-\Lambda\partial_\Lambda a^{(0)}_{\Lambda,p} = \left[X^{(0)}_\Lambda, a^{(0)}_{\Lambda,p}\right] , \tag{4.19}$$

$$-\Lambda\partial_\Lambda a^{(1)}_{\Lambda,p} = \left[X^{(1)}_\Lambda, a^{(0)}_{\Lambda,p}\right] + \left[X^{(0)}_\Lambda, a^{(1)}_{\Lambda,p}\right] . \tag{4.20}$$

We can show that

$$a^{(1)}_{\Lambda,p} = [A_\Lambda, a^{(0)}_{\Lambda,p}] . \tag{4.21}$$

5. Quantum error correction by the renormalization group

5.1 Encoding q-level states

In this subsection, we do *not* restrict ourselves to the free field theory. We use only the properties of the creation and annihilation operators (4.1) and (4.2). In order to realize a q-level system in scalar field theories, following [4], we use coherent states defined by

$$|f\rangle_\Lambda = \exp\left[\int_p \left(f(p)a^\dagger_\Lambda(-p) - f^*(-p)a_\Lambda(p)\right)\right] |\Psi\rangle_\Lambda , \tag{5.1}$$

where f is an arbitrary function. Note that

$$a_{\Lambda,p}|f\rangle_\Lambda = f(p)|f\rangle_\Lambda . \tag{5.2}$$

The inner product between coherent states is given by

$${}_\Lambda\langle f' \mid f\rangle_\Lambda = \exp\left[-\frac{1}{2}\int_p \left(|f'(p)|^2 - 2f'^*(p)f(p) + |f(p)|^2\right)\right] , \tag{5.3}$$

which implies that ${}_\Lambda\langle f|f\rangle_\Lambda = 1$. We construct q-level states by choosing $f = r f_0$ with f_0 being a real function and $r = 0, 1, \ldots, q-1$:

$$|r f_0\rangle_\Lambda = \exp\left[-r\int_p f_0(-p)\left(a_{\Lambda,p} - a^\dagger_{\Lambda,-p}\right)\right]|\Psi\rangle_\Lambda$$
$$= \exp\left[-r\int_p f_0(-p)a^-_{\Lambda,p}\right]|\Psi\rangle_\Lambda \ . \tag{5.4}$$

The inner product between these states is given by

$${}_\Lambda\langle r' f_0|r f_0\rangle_\Lambda = \exp\left[-\frac{1}{2}(r-r')^2\int_p |f_0(p)|^2\right]. \tag{5.5}$$

Note that these states form an orthonormal basis in an approximate way when $\int_p |f_0|^2$ is large enough. When $f_0(x)$ is localized around $x = x_0$, the q-level states are realized locally.

We encode the q-level states into states in the UV region as

$$|r f_0\rangle_{\rm UV} = U^\dagger(\Lambda,\Lambda_{\rm UV})\,|r f_0\rangle_\Lambda\,, \tag{5.6}$$

where $U^\dagger(\Lambda,\Lambda_{\rm UV})$ is the inverse of the unitary operator defined in (3.4). This equation implies that we encode information defined in the IR region with small Λ into the UV region in terms of the inverse of the renormalization group[3].

In what follows, we consider error operators $D[g]$ defined in the IR region

$$D[g] = \exp\left[\int_p g(-p)\left(a_{\Lambda,p} - a^\dagger_{\Lambda,-p}\right)\right] = \exp\left[\int_p g(-p)a^-_{\Lambda,p}\right], \tag{5.7}$$

where g is an arbitrary real function. Note that $D[g]$ is an operator that generates a coherent state in the IR region, namely $D[g]|\Psi\rangle_\Lambda$ is a coherent state. We show that quantum error correction condition or the Knill-Laflamme condition[11]

$${}_{\rm UV}\langle r' f_0|\,D^\dagger[g]D[h]\,|r f_0\rangle_{\rm UV} = M[g,h]\;{}_{\rm UV}\langle r' f_0\,|\,r f_0\rangle_{\rm UV}\,, \tag{5.8}$$

with $M[g,h]$ being a Hermitian matrix on the functional vector space is approximately satisfied in the IR region (exactly satisfied in the $\Lambda \to 0$ limit). Namely, we see that $\mathrm{span}(\{|r f_0\rangle_{\rm UV}\})$ gives a code subspace that is correctable for the errors caused by $D[g]$. In order to show this, it is enough to calculate

$${}_{\rm UV}\langle r' f_0|\,D[g]\,|r f_0\rangle_{\rm UV}\,, \tag{5.9}$$

for any real functions g, because $D^\dagger[g]D[h] = D[h-g]$.

[3]Note that $U(\Lambda,\Lambda_{\rm UV})$ does *not* necessarily give the renormalization group flow of $|r f_0\rangle_\Lambda$, since it is defined as giving that of the ground state. We just define the encoding of $|r f_0\rangle_\Lambda$ by $U^\dagger(\Lambda,\Lambda_{\rm UV})$.

5.2 Free field theory

In this subsection, we show that the error correction condition is satisfied in the free scalar field theory. (5.9) is calculated as follows.

$$
\begin{aligned}
&{}_{\mathrm{UV}}\langle r' f_0|D[g]\,|r f_0\rangle_{\mathrm{UV}} \\
&\quad = {}_{\mathrm{UV}}\langle r' f_0|\exp\left[\int_p g(-p) a^{-(0)}_{\Lambda,p}\right]|r f_0\rangle_{\mathrm{UV}} \\
&\quad = {}_{\mathrm{UV}}\langle r' f_0|\exp\left[\int_p g(-p)\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}} a^{-(0)}_{\mathrm{UV},p}\right]|r f_0\rangle_{\mathrm{UV}} \\
&\quad = {}_{\mathrm{UV}}\langle r' f_0|\exp\left[-\int_p \left(r f_0(-p) - g(-p)\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}}\right) a^{-(0)}_{\mathrm{UV},p}\right]|\Psi\rangle_{\mathrm{UV}} \\
&\quad = {}_{\mathrm{UV}}\left\langle r' f_0 \middle| r f_0 - \sqrt{\frac{\omega_{\mathrm{UV}}}{\omega_\Lambda}} g\right\rangle_{\mathrm{UV}} \\
&\quad = \exp\left[-\frac{1}{2}\int_p \left(-2(r-r')\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}} g(-p) f_0(p) + \frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}|g(p)|^2\right)\right] {}_{\mathrm{UV}}\langle r' f_0\,|\,r f_0\rangle_{\mathrm{UV}}\,,
\end{aligned}
\tag{5.10}
$$

where we used (5.3). The exponent is small enough in the IR region with $\Lambda/m \ll 1$. Thus, we obtain

$$
{}_{\mathrm{UV}}\langle r' f_0|\,D[g]\,|r f_0\rangle_{\mathrm{UV}} \sim {}_{\mathrm{UV}}\langle r' f_0\,|\,r f_0\rangle_{\mathrm{UV}}\ . \tag{5.11}
$$

Namely, the Knill-Laflamme condition is satisfied in an approximate sense[4]. The deviation from the exact condition can be read off from (5.10). In particular, the Knill-Laflamme condition is exactly satisfied in the IR limit, $\Lambda/m \to 0$.

5.3 Perturbation theory

In this subsection, we show that the Knill-Laflamme condition is satisfied to the first order in the perturbation theory. In order to calculate (5.9), we begin with representing the error operator up to the first order in α in terms of $a^{\pm}_{\mathrm{UV},p}$ by using (4.21) and (2.15). The details of the calculation are presented in [6]. The result is

$$
D[g] = \exp\left[\int_p g(-p) a_{-,\Lambda}(p)\right] = \exp\left[X + \alpha Y\right] \tag{5.12}
$$

with

$$
X = \int_p g(-p)\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}} a^{-}_{\mathrm{UV},p}\,, \tag{5.13}
$$

$$
\begin{aligned}
Y = \int_p g(-p)\Big\{ &-\frac{\lambda}{4!}\int_{k_1k_2k_3} C_1(k_1,k_2,k_3;p) a^{-}_{\mathrm{UV},-k_1} a^{-}_{\mathrm{UV},-k_2} a^{-}_{\mathrm{UV},-k_3} \\
&-\frac{\lambda}{8}\int_{k_1k_2k_3} C_2(k_1,k_2;k_3;p) a^{+}_{\mathrm{UV},-k_1} a^{+}_{\mathrm{UV},-k_2} a^{-}_{\mathrm{UV},-k_3} \\
&-\frac{\lambda}{4}\int_k C_3(k;p) a^{+}_{\mathrm{UV},p} - C_4(p) a^{-}_{\mathrm{UV},p}\Big\}\,,
\end{aligned}
\tag{5.14}
$$

[4]When $\int_p |f_0(p)|^2 \to \infty$, ${}_{\mathrm{UV}}\langle r' f_0\,|\,r f_0\rangle_{\mathrm{UV}} \to \delta_{rr'}$. In this case, the Knill-Laflamme condition is exactly satisfied without taking the IR limit. This is true only in the free case.

where

$$
\begin{aligned}
&C_1(k_1,k_2,k_3;p)\\
&\quad=\tilde{\delta}(k_1+\cdots+p)\left(\frac{1}{\omega_{\Lambda,1}+\cdots+\omega_{\Lambda,p}}\left(\prod_{i=1}^{3}\sqrt{\frac{K_i}{2\omega_{\Lambda,i}}}\right)\sqrt{\frac{K_p}{2\omega_{\Lambda,p}}}\sqrt{\frac{\omega_{\mathrm{UV},1}\omega_{\mathrm{UV},2}\omega_{\mathrm{UV},3}}{\omega_{\Lambda,1}\omega_{\Lambda,2}\omega_{\Lambda,3}}}\right.\\
&\qquad\left.-\frac{1}{\omega_{\mathrm{UV},1}+\cdots+\omega_{\mathrm{UV},p}}\left(\prod_{i=1}^{3}\sqrt{\frac{K_i}{2\omega_{\mathrm{UV},i}}}\right)\sqrt{\frac{K_p}{2\omega_{\mathrm{UV},p}}}\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}}\right),
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
&C_2(k_1,k_2;k_3;p)\\
&\quad=\tilde{\delta}(k_1+\cdots+p)\left(\frac{1}{\omega_{\Lambda,1}+\cdots+\omega_{\Lambda,p}}\left(\prod_{i=1}^{3}\sqrt{\frac{K_i}{2\omega_{\Lambda,i}}}\right)\sqrt{\frac{K_p}{2\omega_{\Lambda,p}}}\sqrt{\frac{\omega_{\Lambda,1}\omega_{\Lambda,2}}{\omega_{\mathrm{UV},1}\omega_{\mathrm{UV},2}}}\sqrt{\frac{\omega_{\mathrm{UV},3}}{\omega_{\Lambda,3}}}\right.\\
&\qquad\left.-\frac{1}{\omega_{\mathrm{UV},1}+\cdots+\omega_{\mathrm{UV},p}}\left(\prod_{i=1}^{3}\sqrt{\frac{K_i}{2\omega_{\mathrm{UV},i}}}\right)\sqrt{\frac{K_p}{2\omega_{\mathrm{UV},p}}}\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}}\right),
\end{aligned}
\tag{5.16}
$$

$$
C_3(k;p)=\frac{1}{2\omega_{\Lambda,p}+2\omega_{\Lambda,k}}\frac{K_p}{2\omega_{\Lambda,p}}\frac{K_k}{2\omega_{\Lambda,k}}\sqrt{\frac{\omega_{\Lambda,p}}{\omega_{\mathrm{UV},p}}}-\frac{1}{2\omega_{\mathrm{UV},p}+2\omega_{\mathrm{UV},k}}\frac{K_p}{2\omega_{\mathrm{UV},p}}\frac{K_k}{2\omega_{\mathrm{UV},k}}\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}},
\tag{5.17}
$$

$$
C_4(p)=\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}}\left[\left(\frac{\delta m_\Lambda^2}{2}+\frac{\lambda}{4!}\int_q\frac{6K_q}{2\omega_{\Lambda,q}}\right)\frac{K_p}{2\omega_{\Lambda,p}^2}-\left(\frac{\delta m_{\mathrm{UV}}^2}{2}+\frac{\lambda}{4!}\int_q\frac{6K_q}{2\omega_{\mathrm{UV},q}}\right)\frac{K_p}{2\omega_{\mathrm{UV},p}^2}\right].
\tag{5.18}
$$

Here $\omega_{\Lambda,i}$, $\omega_{\mathrm{UV},i}$ and K_i stand for ω_{Λ,k_i}, ω_{UV,k_i} and K_{k_i}, respectively. Note that $C_1(k_1,k_2,k_3;p)$ is symmetric with respect to the permutation of k_1, k_2 and k_3 while $C_2(k_1,k_2;k_3;p)$ is symmetric with respect to the permutation of k_1 and k_2.

The final result is

$$
\begin{aligned}
&{}_{\mathrm{UV}}\langle r' f_0|\,D[g]\,|r f_0\rangle_{\mathrm{UV}}\\
&={}_{\mathrm{UV}}\left\langle r' f_0\middle| r f_0-\sqrt{\frac{\omega_{\mathrm{UV}}}{\omega_\Lambda}}g\right\rangle_{\mathrm{UV}}\\
&\quad\times\left[1+\alpha\left\{-\frac{\lambda}{4!}\int_{p,k_1,k_2,k_3}g(-p)C_1(k_1,k_2,k_3;p)\right.\right.\\
&\qquad\times\left\{(r-r')^3f_0(-k_1)f_0(-k_2)f_0(-k_3)-3(r-r')^2\sqrt{\frac{\omega_{\mathrm{UV},3}}{\omega_{\Lambda,3}}}f_0(-k_1)f_0(-k_2)g(-k_3)\right.\\
&\qquad\quad+3(r-r')\sqrt{\frac{\omega_{\mathrm{UV},2}\omega_{\mathrm{UV},3}}{\omega_{\Lambda,2}\omega_{\Lambda,3}}}f_0(-k_1)g(-k_2)g(-k_3)\\
&\qquad\quad\left.-\sqrt{\frac{\omega_{\mathrm{UV},1}}{\omega_{\Lambda,1}}}g(-k_1)\sqrt{\frac{\omega_{\mathrm{UV},2}}{\omega_{\Lambda,2}}}g(-k_2)\sqrt{\frac{\omega_{\mathrm{UV},3}}{\omega_{\Lambda,3}}}g(-k_3)\right\}\\
&\quad+\frac{\lambda}{8}\int_{p,k_1,k_2}g(-p)C_1(k_1,k_2,-k_1-k_2;p)\left\{(r-r')f_0(-k_2)-\sqrt{\frac{\omega_{\mathrm{UV},2}}{\omega_{\Lambda,2}}}g(-k_2)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\lambda}{8}\int_{p,k_1,k_2,k_3} g(-p)C_2(k_1,k_2;k_3;p)\left\{(r-r')(r^2+r'^2)f_0(-k_1)f_0(-k_2)f_0(-k_3)\right.\\
&\quad -2(r^2-r'^2)\sqrt{\frac{\omega_{\mathrm{UV},1}}{\omega_{\Lambda,1}}}g(-k_1)f_0(-k_2)f_0(-k_3)\\
&\qquad\qquad -(r+r')^2\sqrt{\frac{\omega_{\mathrm{UV},3}}{\omega_{\Lambda,3}}}f_0(-k_1)f_0(-k_2)g(-k_3)\\
&\quad +2(r+r')\sqrt{\frac{\omega_{\mathrm{UV},2}\omega_{\mathrm{UV},3}}{\omega_{\Lambda,2}\omega_{\Lambda,3}}}f_0(-k_1)g(-k_2)g(-k_3)\\
&\qquad\qquad +(r-r')\sqrt{\frac{\omega_{\mathrm{UV},1}\omega_{\mathrm{UV},2}}{\omega_{\Lambda,1}\omega_{\Lambda,2}}}g(-k_1)g(-k_2)f_0(-k_3)\\
&\quad \left.-\sqrt{\frac{\omega_{\mathrm{UV},1}\omega_{\mathrm{UV},2}\omega_{\mathrm{UV},3}}{\omega_{\Lambda,1}\omega_{\Lambda,2}\omega_{\Lambda,3}}}g(-k_1)g(-k_2)g(-k_3)\right\}\\
&+\frac{\lambda}{4}\int_{p,k_1,k_2,k_3} g(-p)C_2(k_1,k_2;-k_1;p)\left\{(r+r')f_0(-k_2)-\sqrt{\frac{\omega_{\mathrm{UV},2}}{\omega_{\Lambda,2}}}g(-k_2)\right\}\\
&-\frac{\lambda}{8}\int_{p,k_1,k_2,k_3} g(-p)C_2(k_1,-k_1;k_3;p)\left\{(r-r')f_0(-k_3)-\sqrt{\frac{\omega_{\mathrm{UV},3}}{\omega_{\Lambda,3}}}g(-k_3)\right\}\\
&-\frac{\lambda}{4}\int_{p,k} g(-p)C_3(k;p)\left\{(r+r')f_0(p)-\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}}g(p)\right\}\\
&-\int_p g(-p)C_4(p)\left\{(r-r')f_0(p)-\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}}g(p)\right\}\\
&-\frac{\lambda}{4}\int_{p,l,k_2,k_3}\sqrt{\frac{\omega_{\mathrm{UV},p}}{\omega_{\Lambda,p}}}g(-p)g(-\ell)C_2(p,k_2;k_3;\ell)\\
&\quad\times\left\{\left(r^2-r'^2\right)f_0(-k_2)f_0(-k_3)-(r+r')\sqrt{\frac{\omega_{\mathrm{UV},3}}{\omega_{\Lambda,3}}}f_0(-k_2)g(-k_3)\right.\\
&\qquad\quad\left.-(r-r')\sqrt{\frac{\omega_{\mathrm{UV},2}}{\omega_{\Lambda,2}}}f_0(-k_3)g(-k_2)+\sqrt{\frac{\omega_{\mathrm{UV},2}\omega_{\mathrm{UV},3}}{\omega_{\Lambda,2}\omega_{\Lambda,3}}}g(-k_2)g(-k_3)\right\}\\
&+\frac{\lambda}{12}\int_{p,q,\ell,k}\sqrt{\frac{\omega_{\mathrm{UV},q}\omega_{\mathrm{UV},p}}{\omega_{\Lambda,q}\omega_{\Lambda,p}}}g(-p)g(-\ell)g(-q)\\
&\qquad\qquad\qquad\times C_2(p,q;k;\ell)\left\{(r-r')f_0(-k)-\sqrt{\frac{\omega_{\mathrm{UV},k}}{\omega_{\Lambda,k}}}g(-k)\right\}\Bigg\}\Bigg]\,.
\end{aligned}
\tag{5.19}
$$

In the IR region with $\Lambda/m \ll 1$, we see that C_1, C_2 and C_3 are obviously small enough. On the other hand, we need to be careful for C_4. It has δm_Λ^2 which behaves as $(\Lambda_{\mathrm{UV}}/\Lambda)^2$ in the IR region[5]. However, this term also has the factor of $K_p/2\omega_{\Lambda,p}^2$ which behaves as $(\Lambda/\Lambda_{\mathrm{UV}})^2$ which cancel the above factor. Then, due to the pre-factor $\sqrt{\omega_{\mathrm{UV}}/\omega_{\Lambda,p}}$, C_4 is also small enough. Thus, the first-order

[5](2.11) is rewritten in terms of the rescaled quantities as

$$
-\Lambda\frac{\partial}{\partial\Lambda}\delta m^2(\Lambda)=2\delta m^2(\Lambda)-\frac{1}{2}\lambda(\Lambda)\int_p\left(\frac{d}{2\omega_{\Lambda,p}}-\frac{p^2}{2\omega_{\Lambda,p}^3}\right)K_p\,.
$$

.

terms in the perturbation theory are small enough. We therefore obtain

$$ {}_{\mathrm{UV}}\langle r' f_0 |\, D[g]\, | r f_0 \rangle_{\mathrm{UV}} \sim {}_{\mathrm{UV}}\langle r' f_0 \,|\, r f_0 \rangle_{\mathrm{UV}} \tag{5.20} $$

in the IR region. Namely, the Knill-Laflamme condition is satisfied in an approximate sense. In particular, it is exactly satisfied in the IR limit, $\Lambda/m \to 0$ limit. We have shown that up to the first order in the perturbation theory, the code subspace spanned by $\{|r f_0\rangle\}$ is error correctable for $D[g]$, even in the interacting theory.

6. Conclusion and discussion

In this paper, we showed that quantum error correction is realized by the renormalization group in scalar field theories. We constructed a q-level system by using coherent states in the IR region. We encoded it in the UV region by acting on the states in the q-level system the inverse of the unitary operator that gives the renormalization group flow of the ground state. We found that the Knill-Laflamme condition is satisfied in an approximate sense in the IR region and exactly in the IR limit for the operators that create coherent states. We confirmed this to the first order in the perturbation theory.

Some future directions are in order. We would like to extend the analysis in this paper to non-perturbative one, where the ERG equation for wave functionals (2.4) should be useful. It is interesting to consider other realizations of q-level system in the IR region or other classes of error operators. In such cases, other types of the inverse of the renormalization group may be required. Indeed, since the renormalization group is a semi-group, taking the inverse of the renormalization group [12–14] is nontrivial. The above directions should be relevant for revealing a general relationship between the renormalization group and quantum error correction. As well as the non-perturbative analysis, extension to gauge theories is important from the viewpoint of the gauge/gravity correspondence, since the strongly coupled regime at large N in the boundary theory corresponds to classical gravity in the bulk. We hope to report developments in these directions in the near future.

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A. The Knill-Laframme condition

Here we review the Knill-Laflamme condition[11] for quantum error correction This is the sufficient and necessary condition that a code is correctable for errors. In quantum error correction, in order to protect quantum information possessed by span$(\{|i\rangle\})$ from errors, span$(\{|i\rangle\})$ is encoded into a larger Hilbert space $\mathcal{H}$. The encoding map W is defined by

$$|\tilde{i}\rangle = W|i\rangle \ , \tag{A.1}$$

where $|\tilde{i}\rangle$ are elements of $\mathcal{H}$ and $W^\dagger W = I$ is satisfied. span$(\{|\tilde{i}\rangle\})$, which is a subspace of $\mathcal{H}$, is called the code subspace.

Suppose that the errors are described by error operators $\{E_a\}$ which are linear maps in $\mathcal{H}$. The Knill-Laflamme condition [11] is given by

$$\langle\tilde{i}|E_a^\dagger E_b|\tilde{j}\rangle = M_{ab}\,\langle\tilde{i}|\tilde{j}\rangle \ , \tag{A.2}$$

where M_{ab} are elements of a Hermitian matrix[6]. The Knill-Laflamme condition is said to be satisfied in an approximate sense when (A.2) holds up to a small quantity.

B. The Polchinski equation

In this appendix, we briefly review the Polchinski equation[8] for scalar field theories. The ERG equations for scalar field theories have in general the following structure [15–18]

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{-S_\Lambda[\phi]} = \int_p \frac{\delta}{\delta\phi(p_0,p)}\left[G_\Lambda[\phi](p)e^{-S_\Lambda[\phi]}\right], \tag{B.1}$$

where Λ is the effective cutoff and S_Λ is the effective action at the scale Λ. The functional $G_\Lambda[\phi](p)$, which also depends on p, is required to correspond to a continuum blocking procedure and to ensure the UV regularization of the equation. The structure in Eq.(B.1) ensures the physical requirement that the partition function is unchanged under the infinitesimal change of the effective cutoff Λ:

$$-\Lambda\partial_\Lambda Z = -\Lambda\partial_\Lambda\int\mathcal{D}\phi e^{-S_\Lambda[\phi]} = \int_{p_0,p}\mathcal{D}\phi\frac{\delta}{\delta\phi(p_0,p)}\left[G_\Lambda[\phi](p_0,p)e^{-S_\Lambda[\phi]}\right] = 0\,. \tag{B.2}$$

A typical form of $G_\Lambda[\phi](p_0,p)$ is given by

$$G_\Lambda[\phi](p_0,p) = \frac{1}{2}\dot{C}_\Lambda(p_0,p)\frac{\delta}{\delta\phi(-p_0,p)}(S_\Lambda - 2\hat{S})\ , \tag{B.3}$$

where $\dot{C}_\Lambda \equiv -\Lambda\partial_\Lambda C_\Lambda$ is an ERG integration kernel that incorporates the UV regularization and specifies the coarse-graining procedure with $\hat{S}$, which is called the seed action. The Polchinski equation [8] is obtained by setting the seed action $\hat{S}$ to S_0, i.e., the free part of the effective action S_Λ taking the form

$$S_0 = \int_p\frac{1}{2}\phi(p_0,p)C_\Lambda^{-1}(p_0,p)\phi(-p_0,-p)\ . \tag{B.4}$$

It is easy to check that S_0 satisfies Eq.(B.1) with Eq.(B.3). Then, Eq.(B.1) reduces to

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{-S_\Lambda[\phi]} = \int_{p_0,p}\frac{\delta}{\delta\phi(p_0,p)}\left[\frac{1}{2}\dot{C}_\Lambda(p_0,p)\left\{\frac{\delta}{\delta\phi(-p_0,-p)}(S_\Lambda-2S_0)\right\}e^{-S_\Lambda[\phi]}\right]\ . \tag{B.5}$$

[6]Note that the elements of $\{|\tilde{i}\rangle\}$ are linearly independent but not necessarily orthogonal.

By decomposing the effective action into the free part and the interaction part as

$$S_\Lambda = S_0 + S_{\rm int} \,, \tag{B.6}$$

one obtains from Eq.(B.5) a conventional form of the Polchinski equation for $S_{\rm int}$:

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{-S_{\rm int}} = -\frac{1}{2} \int_p \dot{C}_\Lambda(p_0, p) \frac{\delta^2}{\delta\phi(p_0, p)\delta\phi(-p_0, -p)} e^{-S_{\rm int}} \,. \tag{B.7}$$

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