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# A less commutative version of quarkonium masses

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Quarkonium bound states are especially promising candidates to test the probable quantum structure of space-time since they represent a system with reasonably small characteristic distance. The quantum mechanical interaction between the quarks is heuristically described by the Cornell potential. In this contribution, we insert this system in a 3-dimensional rotationally invariant space which is composed of concentric fuzzy spheres of increasing radius called the fuzzy onion in order to extract some consequences of the non-trivial structure on its properties.

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## 1. Introduction

It is widely accepted that the successful unification of quantum field theory and general relativity will introduce to the space-time manifold some kind of discrete, quantum structure [1, 2]. Straightforwardly, this leads to the emergence of non-commutative (NC) spaces [3]. One of the simplest example is the two-dimensional NC plane, as known as the phase space in quantum mechanics (QM) for a particle in one dimension. The other examples are the fuzzy spaces – NC spaces with finite dimension – the most prominent example being the fuzzy sphere [4, 5].

Opposing to the usual manifolds, coordinates in such spaces are described by non-commuting operators. Formulation of quantum mechanics on NC and fuzzy spaces is quite straightforward, we can define the NC versions of all the relevant operators and then look for the solutions of the corresponding Schrödinger equation [6, 7].

The general construction of non-commutative spaces usually follows the procedure of quantization of Poisson manifolds [8]. This works, however, only for even dimensional manifolds, and much more work is required to describe a space with three dimensions. We will follow the construction of a three dimensional space of [9, 10]. Note that although the construction of the non-commutative space is the same in both of the cases, they differ in the mathematical description; we will follow the mathematical formulation of [9]. Here, let us give the common intuitive picture.

The space is constructed as an infinite set of fuzzy spheres with growing radius, which form a layered structure one could call a fuzzy onion. This leads to a different discreteness in the radial coordinate than in the angular coordinates, since the latter is smooth and still possesses the rotational symmetry while the former is rigid. However, the distance of the fuzzy-sphere-layers is the same as the characteristic distance of the NC structure of each layer, and thus the discreteness of the space is compatible throughout the whole 3D space.

This contribution is the curtailed version of the original paper [11] and we will mainly focus on the non-commutative aspects of the physics of quarkonia. We will concentrate on non-relativistic quantum mechanical systems in such a space [12–17], with a plan of extracting at least some consequences of the non-trivial structure on their properties. Our choice, the quarkonium bound states [18] are especially promising candidates to test the quantum structure of space-time, since they represent a system with reasonably small characteristic distance. The two quarks orbit quite close to each other and thus probe any new features of the space-time much better than particles in different, larger systems – e.g. the electrons in atoms [9].

This contribution is divided into two major parts. Both of the parts start with the preliminaries and the basic notions followed by the real research.
In section 2, the classical space is tackled: we introduce the Cornell potential, describe the non-relativistic radial WKB approximation and apply it for the quarkonia.
In section 3, the non-commutative space is tackled: we formulate the mathematical description of the NC QM, show the exact solution for the NC hydrogen atom and determine the non-commutative modifications to the quarkonium mass spectra.



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## 2. Quarkonia in the standard QM

### 2.1 Cornell potential and quarkonium masses

In this work, we will deal with the masses of bound states of two heavy quarks. The analytical treatment is rather phenomenological and the standard approach goes as follows. Describe the two quarks of masses $m_1$ and $m_2$ as point masses with the Cornell potential

$$V_Q(r) = -\frac{C}{r} + B\,r \ . \tag{1}$$

The linear part is responsible for the quark confinement and the $r^{-1}$ part describes the electrostatic-like interaction between charged quarks. Both $B$ and $C$ could be in principle determined from the first principles and the underlying fundamental theory, but we will treat them as free parameters to be fixed by experimental data. We will, however, keep the parameters $B$ and $C$ different for different composition of quarkonium states.

For large masses $m_1$ and $m_2$, the system will be well described as a non-relativistic two body problem in QM. The binding energies – which will be a function of two quantum numbers $n$ and $l$ – are thus given by the Schrödinger equation for a particle of mass $\mu = m_1 m_2/(m_1 + m_2)$ in the potential $V_Q(r)$. The observed mass of the bound state is then[1]

$$M_{nl} = m_1 + m_2 + E_{nl} \ . \tag{2}$$

A significant part of our work will thus be devoted to the determination of the binding energies $E_{nl}$. On one hand, we will look for corrections due to the NC structure of the space, but we will also present a modified way to treat the QM problem with the Cornell potential in the classical space, leading to some new results even with no non-commutativity.

### 2.2 Radial WKB approximation

The WKB method is a technique for obtaining approximate solution to the time-independent Schrödinger equation, see [19, 20].

In the '1D' WKB approximation, the wave function $\psi(x)$ is defined for $x \in \mathbb{R}$, whereas for rotationally invariant potentials the radial wave function $R(r)$ is defined for $r \in \mathbb{R}^+$; hence the 'standard' WKB method directly cannot be used. To avoid this problem, region $0 < r < \infty$ can be mapped to $-\infty < x < \infty$ by the bijective transformation[2] $r = e^x$. The radial wave function is being searched for in the form

$$R\left(r = e^x\right) \longrightarrow R(x) = U(x)e^{f(x)} \ , \tag{3}$$

where the function $f(x)$ is chosen such that the first derivative of $U(x)$ disappears from the radial Schrödinger equation

$$0 = R''(r) + \frac{2}{r}R'(r) + \frac{2\mu}{\hbar^2}\left(E - V(r) - \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}\right)R(r) \ . \tag{4}$$

[1] We will be mostly working with physical quantities in natural units. The usage of the SI units will be emphasized or should be clear from the context.

[2] Since $x$ is an auxiliary variable, we will not pay attention to its physical dimension.

In the present case, $f(x) = -x/2$, thus we get

$$0 = U''(x) + \frac{e^{2x}}{\hbar^2} \underbrace{2\mu \left( E - V(e^x) - \frac{(l+\frac{1}{2})^2\hbar^2}{2\mu e^{2x}} \right)}_{p^2(e^x)} U(x) \; . \tag{5}$$

Hence, the particle's semi-classical momentum $p(r)$ can be picked out:

$$p(r) = \sqrt{2\mu \left( E - V(r) - \frac{(l+\frac{1}{2})^2\hbar^2}{2\mu r^2} \right)} \; , \tag{6}$$

to be substituted to the WKB condition

$$\frac{1}{\hbar} \int_{r_1}^{r_2} \mathrm{d}r \, p(r) = \left( n + \frac{1}{2} \right) \pi \; , \; n \in \mathbb{Z}_0^+ \; , \tag{7}$$

where the limits of the integration $\{r_1, r_2 : 0 < r_1 < r_2\}$ can be found utilizing the relation $p(r_1) = p(r_2) = 0$. Equation (7) is a condition on energy levels $E$.

### 2.3 Mass spectrum of the quarkonia in the classical space

This section is based on [11]. By the application of the WKB condition (7) on the Cornell potential (1), we get the equation

$$\int_{\zeta_1}^{\zeta_2} \mathrm{d}\zeta \sqrt{\epsilon + \frac{c}{\zeta} - b\zeta - g\zeta^2 - \frac{(l+\frac{1}{2})^2}{\zeta^2}} = \left( n + \frac{1}{2} \right) \pi \; , \; 0 < \zeta_1 < \zeta < \zeta_2 \; , \tag{8}$$

where $\zeta = r/r_Q$ is the dimensionless radial coordinate, $r_Q = \sqrt{C/B}$ is the typical distance of the Cornell potential, $\epsilon = 2\mu E r_Q^2/\hbar^2$ is the dimensionless energy, $c = 2\mu C r_Q/\hbar^2$ is the dimensionless Coulombic part[3] and $b = 2\mu B r_Q^3/\hbar^2$ is the dimensionless linear confinement term. For the charmonium and the bottomonium systems $\mu = m_q m_{\bar{q}}/(m_q + m_{\bar{q}}) = m_q/2$ where $m_q = m_{\bar{q}}$ is the mass of the considered (anti)quark.

To solve the integral (8), we use the Pekeris-type approximation, see [23–25]. The original idea is based on the expansion of the terms under the square root around the dimensionful quantity $1/r_Q$ in $(1/r)$-space, see [22]. In paper [21], the expansion point was left as a free parameter, but here, we chose it to be the characteristic distance of the Cornell potential $r_Q$. This is the subtle but important point where our approach differs from previous works.

After the calculations [11], we get the mass of the quarkonia in the commutative classical space:

$$M_{nl} = (m_1 + m_2) - \frac{2\mu}{\hbar^2} \left[ \frac{2C}{n + \frac{1}{2} + \sqrt{\frac{2\mu}{\hbar^2} C \sqrt{\frac{C}{B}} + \left( l + \frac{1}{2} \right)^2}} \right]^2 + 3\sqrt{BC} \; . \tag{9}$$

[3] In those few cases where the SI units are adopted, the speed of the light is denoted $c$. In those cases where the dimensionless Coulombic part $c$ is used, we will not adopt the SI units. Thus, the two different interpretations of the denotation $c$ will be perceptible.



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The free parameters $B$ and $C$ are fixed with the help of the actual experimental data [26, 27]. We substitute the experimental data for 1S and 2S states to equation (9), hence we obtain the free parameters by solving the two equations numerically for the given system. From this moment on, we consider them as constants – even though different for different quark composition of the states – the mass of other excited states will be obtained and the results will be compared with the available experimental data.

### Charmonium mass spectrum

The results for the charmonium states are given by the following table.

| $c\bar{c}$ meson | $m_q = 1.27$ GeV | $B = 0.322$ GeV$^2$ | $C = 0.891$ |
|---|---|---|---|
| state | particle | present work $M_{nl}$ [GeV] | experimental data $M_{nl}$ [GeV] |
| 1S | $J/\psi(1S)$ | used for $B$, $C$ | 3.097 |
| 2S | $\psi(2S)$ | used for $B$, $C$ | 3.686 |
| 3S | $\psi(4040)$ | 3.889 | 4.039 |
| 4S | $\psi(4230)$ | 3.982 | 4.223 |
| 1P | $\chi_{C1}(1P)$ | 3.518 | 3.511 |
| 2P | $\chi_{C2}(3930)$ | 3.823 | 3.923 |
| 1D | $\psi(3770)$ | 3.787 | 3.774 |

From the known states for the given values of $n$ and $l$ that differ by other quantum numbers, we use the same states as the authors of [21]. This is mainly motivated by the experimental accessibility of these states in particle colliders.
Let us point out that our results for the masses are in good agreement with the experimental data and stand their ground among other, arguably more sophisticated models [28–32]. It is interesting that such a simple model and approach lead to these rather precise numbers. The data for the bottomonium and the bottom-charmed meson states were collected to analogous tables, we would like to refer the interested readers to [11].

### Typical distance of the Cornell potential

Up to now we have been working with natural units. Finding the correspondent value of the obtained parameters $C$ and $B$ in SI units, we acquire the typical distance $r_Q$ for all the three probed mesons too.

| quarkonium | $\mu$ [GeV c$^{-2}$] | $B$ [GeV fm$^{-1}$] | $C$ [GeV fm] | $r_Q$ [$10^{-16}$ m] |
|---|---|---|---|---|
| $c\bar{c}$ | 0.64 | 1.633 | 0.175 | 3.28 |
| $b\bar{b}$ | 2.09 | 6.425 | 0.068 | 1.03 |
| $c\bar{b}$ | 0.97 | 3.067 | 0.119 | 1.97 |

We have obtained $r_Q$ at the order of $10^{-16}$ m, which is known to be roughly the size of the quark bound states [33].

## 3. Quarkonia in the NC QM

### 3.1 Construction of the NC space

In this section, we will review the construction of a 3D noncommutative space based on [9]. In standard QM, physical quantities are represented by hermitian operators acting on the states of the Hilbert space $\mathcal{H}$ – on the wave functions. In QM with NC coordinates, this idea is still preserved, but the NC wave functions are composed of operators acting in the auxiliary Fock space. The NC space $\mathbb{R}^3_\lambda$ can be described by a model of concentric fuzzy spheres with increasing radius. The commutator of coordinates is defined as

$$\left[x_i, x_j\right] = 2\mathrm{i}\lambda\varepsilon_{ijk}x_k \ , \tag{10}$$

where the parameter $\lambda$ describes the fuzziness of the space structure[4]. The Fock space is accompanied by two sets of creation $a^\dagger$ and the annihilation $a$ operators; their commutation relations are

$$\left[a_\alpha, a^\dagger_\beta\right] = \delta_{\alpha\beta} \ , \ \left[a_\alpha, a_\beta\right] = \left[a^\dagger_\alpha, a^\dagger_\beta\right] = 0 \ , \ \alpha, \beta \in \{1, 2\} \ . \tag{11}$$

The appealing fact is that all the operators of all the relevant physical quantities can be constructed with this choice of the operators $a^\dagger$ and $a$. The definition of position operator[5]

$$x_j = \lambda\sigma^j_{\alpha\beta}a^\dagger_\alpha a_\beta \ , \ j \in \{1, 2, 3\} \tag{12}$$

obeys commutator (10), $\sigma^j$ is the corresponding Pauli matrix. The magnitude of the position vector is defined as

$$r = \lambda\left(a^\dagger_\alpha a_\alpha + 1\right) \ ; \tag{13}$$

for further calculations we take the advantage of the operator

$$\tilde{r} = \lambda\left(a^\dagger_\alpha a_\alpha\right) \ . \tag{14}$$

The wave functions $\Psi(x_j)$ can be expressed instead of $x_j$ in terms of $a^\dagger$ and $a$ in the way:

$$\Psi = \sum C_{m_1 m_2 n_1 n_2}(a^\dagger_1)^{m_1}(a^\dagger_2)^{m_2}(a_1)^{n_1}(a_2)^{n_2} \ . \tag{15}$$

The number of $a^\dagger$'s and $a$'s are equal in the definition of $x_j$, thus they must be equal in the expression for $\Psi$, too, so $m_1 + m_2 = n_1 + n_2$, where $m_1, m_2, n_1, n_2 \in \mathbb{Z}^+_0$. Moreover, for the arrangement of operators $a^\dagger$ and $a$, the normal ordering is imposed by hand, i.e. all the creation operators $a^\dagger$ are to the left of any annihilation operators $a$.

The NC analogy of the method of separation the variables in standard QM leads to the separated form

$$\Psi_{lm} = \lambda^l \sum_{lm} \frac{(a^\dagger_1)^{m_1}(a^\dagger_2)^{m_2}}{m_1!m_2!} : \mathcal{K}_l(\tilde{r}) : \frac{a_1^{n_1}(-a_2)^{n_2}}{n_1!n_2!} \ , \tag{16}$$

[4]Note that we would have a similar looking commutation relation for a single fuzzy sphere. The difference is that in that case, operator (13) would be restricted to have a fixed value, restricting us onto a subset of the Fock space. Here, we allow any value of $r$, leading to spheres of all radii.

[5]The operators acting on the states of the Fock space are marked without hat.

where $\mathcal{K}_l(\tilde{r})$ is the NC analogy of the radial wave function[6] $K_l(r)$, constructed in the same way as (15), and between the colon marks normal ordering is needed. The summation goes over all quantum numbers $l = m_1 + m_2 = n_1 + n_2$ and $m = 1/2\ (m_1 - m_2 - n_1 + n_2)$.
The action of the operator $\hat{r}$ from (13) on the wave function[7] in (16) can be calculated:

$$\hat{r}\Psi_{lm} = \lambda^l \sum_{lm} \frac{(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}}{m_1!m_2!} :\ [(\tilde{r} + \lambda l + \lambda)\mathcal{K} + \lambda\tilde{r}\mathcal{K}'] : \frac{a_1^{n_1}\,(-a_2)^{n_2}}{n_1!n_2!}\ . \tag{17}$$

The definition of the Laplace operator in $\mathcal{H}_\lambda$ is[8]

$$\hat{\Delta}_\lambda\Psi = -\frac{1}{\lambda r}\left[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \Psi]\right]\ ; \tag{18}$$

the action of the operator $\left[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha,\ \cdot\ ]\right]$ on the wave function in (16) is

$$\left[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \Psi_{lm}]\right] = \lambda^l \sum_{lm} \frac{(a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}}{m_1!m_2!} :\ [-\lambda\tilde{r}\mathcal{K}'' - 2(l+1)\lambda\mathcal{K}'] : \frac{a_1^{n_1}\,(-a_2)^{n_2}}{n_1!n_2!}\ , \tag{19}$$

where $\mathcal{K} \equiv \mathcal{K}_l(\tilde{r})$ and similarly for derivatives.
A different construction of a 3D non-commutative space has been presented in [34]. While it preserves the symmetries of the space, the Hamiltonian is constructed in a way that breaks the azimuthal symmetry of the problem, leading to results different from the ones presented here. In this framework, the hydrogen atom [35] has been considered, and recently also the spectrum of the $b\bar{b}$ [36] and $c\bar{c}$ [37] states. We will comment on these results where appropriate.
Putting together our knowledge of the NC potential and the NC Laplace operator, we acquire the Hamiltonian operator $\hat{H}_\lambda$ in the form

$$\hat{H}_\lambda\Psi = \left[-\frac{\hbar^2}{2\mu}\hat{\Delta}_\lambda + V(\hat{r})\right]\Psi\ . \tag{20}$$

This leads to the eigenvalue problem for the Hamiltonan of the form

$$\frac{\hbar^2}{2\mu\lambda}\left[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \Psi]\right] + \hat{r}V(\hat{r})\Psi = E\hat{r}\Psi\ . \tag{21}$$

Note that due to the presence of $\lambda$ in operator $\hat{r}$ on the RHS of the above equation and due to (17) the energy and the non-commutative length scale mix, and the effect of non-commutativity cannot be easily written as a small perturbation Hamiltonian. This means that the standard perturbation theory of quantum mechanics is not going to be applicable, and we will have to use other methods.

### 3.2 NC hydrogen atom

This section is based on [9]. In light of (20), the NC analog of the Schrödinger equation with the Coulomb potential $V_C(r) = -e^2/(4\pi\varepsilon_0 r)$ in $\mathbb{R}^3_\lambda$ is

$$\frac{\hbar^2}{2m_e\lambda r}\left[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \Psi]\right] - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}\Psi = E\Psi\ . \tag{22}$$

[6]The relation between the radial wave function $R_l(r)$ and $K_l(r)$ is of the form $R_l(r) = K_l(r)r^l$ where the term $r^l$ describes the behaviour of $R_l(r \to 0)$.
[7]The operators acting on the states of the Hilbert space $\mathcal{H}_\lambda$ are marked with hat.
[8]Expression $\frac{1}{r}$ is understood as the inverse operator to the operator $\hat{r}$.



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Multiplying the previous equation with $\hat{r}$ and inserting (19) and (17), we get

$$: \tilde{\rho}\mathcal{K}'' + [-\epsilon\sigma\tilde{\rho} + 2(l+1)]\mathcal{K}' + [-\epsilon\tilde{\rho} - \epsilon\sigma(l+1) + 2]\mathcal{K}: \; = 0 \,, \tag{23}$$

where $\epsilon = E/\left(-\frac{1}{2}m_e c^2\alpha^2\right)$ is the dimensionless energy, $\tilde{\rho} = \tilde{r}/a_B$ is the dimensionless radial coordinate, $\sigma = \lambda/a_B$ is the dimensionless non-commutative parameter, $a_B = \hbar/(m_e c\alpha)$ is the Bohr radius, $\alpha = e^2/(4\pi\varepsilon_0\hbar c)$ is the fine-structure constant and $m_e$ is the mass of the electron. Equation (23) is the analog of the radial Schrödinger equation in NCQM expressed with $\mathcal{K}(\tilde{\rho})$ instead of $\mathcal{R}(\tilde{\rho})$. It can be interpreted in the way that the configuration of the operators on the left side results in a zero operator. In the case we remove the colon marks, we get the differential equation

$$\rho K'' + [-\epsilon\sigma\rho + 2(l+1)]K' + [-\epsilon\rho - \epsilon\sigma(l+1) + 2]K = 0 \,, \tag{24}$$

for an ordinary function $K$ of an ordinary variable $\rho$. The claim is that one obtains solution to (23) by replacing $\rho$ in the solution to (24) with operator $\tilde{\rho}$, see [12].

The solution of (24) for the radial wave function $K(\rho)$ results in hypergeometric function, and the exact energy spectrum of the bound states with SI units is

$$E_N^\lambda = \frac{\hbar^2}{m_e\lambda^2}\left(1 - \sqrt{1 + \frac{m_e^2c^2\alpha^2}{\hbar^2}\frac{\lambda^2}{N^2}}\right) . \tag{25}$$

One can see that the energy levels calculated in standard QM increase due to the effect of the NC space structure. Taking the limit $\lambda \to 0$, we recover the energy spectrum of the Coulomb potential from the standard QM.

### 3.3 Mass spectrum of the quarkonia in the non-commutative space

This section is based on [11]. In light of (20), the NC analog of the Schrödinger equation with the Cornell potential $V_Q(\hat{r})$ from (1) in $\mathbb{R}^3_\lambda$ is

$$\frac{\hbar^2}{2\mu\lambda r}\left[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \Psi]\right] + \left(-\frac{C}{r} + B\hat{r}\right)\Psi = E\Psi \,. \tag{26}$$

This equation yields the radial Schrödinger equation in the NC configuration space

$$\begin{aligned} R'' + \frac{2}{\zeta}R' - \frac{l(l+1)}{\zeta^2}R + \left(\frac{c}{\zeta} - b\zeta\right)R + \epsilon R + \sigma\left(\epsilon R' + \frac{\epsilon}{\zeta}R - 2b\zeta R' - 3bR\right) + \\ + \sigma^2\Big(-b\zeta R'' - 3bR' - \frac{b}{\zeta}R\Big) = 0 \,, \end{aligned} \tag{27}$$

where $\sigma = \lambda/r_Q$ is the dimensionless parameter. In the case we take the limit $b \to 0$ in equation (27), we recover the radial Schrödinger equation for the NC Coulomb potential (24). In contrast to (24), this Schrödinger equation (27) cannot be solved analytically. In real, it could not be solved analytically even in the standard QM. This was the reason of utilization of the WKB approximation, and we will use this method again to tackle (27).

Nevertheless, we encounter some technical issues regarding the calculation of the integral in the WKB condition. The parameter $\sigma$ is deemed to be small, since $r_Q$ is expected to be at the order of the size of the quarkonium states[9], which is roughly $10^{-16}$ m [33], and $\lambda$ is expected to be at the Planck scale, i.e. $10^{-35}$ m, so their ratio gives $\sigma \approx 10^{-19}$. This means we do not cause a great deal of inaccuracy if all the terms multiplied by the 3rd or higher power of $\sigma$ become neglected.

Considering all the terms up to the 2nd order of $\sigma$, the WKB condition is attained

$$\left(n+\frac{1}{2}\right)\pi = \int_{\zeta_1}^{\zeta_2} \mathrm{d}\zeta \sqrt{\begin{array}{l}\left[\epsilon + \sigma^2\left(cb - \frac{\epsilon^2}{4}\right)\right] + \left[c - \sigma^2 bl(l+1)\right]\frac{1}{\zeta} - \\ -b\left(1 - 2\sigma^2\epsilon\right)\zeta - 2\sigma^2 b^2\zeta^2 - \frac{\left(l+\frac{1}{2}\right)^2}{\zeta^2}\end{array}} + O(\sigma^2)\,. \tag{28}$$

This is the non-commutative extension of the WKB condition (8), although, in this case, the coefficients standing before the different powers of $\zeta$ contain the effect of the blur of the space. Using again the Pekeris-type approximation, the calculation gives us the mass spectrum of the quarkonia comprehending the non-commutativty of the space

$$\begin{aligned} M_{nl}^{\sigma} &= M_{nl} + \sigma^2 M_{nl}^{(2)} + O(\sigma^2) = \\ &= \left( (m_1+m_2) - \frac{2\mu}{\hbar^2}\left[\frac{2C}{n+\frac{1}{2}+\sqrt{\frac{2\mu}{\hbar^2}C\sqrt{\frac{C}{B}}+\left(l+\frac{1}{2}\right)^2}}\right]^2 + 3\sqrt{BC}\right) + \\ &\quad + \sigma^2\frac{\hbar^2}{2\mu}\frac{B}{C}\left(\frac{b\left(105b^2+62bc+9c^2\right)+4b(c+3b)l(l+1)}{8\left[n+\frac{1}{2}+\sqrt{b+(l+\frac{1}{2})^2}\right]^2} - \frac{b}{4}(15b+4c) + \right. \\ &\quad \left. + \frac{b(c+3b)^4}{8\sqrt{b+(l+\frac{1}{2})^2}\left[n+\frac{1}{2}+\sqrt{b+(l+\frac{1}{2})^2}\right]^5} - \frac{(45b-c)(c+3b)^3}{64\left[n+\frac{1}{2}+\sqrt{b+(l+\frac{1}{2})^2}\right]^4}\right) + O(\sigma^2)\,. \end{aligned} \tag{29}$$

We calculate this modification of the mass spectrum $\sigma^2 M_{nl}^{(2)}$ for all the examined states of all the three quarkonia. Since the potential effect of the non-commutativity of the space is on the 39th decimal place in the mass spectrum of the mesons, these modifications are beyond the limit of accuracy of any current measurement. For this reason we used the non-perturbed mass spectra to find the parameters $B$ and $C$, and with the help of these parameters give an estimation for the leading correction in the non-commutative case.

**Charmonium mass spectrum modifications**

The setup of [36, 37] allowed for a direct use of the standard perturbation theory in quantum mechanics, without any need of WKB or Pekeris-type approximations. Both of these works obtained leading correction to the mass spectrum at the second order in non-commutative parameter, as did we, but with a different $n$ and $l$ dependence. Interestingly, for some values of the quantum numbers

[9]This is an assumption to be checked *a posteriori*.

the correction has diverged, whereas our results are under control for all values. Let us remind the reader that the setup of that work uses a different version of the NC Hamiltonian than (20).
We obtain corrections for the charmonium meson given by the following table.

| NC $c\bar{c}$ meson | $b = c = 1.883\,,\ \ \sigma^2 \approx 0.93\times 10^{-39}$ |
|---|---|
| state | correction to the mass spectrum $\sigma^2 M_{nl}^{(2)}$ [GeV] |
| 1S | $0.522\,\sigma^2$ |
| 2S | $-1.422\,\sigma^2$ |
| 3S | $-2.613\,\sigma^2$ |
| 4S | $-3.301\,\sigma^2$ |
| 1P | $-0.456\,\sigma^2$ |
| 2P | $-1.936\,\sigma^2$ |
| 1D | $-1.062\,\sigma^2$ |

The non-commutative modifications for the bottomonium and the bottom-charmed meson states were collected to analogous tables, we would like to refer the interested readers to [11].

**Maximum possible value of the of the non-commutative parameter $\lambda$**

Although, we have claimed that $\lambda$ is expected to be at the Planck scale, i.e. $10^{-35}$ m, it is just a hypothesis which cannot be obtained from experimental data, since no effect of the potential discrete structure of the space-time has been observed so far. We therefore take the uncertainty of the measurement into account: the uncertainty of the mass $M_{nl}$ must be greater than any effect of the non-commutativity. In other words, the non-commutative mass spectrum correction $\sigma^2 M_{nl}^{(2)}$ must be concealed by the experimental uncertainty of the mass $M_{nl}$. From this inequality we acquire the maximum possible value of $\lambda$.

Among all the mentioned particles in this paper the mass of the $J/\psi(1S)$ particle – a $c\bar{c}$ meson – is measured the most precisely: $M_{00} = (3096.900 \pm 0.006)$ MeV, see [26]. So, we use its uncertainty to determine the upper value of $\lambda$. With the help of the correction to the mass spectrum $\sigma^2 M_{00}^{(2)}$ for the $c\bar{c}$ meson we get the following inequality

$$0.522\left(\lambda^2 \frac{B}{C}\right)\,\mathrm{GeV} \le 0.006\,\mathrm{MeV} \tag{30}$$

whence we get

$$\lambda \le 1.11\times 10^{-18}\,\mathrm{m}\,. \tag{31}$$

Both [36] and [37] have considered the hyper-fine splitting of the quarkonium states to determine the upper bound on the characteristic distance of the non-commutativity, obtaining a value greater by two orders of magnitude than (31). Since the experimental error in the measurement of $J/\psi(1S)$ is much smaller than the splitting, we obtain a stricter bound on $\lambda$. However, even the bound we obtain is still a very mild one.

## 4. Conclusions

We have presented results regarding the consequence of non-commutativity of space on the masses of bound states of heavy quarks. Working under the non-relativistic assumption and using the WKB and slightly modified Pekeris-type approximations, we have described the derivation the masses of the considered mesons (as the commutative $0^{\text{th}}$ order term) and the first non-trivial correction due to the non-commutative structure. Our model has two free parameters, which have been fixed using the experimentally observed masses and gave the rest of the masses in a good agreement with the experiment. The leading NC correction is proportional to $(\lambda/r_Q)^2$, where $\lambda$ is the characteristic distance of the space non-commutativity and $r_Q$ is the characteristic distance of the quarkonium state.

For $\lambda$ at the order of Planck length, where the quantum structure of space-time is expected to play a significant role from the quantum gravity considerations, we obtained relative correction of the mass at the order of $10^{-39}$. Considering the most precisely measured mass, we have obtained the upper bound on $\lambda$ at the order of $10^{-18}$ m.

Our original hope was that the quarkonium states will turn out to be a (relatively) good probe for the quantum structure of spacetime. Since the typical radius of the states $r_Q$ is much smaller than the Bohr radius, we do get a significantly larger shift than in the case of hydrogen atom [12], but we are still far from any reasonably measurable contribution. The main issue is that the relative corrections to the masses turned out to be of the second order in $\lambda/r_Q$. So, in the future, it would be interesting to look further for a system where the correction is of the first order.

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