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The fractional quantum Hall effect on a sphere and the Atiyah-Singer index theorem

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Most analyses of the ground state wavefunctions of the QHE ignore electron spin and simply use anti-commuting scalars: either because strong magnetic fields split the spins and the ground state is completely spin polarised or because the ground state is effectively spin degenerate so the only effect of spin is to double the degeneracy of the ground state. A useful approach is that of the Haldane sphere where the system is put on a 2-dimensional sphere and a normal magnetic field is generated by placing a magnetic monopole at the centre of the sphere. The analysis of the fractional quantum Hall effect in this approach is somewhat subtle, as the ground state wavefunction of the unperturbed Hamiltonian is not unique in Haldane's approach. When the fermionic nature of the charged particles is fully incorporate into this picture the analysis is modified and we show that, incorporating Jain's composite fermion picture into the Haldane sphere leads to a unique unperturbed ground state for the fractional quantum Hall effect on a sphere. A mass gap then ensures stability under perturbations. An important tool in the analysis is the Atiyah-Singer index theorem.

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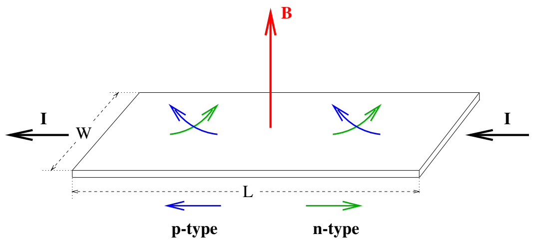
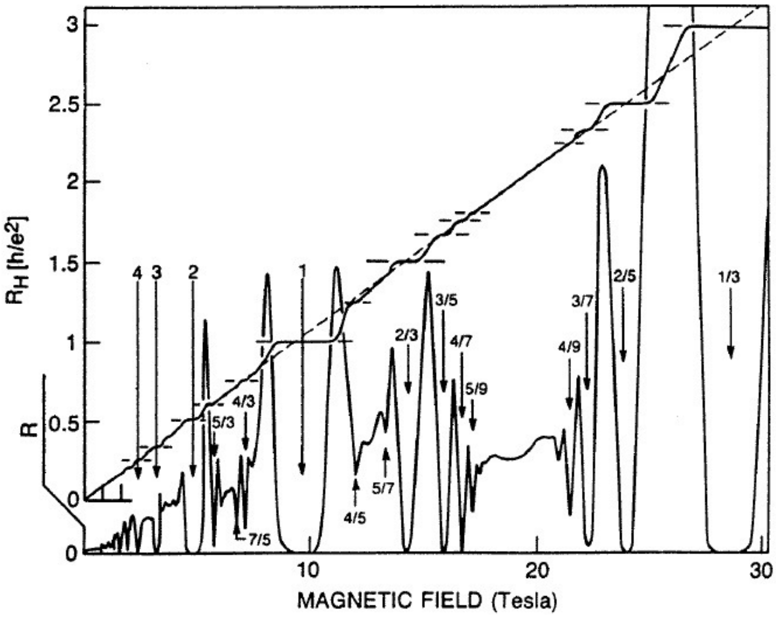
1. The quantum Hall Effect (QHE)

The quantum Hall effect is a fascinating phenomenon not just from an experimental and practical point of view, but also because the theoretical ideas that it has spawned.

1.1 Experiment



When a very pure thin slab of semi-conducting material (with a thickness that is less than the Dec Broglie wavelength of the electrons, so it is effectively 2-dimensional) is placed in a strong magnetic field, at cryogenic temperatures, the Hall resistance no longer follows the classical, linear dependence on the magnetic field, but develops strong plateaux that are quantised to a very high level of accuracy. On these plateau the Ohmic resistance drops to zero, as shown below (from [1]):



1.2 The integer QHE

In 2-dimensions resistance has units $\frac{h}{e^2}$, with $\frac{h}{e^2}$ corresponding to 25.81280745kΩ. In the integer QHE [2] the Hall resistance is 1 over an integer, in units with $\frac{e^2}{h} = 1$. A key ingredient in understanding the Hall quantisation is the notion of Landau levels. The Hamiltonian for a free electron moving in a 2-dimensional plane with a uniform, transverse magnetic field B is equivalent to that of a harmonic oscillator, so the energy levels are quantised and equally spaced. A system of N such non-interacting electrons has the Hamiltonian

$$H = \frac{\hbar^2}{2m} \sum_{i=1}^{N} \left(-i\nabla_i - \frac{e}{\hbar} A(x_i, y_i) \right)^2, \qquad A = \frac{B}{2}(x dy - y dx). \tag{1}$$

The energy levels (Landau levels) are degenerate, and the degeneracy $\frac{e\mathcal{A}}{B}$, increases with the area of the sample $\mathcal{A}$. If the Fermi energy ϵ_F falls between two Landau levels then all the Landau levels below ϵ_F are fully occupied and all those above are completely empty, there is therefore a mass gap. Landau levels are exactly filled when the magnetic flux per unit area is an integer multiple of the particle density (and so the total magnetic flux is an integer multiple of the total number of charge carriers in the sample), this integer measures the number of full Landau levels and is called the filling factor.

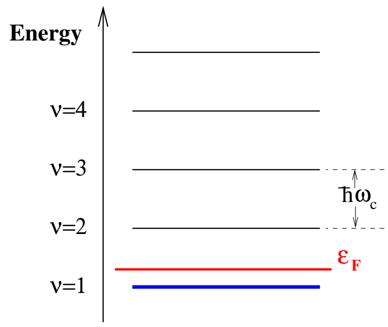


Figure 1: Landau levels for a charged particle in 2-d with a transverse magnetic field.

For example, it the number of particles is 3 and the total magnetic flux is $\Phi = B\mathcal{A} = 3\Phi_0$, (with $\Phi_0 = \frac{h}{e}$), as in the figure below, then the filling factor is $\nu = \frac{en}{B} = 1$ (in real samples, particle densities can be $n \sim 10^{11}\mathrm{cm}^{-2}$).

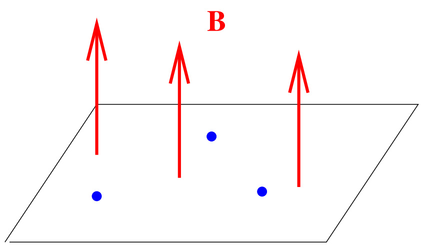


Figure 2: the completely filled lowest Landau level, filling factor 1.

On the plateaux the Ohmic conductivity $\sigma_{xx} = 0$ and the Hall conductivity $\sigma_{xy} = \nu = 1, 2, \dots$, in units with $\frac{e^2}{h} = 1$.

1.3 The fractional QHE

Two years after the integral QHE effect was discovered, with even purer samples, states with $\sigma_{xy} = \frac{p}{q}$ (p and q integral, with q odd) were found [3]. One way to view these states is to view the quasi-particles as composite fermions [4].

Take the wave function $\Psi(z_1, \dots, z_N)$ for a system consisting of N particles at positions $z_i = x_i + iy_i$, $i = 1, \cdots, N$ in terms of complex coordinates. Now perform the gauge transformation

$$\widetilde{\Psi}(z_1, \dots, z_N) = e^{\frac{i\theta}{\pi}(\sum_{i<j}\phi_{ij})}\Psi(z_1, \dots, z_N) \tag{2}$$

where θ is a constant and $\phi_{ij} = \arg(z_i - z_j)$. Graphically

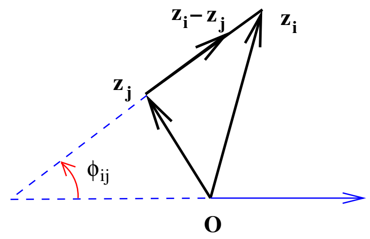


Figure 3: definition of the statistical angle ϕ_{ij}.

Let $\phi(z_1, \cdots, z_N) = \sum_{i<j} \phi_{ij}$, if any two particles are interchanged $i \leftrightarrow j$, then $\phi_{ij} \to \phi_{ij} + \pi$ and the phase changes by θ. If θ is an integral multiple of π,

$$\theta = \pi s, \qquad s \in \mathbf{Z}$$

then the wavefunction will change sign under this interchange if s is even, because the particles are fermions, but if s odd there is an extra minus sign and the wavefunction does not change sign under particle interchange: the gauge transformation has mapped a fermionic system to a bosonic system.

In the Hamiltonian $-i\hbar\nabla - e\mathbf{A} \to -i\hbar\nabla - e(\mathbf{A} + \mathbf{a})$ with

$$a_\alpha(z_i) = -\frac{\hbar\theta}{e\pi} \sum_{j \neq i} \nabla^{(i)}_\alpha \phi_{ij} \Rightarrow \epsilon^{\beta\alpha} \nabla^{(i)}_\beta a_\alpha(z_i) = -\frac{2\hbar\theta}{e} \sum_{j \neq i} \delta(z_i - z_j),$$

with $\alpha = 1, 2$.

In the gauge transformation $A \to A + a$, $a = -\frac{s\hbar}{e} d\phi$ is called the *Statistical Gauge Field*, because it can change the statistics of the multiparticle wavefunction. It gives rise to a statistical magnetic field.

$$b = da = -\frac{2\pi s\hbar}{e} \sum_{i \neq j} \delta(z_i - z_j) = -s \sum_{i<j} \delta(z_i - z_j) \frac{h}{e}.$$

If the number of particles is large, we can chose the origin to coincide with the position of one of the particles and use a fluid dynamical picture with particle density $n(z)$ to write

$$b(z) := \epsilon^{\beta\alpha} \nabla_\beta a_\alpha(z) = -\frac{2\hbar\theta}{e} n(z) = -\frac{sh}{e} n(z).$$

The gauge transformation (2) is singular if any two particles sit at the same point, giving rise to δ-functions in $b(z)$, but we have ignored Coulomb repulsion: charged particles with the same charge can never be at the same point, so (2) is a perfectly good gauge transformation for a system of charged particles, and Coulomb repulsion is an essential ingredient in the understanding of the fractional QHE.

For integral s this gauge transformation effectively attaches s units of statistical magnetic flux to each particle, and so goes under the name of *flux attachment*. The picture for $s = 2$, with filling factor $\frac{en}{B} = \frac{1}{2}$, is shown below: The statistical gauge field exactly cancels the background field and filling factor $\frac{1}{2}$ with $s = 2$ is equivalent to a system of composite fermions with no magnetic field: it is a standard fermi liquid, albeit of composite fermions.

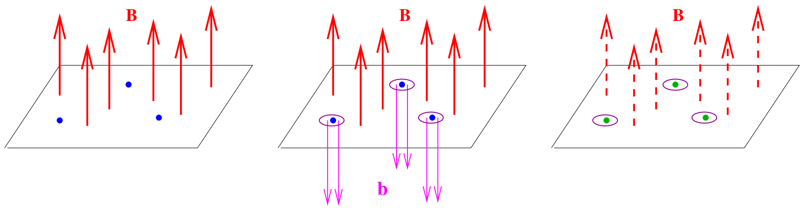


Figure 4: composite fermions in effectively zero field

If we now crank up the external field slightly, as shown in figure 5,

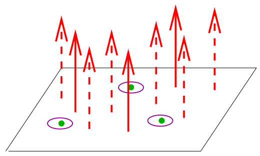


Figure 5: filling factor $\nu = \frac{1}{3}$ is the integer QHE for composite fermions.

we have a system of composite fermions, effectively with $\nu = 1$, although with 9 units of external magnetic field and 3 electrons the experimenter would measure $\nu = \frac{1}{3}$, the fractional QHE. This is Jain's composite fermion picture [4]: the fractional QHE is the integral QHE with composite fermions. The notion that the quasi-particles underpinning the fractional QHE are composite fermions is reminiscent the Seiberg-Witten low energy effective action for $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in 4-dimensions [5, 6], in which the effective degrees of freedom at low energy are also composite fermions: in this case dyons consisting of quarks attached to an even number of magnetic monopoles with unit charge, [7].

2. Laughlin wavefunctions

The Hamiltonian (1) does not capture all of the physics of the QHE, any real crystal has impurities and there is Coulomb repulsion between the electrons. In 1983 Laughlin suggested an analytic form for a trial ground state wavefunction for the full Hamiltonian that is found to have very good overlap with the numerically determined true ground state [8]. In an infinite plane Laughlin's wavefunction for $\nu = 1$ is

$$\begin{aligned}\Psi(z_1,\dots,z_N) &= \prod_{i<j}(z_i - z_j)e^{-\frac{1}{4l_B^2}\sum_i |\bar{z}_i z_i|} \qquad (l_B^2 = \tfrac{\hbar}{eB})\\ &= e^{i\Phi}\prod_{i<j}|z_i - z_j|e^{-\frac{1}{4l_B^2}\sum_i |\bar{z}_i z_i|},\end{aligned}$$

where $\Phi = \sum_{i<j}\phi_{ij}$, which is equivalent to $\theta = \pi$, $s = 1$ with a statistical gauge field. This can be interpreted as a condensation of composite *bosons* in zero field,

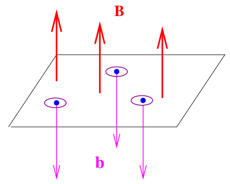


Figure 6: the Laughlin ground state wave function for $\nu = 1$: a Bose condensate of composite bosons.

Laughlin also proposed trial wavefunctions for the fractional effect:

$$\Psi(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^3 e^{-\frac{1}{4l_B^2}\sum_i |\bar{z}_i z_i|},$$

for $\nu = \frac{1}{3}$, or, more generally,

$$\Psi(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^{2k+1} e^{-\frac{1}{4l_B^2}\sum_i |\bar{z}_i z_i|},$$

for $\nu = \frac{1}{q}$, with $q = 2k+1$ an odd integer. The exponential factors serve to localise the wavefunction near the origin.

3. The quantum Hall effect on a sphere

Alternatively one can use a spherical geometry with the normal magnetic field generated by a magnetic monopole $M\Phi_0$ at the centre of a 2-dimensional sphere, with $\Phi_0 = \frac{h}{e}$ and M an integer, [9]. The Hamiltonian for a charged particle of mass m moving on the surface of a sphere of radius R is

$$H = \frac{1}{2mR^2}\sum_{i=1}^{N} |\vec{\Lambda}_i|^2, \qquad \vec{\Lambda} = \vec{r} \times (-i\hbar\vec{\nabla} - e\vec{A}(\vec{r})). \tag{3}$$

Without loss of generality we can choose $M > 0$. The gauge potential cannot be defined globally, but different gauges can be used on the northern and southern hemispheres in polar coordinates (ϑ, φ),

$$A^{(\pm)} = \frac{M\hbar}{2e}(\pm 1 - \cos\vartheta)d\varphi. \tag{4}$$

Haldane calculated the energy eigenvalues,

$$E_n = \frac{\hbar^2}{2mR^2}\left\{n(n+1) + \big(n + 1/2\big)M\right\}$$

and showed that the ground state, $n = 0$, has degeneracy $M + 1$.

In Dirac's original description of a magnetic monopole he used $A^{(+)}$ on the whole sphere and introduced a vortex at the south pole which was a gauge artefact — the famous "Dirac string". Using the complex coordinate $z = \tan\left(\frac{\vartheta}{2}\right) e^{-i\varphi}$, in the gauge

$$A^{(+)} = \frac{iM\hbar}{2e}\left(\frac{\bar{z}dz - zd\bar{z}}{1 + \bar{z}z}\right), \tag{5}$$

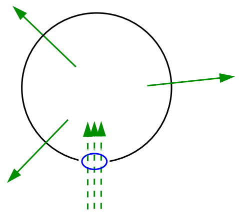


Figure 7: Dirac string.

$A^{(+)}$ is actually well defined everywhere, except at the south pole where there is a (fictitious in Dirac's picture) vortex,

The field strength is

$$F = dA = -\left(\frac{iM\hbar}{e}\right)\frac{dz\wedge d\bar{z}}{(1+\bar{z}z)^2} = \left(\frac{M\hbar}{2e}\right)\sin\theta d\theta\wedge d\phi.$$

A basis for the ground state wave function of (3), in the gauge (5) away from the S-pole, is

$$\psi_p(z) = \frac{z^p}{(1+\bar{z}z)^{\frac{M}{2}}},$$

with $p = 0,\dots,M$, [10].

For a system of N particles the most general ground state wavefunction of the Hamiltonian (3) is

$$\Psi(z_1,\dots,z_N) = \prod_{i=1}^{N}\frac{1}{(1+\bar{z}_iz_i)^{\frac{M}{2}}}\sum_{p_i=0}^{M} C_{p_1\dots p_N} z_1^{p_1}\cdots z_N^{p_N},$$

with $C_{p_1\dots p_N}$ arbitrary constants. The ground state is not unique, but for fermions $C_{p_1\dots p_N}$ should be anti-symmetric and, when $N = M+1$, it must be proportional to the $(M+1)$-dimensional ϵ-tensor. For the special case $N = M+1$ the normalised ground state is unique [9]

$$\Psi(z_1,\dots,z_N) = \prod_{i=1}^{N}\frac{1}{(1+\bar{z}_iz_i)^{\frac{M}{2}}}\prod_{i<j}(z_i - z_j),$$

and there is a mass gap. For $N = M+1$ the filling factor is $\nu = \frac{N}{M} = \frac{N}{N-1}$ and. For a large number of particles $\lim_{N\to\infty}\nu = 1$ and we recover the integer QHE, with $\nu = 1$ (in numerical analyses, with N finite, the numerator in this expression is usually augmented by one manually in order to get $\nu = 1$ — a process known as "the shift"). Of course this analysis has ignored interactions, but the mass gap ensures stability of the Haldane ground state under perturbations.

The fractional effect is not so clear. When $N = \frac{M}{q}+1$ the filling factor is $\nu = \frac{N}{M} = \frac{N}{(N-1)q}$, with $\nu = \lim_{N\to\infty} = \frac{1}{q}$, and

$$\Psi(z_1,\dots,z_N) = \prod_{i=1}^{N}\frac{1}{(1+\bar{z}_iz_i)^{\frac{M}{2}}}\prod_{i<j}(z_i - z_j)^q, \tag{6}$$

with $q \le (N-1)M$, is a ground state, but it is not unique if $q > 1$. For example, for $q = M = 3$, $N = 2$ the most general anti-symmetric ground state is

$$\Psi = \frac{1}{\{(1+\bar{z}_1z_1)(1+\bar{z}_2z_2)\}^{\frac{3}{2}}}\sum_{p_i=0}^{3} C_{[p_1p_2]} z_1^{p_1} z_2^{p_2}$$

with $C_{[p_1p_2]}$ a 4×4 anti-symmetric matrix, so this ground state is 6-fold degenerate. However there is an angular momentum operator [9] that commutes with the Hamiltonian and which can be used to classify the ground states: for example $q = 3$ in (6) is $J = 0$, while $q = 1$ is $J = 2$, $J_3 = -2$. Haldane argues that perturbations that affect quantum hall systems will lift the degeneracy of states with $J \neq 0$, but will leave the $J = 0$ sates unaffected, so that the true ground state will be the $J = 0$ singlet.

In the following section the fermionic nature of the charge carriers is incorporated into the picture.

3.1 The integer QHE: monopoles and the index theorem

For a system of non-interacting fermions on a sphere the non-relativistic Hamiltonian can be written in terms of the Dirac operator,

$$H = -\frac{\hbar^2}{2mR^2}\sum_{i=1}^{N} \not{D}_i^2, \qquad i\not{D} = \sigma^\alpha\big(i\nabla_\alpha - \frac{e}{\hbar}A_\alpha(\vec{r})\big). \tag{7}$$

The energy levels differ from those of (3),

$$E_n = \frac{\hbar^2}{2mR^2}n(n+M),$$

and the ground state $n = 0$ now has degeneracy M, rather than $M + 1$ as in Haldane's picture. Indeed the ground state degeneracy is determined by the Atiyah-Singer index theorem: a magnetic monopole of charge M gives $n_+ - n_-$ zero modes, where n_+ is the number of positive chirality zero modes and n_- the number of negative chirality zero modes, with

$$n_+ - n_- = \int_{S^2} F = M$$

in units with $\Phi_0 = \frac{h}{e} = \frac{e^2}{h} = 1, \hbar = \frac{1}{2\pi}$. On a sphere there are no zero modes without a magnetic monopole and no negative chirality zero modes when $M > 0$. Ground state wave functions for a single particle are[1]

$$\psi_{+,p}(z) = \frac{z^p}{(1+\bar{z}z)^{\frac{M-1}{2}}}, \qquad \psi_- = 0, \qquad p = 0, \dots, M-1.$$

For a system of N particles there is a unique ground state, with a mass gap $\sim \frac{\hbar^2}{mR^2}$, if and only if $N = M$, in which case

$$\Psi(z_1, \dots, z_N) = \prod_{i=1}^{N} \frac{1}{(1+\bar{z}_i z_i)^{\frac{M-1}{2}}} \prod_{i<j}(z_i - z_j).$$

This has filling factor $\nu = \frac{N}{M} = 1$ for any finite N. In contrast to Haldane's model, there is no need to perform a "shift" to get the integer QHE when the spinorial nature of the electrons is taken into account, the shift is accounted for geometrically — it arises from the spin connection for fermions moving on a sphere.

[1]More details are given in [10], though z here is $\bar{z}$ in that reference, z is used here for a slight notational simplicity.

3.2 The fractional QHE: vortices and the index theorem

To analyse the fractional QHE on a sphere we invoke Jain's composite fermion picture. A uniform monopole field of charge $-\mathrm{v}$ plus a vortex at a point z_i is generated by a gouge potential

$$a(z) = \frac{i\mathrm{v}}{4\pi}\left(\frac{dz}{z - z_i} - \frac{d\bar{z}}{\bar{z} - \bar{z}_i}\right) - \frac{i\mathrm{v}}{4\pi}\left(\frac{\bar{z}dz - zd\bar{z}}{1 + \bar{z}z}\right).$$

The total magnetic flux vanishes of course,

$$\int_{S^2} f = \int_{S^2} da = 0,$$

in particular the vortex is real, it is not taken to be a ghostly gauge artefact.

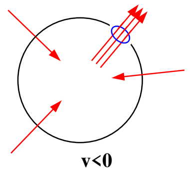


Figure 8: a dressed vortex

In 3-dimensions there is a real string, not a Dirac string. This real string is on the same footing as that associated with effective magnetic monopoles that can be created in spin ice in the laboratory [11]. In contrast to spin ices, we wish to focus on the vortex rather than the uniform background, so this configuration will be referred to as a *dressed vortex*.

Now consider a true Dirac monopole with charge M', and N charged particles at points z_i, with dressed vortices attached to the particles. Including the statistical gauge field $f = da$ the total magnetic flux through the sphere is

$$\int_{S^2} (F + f) = M'.$$

Particles are influenced by a uniform background field $M = M' + N\mathrm{v}$, together with N vortices each with magnetic charge v. For example, the configuration with $\mathrm{v} = -s = -2$, $N = M' = 3$ is shown in figure 9

We now invoke the Atiyah-Singer index theorem to deduce that, in general,

$$n_+ - n_- = M' = M - 2kN,$$

where $\mathrm{v} = -2k$. Again, $n_- = 0$ if $M > 0$, and the ground state is [10]

$$\Psi(z_1, \dots, z_N) = \prod_{i=1}^{N} \frac{1}{(1 + \bar{z}_i z_i)^{\frac{M-1}{2}}} \prod_{i<j} |z_i - z_j|^{2k}(z_i - z_j),$$

which is unique if $N = M'$.

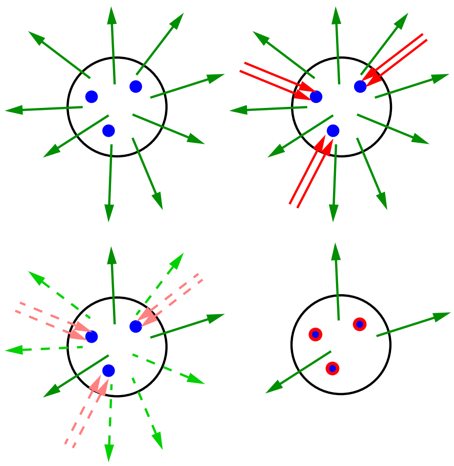


Figure 9: dressed vortices with $s = 2$, $N = M' = 3$, $M = 9$.

Since $M = M' + 2kN = N(2k + 1)$ we have

$$\nu = \frac{1}{2k+1}.$$

The ground state is unique and there is a mass gap, making this composite fermion ground state stable under any perturbation, not just the $J = 0$ perturbations that Haldane's analysis requires.

Wave functions with this analytic form, albeit with an exponential factor for planar geometry rather than $\frac{1}{1+\bar{z}_i z_i}$ factors for spherical geometry, were introduced in [12]. There spherical analogues were analysed numerically in [13] for $N = 10$ and were found to give very good overlap with the true ground state of the full Hamiltonian.

4. Conclusions

It has been shown that including the spin connection for charged fermions confined to the surface of a sphere with a magnetic monopole at the centre tidies up Haldane's analysis in a number of ways:

- The Atiyah-Singer index theorem allows the degeneracy of the ground state to be determined by topological methods alone.
- In contrast to the work of Haldane, when vortices are introduced in order to use Jain's composite fermion model to describe the fractional quantum Hall effect, the ground state is the unique $J = 0$ state, there is no $J \neq 0$ contamination and there is no need to use angular momentum arguments to argue that perturbations will lift any $J \neq 0$ contribution to the ground state.
- The shift that is necessary for a finite number of scalar particles on a sphere is not necessary for fermions, it is accounted for by the spin connection associated with the curved geometry.

The vortices used in the above description of Jain's construction are taken to be δ-function singularities for simplicity, but these could be smeared out to 2-dimensional Skyrmions without affecting the index theorem.

A non-abelian higher dimensional quantum Hall effect was proposed in [14]. The basic idea is that an $SU(2)$ instanton over a 4-sphere has a similar fibre bundle structure to a magnetic monopole inside a sphere S^2:

$$U(1): \quad \begin{array}{ccc} S^1 & \rightarrow & E \\ & & \downarrow \\ & & S^2 \end{array} \qquad \Bigg| \qquad SU(2): \quad \begin{array}{ccc} S^3 & \rightarrow & E \\ & & \downarrow \\ & & S^4 \end{array}$$

and Zhang constructed a ground state wave function for fermions on S^4 interacting with an $SU(2)$ instanton (the abelian QHE in 4-dimensions was developed in [15] using the complex projective space $\boldsymbol{CP}^2$, analogous to $S^2 \approx \boldsymbol{CP}^1$), It would be interesting to investigate whether one could get a non-abelian fractional higher dimensional quantum Hall effect by introducing a second gauge field with point-like (δ-function) $SU(2)$ Skyrmions on S^4.

Effectively 4-dimensional systems, using bosons in an optical lattice to mimic to create two 2-dimensional systems coupled by a weak magnetic field to mimic a 4-dimensional torus, were proposed in [16] and observed in [17].

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PoS(CORFU2023)263