On Formulating the Non-Geometric Scalar Potentials

George K. Leontaris[a,*] **and Pramod Shukla**[b]

[a]*Physics Department, University of Ioannina*
University Campus, Ioannina 45110, Greece

[b]*Department of Physiccal Sciences, Bose Institute,*
Unified Academic Campus, EN 80, Sector V, Bidhannagar, Kolkata 700091, India

E-mail: leonta@uoi.gr, pshukla@jcbose.ac.in

In the context of four-dimensional type II supergravities, the successive application of various S/T-dualities leads to a generalized notion of fluxes, which includes certain (non-)geometric fluxes along with the standard RR and NS-NS p-form fluxes. These fluxes induce a diverse set of superpotential couplings leading to scalar potentials with a very rich structure, which may possibly result in a vast landscape of physical vacua. However such scalar potentials typically consist of a huge number of terms and in order to make any attempt for phenomenological model building with some analytic understanding/control it is necessary to formulate them in some compact and concise form. Along these lines we review various equivalent methods of deriving the same scalar potential through a set of master formulae, which may open up a new avenue for model building using non-geometric fluxes.

---

*Speaker

1. Introduction

In the context of four-dimensional type IIB supergravities, the study of non-geometric fluxes has attracted a huge amount of interest in last two decades. The existence of such fluxes are supported through the presence of various dualities which, in general, have played a crucial role in connecting different limits of string theories, providing supplemental interpretations and new perspectives. For instance, considering the NS-NS three-form flux H_{mnp} of the type II supergravities, a chain of successive T-dualities leads to various (non-)geometric fluxes [1, 2],

$$H_{mnp} \rightarrow \omega_{np}{}^m \rightarrow Q_p{}^{mn} \rightarrow R^{mnp}, \tag{1}$$

where $\omega_{np}{}^m$ denotes the geometric fluxes (also known as metric flux) while $Q_p{}^{mn}$ and R^{mnp} denote the set of non-geometric fluxes. For type IIB orientifold models one can choose the holomorphic involutions such that the geometric flux $\omega_{np}{}^m$ and the non-geometric flux R^{mnp} are absent [3–6], and one is left with a pair of S-dual flux RR and NS-NS three-form fluxes (F, H), and the non-geometric Q-flux. However after adding these non-geometric fluxes, the underlying modular invariance in broken, and it can only be restored through the inclusion of a new kind of non-geometric flux, the so-called P-flux, which is S-dual to the non-geometric Q-flux [5, 7–12]. Subsequently if one demands a successive application of such S/T-dualities, one needs two additional S-dual pairs of fluxes, denoted as (P', Q') and (H', F') [5, 10, 13–15]. The presence of such four S-dual pairs of 'generalized' fluxes, namely (F, H), (Q, P), (P', Q') and (H', F'), induces a wide range of superpotential couplings which can subsequently result in a very rich structure for the four-dimensional effective scalar potential. However, how many and which type of fluxes can be simultaneously turned-on in a given concrete construction remains an open challenge for the most generic models. Some initiatives have been taken in this direction leading to a set of flux constraints arising in the form of Bianchi identities and tadpole cancellation conditions [5, 7, 10, 14–17].

The study of models based on non-geometric flux compactifications has been motivated along (but not limited to) the following points:

- One of the main attractive features of non-geometric flux models is the fact that one can in principle stabilize all types of moduli [18–30] without invoking non-perturbative effects in the superpotential, or utilizing any corrections of the Kähler potential. In this regard, it should be emphasized that this approach of stabilization includes also the Kähler moduli which, in conventional flux compactifications, are protected by the so-called "no-scale structure". From this point of view, the subsequent consideration of non-geometric fluxes which can generically induce the superpotential couplings for the Kähler moduli as well, has emerged as an important ingredient in the area of moduli stabilization and generic model building [18–34].

- However, as we have earlier argued, although the inclusion of the new (non-geometric) fluxes may greatly facilitate the presence of new flux vacua, it is also true that it introduces numerous inevitable complexities due to the huge number of flux-induced terms in the scalar potential. For example, in the context of type IIB on $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orientifold- it has been found that the resulting four-dimensional scalar potential is very often so huge that it gets hard to analytically solve the extremization conditions. Thus, one has to look either for a simplified Ansatz by

switching off certain flux components at a time, or else one has to opt for a rather involved numerical analysis; for phenomenological model building attempts with (non-)geometric fluxes see [7, 8, 10, 21, 23–25, 34–38, 38–44].

- Apart from the phenomenological applications, understanding the higher dimensional origin of the four-dimensional scalar potential induced from the generalized flux superpotential has helped in exploiting the applications of symplectic geometries [40, 45–51].

- The natural incorporation of non-geometric fluxes in the context of Double Field Theory and its subsequent connection with the non-geometric type II models arrived at after dimensional reduction has opened a completely new avenue to explore the higher dimensional origins of the generalized flux superpotentials [50, 52, 53].

- In addition to the phenomenological model building motivations, the interesting connections among the ingredients of superstring flux-compactifications and those of the gauged super-gravities have significant relevance in understanding both sectors as fluxes in one setting are related to the gauging in the other one [1, 5, 18–20, 54–59].

The main aim of this work is to present a brief review about the scalar potential formulations which is a crucial step towards performing any model building attempts using non-geometric setups. The F-term scalar potential governing the dynamics of the $\mathcal{N} = 1$ low energy effective supergravity can be computed from the Kähler potential (K) and the flux induced holomorphic superpotential (W) by considering the following well known formula,

$$V = e^K \left(K^{A\overline{B}} \left(D_A W \right) \left(\overline{D}_{\overline{B}} \overline{W} \right) - 3|W|^2 \right),\tag{2}$$

where the covariant derivatives for a chiral coordinate A is defined as $D_A W = \partial_A W + (\partial_T K) W$. This general expression (2) has resulted in a series of the so-called "master-formulae" for the scalar potential for a given set of Kähler- and the super-potentials; e.g. see [13, 17, 42, 49, 51, 53, 60–67] Along these lines we plan to briefly recall the iterative steps taken in the literature to understand the insights of the (generalized) flux superpotential via the understanding of the effective scalar potential. In this regard, the initial model building studies have been performed by considering the 4D effective potential derived by merely knowing the Kähler- and the super-potentials [6, 16, 21–24, 26, 52, 59, 68], and without having a complete understanding of their ten-dimensional origin. This issue has been taken into consideration in recent years via two approaches; the first one being through the Double Field Theory (DFT) [50, 69, 70] and the second approach being through the study of the underlying supergravity theories [49, 52, 59, 68, 71–76]. Some of the timelines about exploring the 10D origin of the 4D effective potential can be recalled as below:

- **Step-0:** In the context of standard type IIB flux compactification with the usual NS-NS and RR fluxes, H_3 and F_3, the four-dimensional scalar potential induced by the flux superpotential [77, 78] has been compactly derived through the dimensional reduction of the 10D kinetic pieces [46, 47]. In this case the flux superpotential takes the following schematic form:

$$W_{\text{FH}} = p_1(U^i) + S\, p_2(U^i),\tag{3}$$

where p_1, p_2 are some cubic polynomials in terms of U^i moduli. This leads to the so-called "no-scale-structure" as there is no dependence on the Kähler moduli in V defined in (2), except from a $\mathcal{V}^{-2}$ dependence arising from the overall factor e^K.

- **Step-1:** Motivated by the study of 4D effective scalar potential in a type IIA flux compactification setup with geometric flux [59], a rearrangement of the scalar potential induced through a generalized flux superpotential with non-geometric Q-fluxes on top of having the standard H_3/F_3 within a type IIB non-geometric framework, has been presented in [52]. This "rearranged" scalar potential has a "suitable" form which helps in anticipating the 10D origin of the 4D pieces, a process called as "dimensional oxidation" of the non-geometric flux superpotential [52].

In this case the flux superpotential takes the following schematic form:

$$W_{\mathrm{FHQ}} = p_1(U^i, T_\alpha) + S\, p_2(U^i), \tag{4}$$

where p_1, p_2 are some cubic polynomials in terms of U^i moduli. In addition, the polynomial p_1 has a linear dependence on the Kähler moduli T_α which breaks the no-scale structure and creates the possibilities of stabilizing all moduli without including any (non-)perturbative effects from string-loops or instanton series.

- **Step-2:** In order to restore the S-duality invariance broken by including the non-geometric Q-flux in the type IIB $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$-orientifold setup, the proposal of [52] was further generalized in [68] via the inclusion of the so-called P-flux which is S-dual to the Q-flux. In the meantime the prescription was further extended for the odd axion models within a type IIB compactification on $\mathbb{T}^6/\mathbb{Z}_4$-orientifold in [71, 72].

In this case the flux superpotential takes the following schematic form:

$$W_{\mathrm{FHQP}} = p_1(U^i, T_\alpha) + S\, p_2(U^i, T_\alpha), \tag{5}$$

where p_1, p_2 are some cubic polynomials in terms of U^i moduli whereas they both have only a linear dependence on the Kähler moduli T_α. The scalar potential induced through this superpotential beyond the toroidal case has been studied in [13].

- **Step-3:** One can further extend the S/T dual completion arguments to arrive at a more general flux superpotential of the following form, via including two more S-dual pairs of fluxes, usually denoted as (P', Q') and (H', F') [5, 10, 14, 15, 79]

$$W_{\mathrm{FHQPP'Q'H'F'}} = p_1(U^i, T_\alpha) + S\, p_2(U^i, T_\alpha), \tag{6}$$

where now p_1, p_2 are some cubic polynomials in terms of U^i as well as T_α moduli. Let us note that such a U-dual completed flux superpotential has been explicitly known since quite some time, and for the toroidal type IIB $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orientifold based model it has a total of 128 flux parameters and 7 complexified variable [5]. However its insights (or any phenomenological application in model building) have not been explored in detail because of the huge size of the scalar potential which was recently found to have a total of 76276 terms [17, 51]!

4

In the current work we plan to briefly elaborate on these iterative steps taken to understand the deeper insights of the four-dimensional scalar potentials in the presence of non-geometric fluxes. For this task we basically follow two approaches: (i). To formulate the scalar potential using the metric of the internal toroidal background [52, 68, 71], and (ii). To formulate the scalar potential using the symplectic ingredients [13, 17, 49–51, 62, 67, 80, 81]. These beyond toroidal formulations are generically valid for arbitrary number of complex structure moduli as well as Kähler moduli, and on top of it, do not need the knowledge of internal background metric.

2. Internal metric formulation

Most of the studies related to the 4D type II effective theories in a non-geometric flux compactification framework have been centered around toroidal examples, and in particular using a $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orientifold. An obvious reason is its relatively simpler structure to perform explicit and concrete computations, which have led toroidal setups to serve as promising toolkit in studying concrete examples. In this regard, one useful ingredient is the metric of the toroidal background which is explicitly known unlike the case of generic CY backgrounds, and we will demonstrate the use of internal metric in writing down the scalar potential pieces a la 'dimensional oxidation' approach of [17, 52, 68, 72].

2.1 Type IIB model using a $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orientifold

Let us start by briefly revisiting the relevant features of a concrete setup in the framework of the type IIB orientifold compactification using the well studied $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold, where the two $\mathbb{Z}_2$ actions are defined as,

$$\theta : (z^1, z^2, z^3) \rightarrow (-z^1, -z^2, z^3) \tag{7}$$
$$\overline{\theta} : (z^1, z^2, z^3) \rightarrow (z^1, -z^2, -z^3) \,.$$

Next, the orientifold action is defined via $O \equiv \Omega_p \, I_6 \, (-1)^{F_L}$ where Ω_p is the worldsheet parity, F_L is left-fermion number and I_6 denotes the holomorphic involution defined as,

$$I_6 : (z^1, z^2, z^3) \rightarrow (-z^1, -z^2, -z^3) \,, \tag{8}$$

which subsequently results in a setup of $O3/O7$-type. The complexified coordinates (z^i) on the six-torus $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$ are defined as below,

$$z^1 = x^1 + U^1 x^2, \; z^2 = x^3 + U^2 x^4, \; z^3 = x^5 + U^3 x^6, \tag{9}$$

where the three complex structure moduli U^i's can be written as $U^i = v^i - i \, u^i, \; i = 1, 2, 3$. Now, the holomorphic three-form $\Omega_3 = dz^1 \wedge dz^2 \wedge dz^3$ can be expanded as,

$$\begin{aligned} \Omega_3 &= \alpha_0 + U^1 \, \alpha_1 + U^2 \, \alpha_2 + U^3 \alpha_3 \\ &\quad + U^1 \, U^2 \, U^3 \, \beta^0 - U^2 \, U^3 \, \beta^1 - U^1 \, U^3 \, \beta^2 - U^1 \, U^2 \, \beta^3 \,, \end{aligned} \tag{10}$$

where we have chosen the following basis of the closed three-forms,

$$\alpha_0 = 1 \wedge 3 \wedge 5 \,, \; \alpha_1 = 2 \wedge 3 \wedge 5 \,, \; \alpha_2 = 1 \wedge 4 \wedge 5 \,, \; \alpha_3 = 1 \wedge 3 \wedge 6 \,, \tag{11}$$
$$\beta^0 = 2 \wedge 4 \wedge 6 \,, \; \beta^1 = -1 \wedge 4 \wedge 6 \,, \; \beta^2 = -2 \wedge 3 \wedge 6 \,, \; \beta^3 = -2 \wedge 4 \wedge 5 \,.$$

In the above we use the shorthand notations such as $1 \wedge 3 \wedge 5 = dx^1 \wedge dx^3 \wedge dx^5$ etc. along with the normalization $\int \alpha_\Lambda \wedge \beta^\Delta = -\delta_\Lambda{}^\Delta$. Using these ingredients, the holomorphic three-form can also be expressed in terms of the symplectic period vectors $(\mathcal{X}^\Lambda, \mathcal{F}_\Lambda)$ as $\Omega_3 \equiv \mathcal{X}^\Lambda \alpha_\Lambda - \mathcal{F}_\Lambda \beta^\Lambda$, where the complex structure moduli dependent prepotential $\mathcal{F}$ is given as,

$$\mathcal{F} = \frac{\mathcal{X}^1 \mathcal{X}^2 \mathcal{X}^3}{\mathcal{X}^0} = U^1 U^2 U^3 \tag{12}$$

which results in the following period-vectors,

$$\mathcal{X}^0 = 1, \quad \mathcal{X}^1 = U^1, \quad \mathcal{X}^2 = U^2, \quad \mathcal{X}^3 = U^3,$$
$$\mathcal{F}_0 = -U^1 U^2 U^3, \quad \mathcal{F}_1 = U^2 U^3, \quad \mathcal{F}_2 = U^3 U^1, \quad \mathcal{F}_3 = U^1 U^2. \tag{13}$$

Now, using the same shorthand notations we choose the following bases for the orientifold even two-forms μ_α, and their dual four-forms $\tilde{\mu}^\alpha$,

$$\mu_1 = 1 \wedge 2, \quad \mu_2 = 3 \wedge 4, \quad \mu_3 = 5 \wedge 6;$$
$$\tilde{\mu}^1 = 3 \wedge 4 \wedge 5 \wedge 6, \quad \tilde{\mu}^2 = 1 \wedge 2 \wedge 5 \wedge 6, \quad \tilde{\mu}^3 = 1 \wedge 2 \wedge 3 \wedge 4, \tag{14}$$

Let us mention that for this toroidal orientifold construction there are no two-forms which are anti-invariant under the orientifold projection, i.e. $h_-^{1,1}(X) = 0$, and similarly there dual four-forms are also trivial, and therefore no B_2 and C_2 moduli as well as no geometric-flux components will be present in this model; for the construction of concrete type IIB orientifold models with odd moduli, e.g. see [65, 82–87].

The other chiral variables are the so-called axio-dilaton S and the complexified Kähler moduli which are defined as,

$$S = C_0 + i\, e^{-\phi} \qquad \mathcal{J} = C^{(4)} - \frac{i}{2} J \wedge J \equiv T_\alpha \tilde{\mu}^\alpha, \tag{15}$$

where $J = t^\alpha \mu_\alpha$ is the Kähler form involving the (Einstein-frame) two-cycle volume moduli t^α while the moduli $T_\alpha = \rho_\alpha - i\, \tau_\alpha$ consist of RR axions $C_{ijkl}^{(4)}$ and the four-cycle volume moduli τ_{ijkl} which in terms of six-dimensional components are given as follows,

$$T_1 = C_{3456}^{(4)} - i\, \tau_{3456}, \qquad T_2 = C_{1256}^{(4)} - i\, \tau_{1256}, \qquad T_3 = C_{1234}^{(4)} - i\, \tau_{1234}, \tag{16}$$

where $\tau_1 = t^2 t^3$, $\tau_2 = t^3 t^1$, $\tau_3 = t^1 t^2$ are expressed in the Einstein-frame. The overall volume $(\mathcal{V})$ of the sixfold (in the Einstein-frame) can be given as,

$$\mathcal{V} = t^1 t^2 t^3 = \sqrt{\tau_1 \tau_2 \tau_3}, \qquad \tau_\alpha = \frac{\partial \mathcal{V}}{\partial t^\alpha}, \tag{17}$$

where a useful relation between the two-cycle volumes t^α and the four-cycles volumes τ_α can be given as below,

$$t^1 = \sqrt{\frac{\tau_2 \tau_3}{\tau_1}}, \qquad t^2 = \sqrt{\frac{\tau_1 \tau_3}{\tau_2}}, \qquad t^3 = \sqrt{\frac{\tau_1 \tau_2}{\tau_3}}. \tag{18}$$

Another crucially relevant ingredient in our current study is the information about the internal metric of the toroidal sixfold. It turns out that the internal metric g_{ij} is block-diagonal and has the following non-vanishing components,

$$g_{11} = \frac{t^1}{u^1}, \qquad g_{12} = \frac{t^1 v^1}{u^1} = g_{21}, \qquad g_{22} = \frac{t^1((u^1)^2 + (v^1)^2)}{u^1},$$

$$g_{33} = \frac{t^2}{u^2}, \qquad g_{34} = \frac{t^2 v^2}{u^2} = g_{43}, \qquad g_{44} = \frac{t^2((u^2)^2 + (v^2)^2)}{u^2}, \qquad (19)$$

$$g_{55} = \frac{t^3}{u^3}, \qquad g_{56} = \frac{t^3 v^3}{u^3} = g_{65}, \qquad g_{66} = \frac{t^3((u^3)^2 + (v^3)^2)}{u^3}.$$

These internal metric components can be written out in a more suitable form, to be utilized later, by using the four-cycle volumes τ_i's and the same is given as below,

$$g_{11} = \frac{\sqrt{\tau_2}\,\sqrt{\tau_3}}{u^1\,\sqrt{\tau_1}}, \quad g_{12} = \frac{v^1\,\sqrt{\tau_2}\,\sqrt{\tau_3}}{u^1\,\sqrt{\tau_1}} = g_{21}, \quad g_{22} = \frac{((u^1)^2 + (v^1)^2)\,\sqrt{\tau_2}\,\sqrt{\tau_3}}{u^1\,\sqrt{\tau_1}},$$

$$g_{33} = \frac{\sqrt{\tau_1}\,\sqrt{\tau_3}}{u^2\,\sqrt{\tau_2}}, \quad g_{34} = \frac{v^2\,\sqrt{\tau_1}\,\sqrt{\tau_3}}{u^2\,\sqrt{\tau_2}} = g_{43}, \quad g_{44} = \frac{((u^2)^2 + (v^2)^2)\,\sqrt{\tau_1}\,\sqrt{\tau_3}}{u^2\,\sqrt{\tau_2}}, \qquad (20)$$

$$g_{55} = \frac{\sqrt{\tau_1}\,\sqrt{\tau_2}}{u^3\,\sqrt{\tau_3}}, \quad g_{56} = \frac{v^3\,\sqrt{\tau_1}\,\sqrt{\tau_2}}{u^3\,\sqrt{\tau_3}} = g_{65}, \quad g_{66} = \frac{((u^3)^2 + (v^3)^2)\,\sqrt{\tau_1}\,\sqrt{\tau_2}}{u^3\,\sqrt{\tau_3}}.$$

For the current toroidal setup, the (tree level) Kähler potential takes the following form in terms of the S, T and U moduli,

$$K = -\ln\left(-i(S - \overline{S})\right) - \sum_{j=1}^{3} \ln\left(i(U^j - \overline{U}^j)\right) - \sum_{\alpha=1}^{3} \ln\left(\frac{i\,(T_\alpha - \overline{T}_\alpha)}{2}\right). \qquad (21)$$

2.2 U-dual completion of the flux Superpotential

The four-dimensional effective scalar potential generically has an S-duality invariance following from the underlying ten-dimensional type IIB supergravity. This corresponds to the following $SL(2, \mathbb{Z})$ transformation,

$$S \to \frac{a\,S + b}{c\,S + d} \qquad \text{where} \quad ad - bc = 1\,; \quad a,\ b,\ c,\ d \in \mathbb{Z} \qquad (22)$$

Under this $SL(2, \mathbb{Z})$ transformation, the complex-structure moduli (U^i) and the Einstein-frame internal volume $(\mathcal{V})$ are invariant. Moreover, the Einstein-frame chiral coordinate T_α is S-duality invariant, without orientifold odd axions, i.e. $h^{11}_-(X_6/O) = 0$ [88]. Subsequently, it turns out that the tree level Kähler potential given in eqn. (46) transforms as:

$$e^K \longrightarrow |c\,S + d|^2\,e^K. \qquad (23)$$

This subsequently implies that the S-duality invariance of the physical quantities (such as gravitino mass-square $m^2_{3/2} \propto e^K |W|^2$) suggests that the holomorphic superpotential, W should have a modularity of weight -1, which means the following [88–90]

$$W \to \frac{W}{c\,S + d}. \qquad (24)$$

As we will discuss in the upcoming sections, a generic holomorphic superpotential, respecting the modular weight being -1, can have four S-dual pairs of fluxes denoted as (F, H), (Q, P), (P', Q') and (H', F') [5, 10, 14, 15, 79]. This set of eight fluxes transforms in the following manner under the $SL(2, \mathbb{Z})$ transformations,

$$
\begin{pmatrix} F \\ H \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ H \end{pmatrix}, \qquad \begin{pmatrix} Q \\ P \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix},
$$
$$
\begin{pmatrix} H' \\ F' \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} H' \\ F' \end{pmatrix}, \qquad \begin{pmatrix} P' \\ Q' \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P' \\ Q' \end{pmatrix}. \tag{25}
$$

Under the $SL(2, \mathbb{Z})$ transformations, the various fluxes can readjust themselves to respect the modularity condition (24) in the following two ways [5],

$$
(i). \quad S \rightarrow S + 1, \qquad (ii). \quad S \rightarrow -\frac{1}{S}. \tag{26}
$$

Note that the first case simply corresponds to a shift in the universal axion $C_0 \rightarrow C_0 + 1$ which amounts to having a constant rescaling of the Kähler potential as $e^K \rightarrow |d|^2 e^K$, and the super-potential as $W \rightarrow W/d$. This follows from Eqs. (23)-(24) due to the fact that $S \rightarrow S + 1$ simply corresponds to $c = 0$ case in the $SL(2, \mathbb{Z})$ transformation (22). The second case is quite peculiar in the sense that it corresponds to the following transformation of the universal axions and the dilaton,

$$
C_0 \rightarrow -\frac{C_0}{s^2 + C_0^2}, \qquad s \rightarrow \frac{s}{s^2 + C_0^2}, \tag{27}
$$

which takes $g_s \rightarrow g_s^{-1}$ and hence is known as strong-week duality or S-duality. This relation (27) shows that C_0/s flips sign under S-duality, something which will be useful in understanding the modular completion of the scalar potential later on. From now onwards we will focus only on the second case, i.e. on strong/weak duality. This means that under the $SL(2, \mathbb{Z})$ transformation of the second type which simply takes the axio-dilaton $S \rightarrow -1/S$, the fluxes can be considered to transform as below,

$$
H \rightarrow F, \quad F \rightarrow -H, \quad Q \rightarrow -P, \quad P \rightarrow Q, \tag{28}
$$
$$
F' \rightarrow H', \quad H' \rightarrow -F', \quad P' \rightarrow -Q', \quad Q' \rightarrow P'.
$$

Now let us mention that the complete set of fluxes, including the so-called prime fluxes P', Q', H' and F' which are some mixed-tensor quantities, have the following index structure [5, 10, 14, 15, 79],

$$
F_{ijk}, \qquad H_{ijk}, \qquad Q_i{}^{jk}, \qquad P_i{}^{jk}, \tag{29}
$$
$$
P'^{i,jklm}, \quad Q'^{i,jklm}, \quad H'^{ijk,lmnpqr}, \quad F'^{ijk,lmnpqr}.
$$

Subsequently, one can understand (P', Q') flux as a $(1, 4)$ tensor such that only the last four-indices are anti-symmetrized, while (H', F') flux can be considered as a $(3, 6)$ tensor where first three indices and last six indices are separately anti-symmetrized. These can also be understood as

$$
P'_{ij}{}^k = \frac{1}{4!} \epsilon_{ijlmnp} P'^{k,lmnp}, \qquad Q'_{ij}{}^k = \frac{1}{4!} \epsilon_{ijlmnp} Q'^{k,lmnp}, \tag{30}
$$
$$
H'^{ijk} = \frac{1}{6!} \epsilon_{lmnpqr} H'^{ijk,lmnpqr}, \qquad F'^{ijk} = \frac{1}{6!} \epsilon_{lmnpqr} F'^{ijk,lmnpqr}.
$$

Using (30) for prime fluxes, the number of flux parameters consistent with our toroidal orientifold for (P', Q') are 24 each, while those of (H', F') are 8 each. Subsequently, one has a total of 8+8+24+24+24+24+8+8 = 128 flux parameters allowed by the orientifold action, however these flux parameters are not all independent, since they are subject to restrictions derived from the Bianchi identities and tadpole cancellation conditions [6, 26, 40, 91, 92]. Using generalized geometry motivated through toroidal constructions, it has been argued that the type IIB superpotential governing the dynamics of the four-dimensional effective theory which respects the invariance under $SL(2, \mathbb{Z})^7$ symmetry can be given as [5, 10, 14, 15, 79],

$$W = \int_{X_3} (f_+ - S f_-) \cdot e^{\mathcal{J}} \wedge \Omega_3 \,, \tag{31}$$

where $\mathcal{J}$ denotes the complexified four-form defined in Eq. (15), and one has the following expansions for the quantities $f_\pm$,

$$
\begin{aligned}
f_+ \cdot e^{\mathcal{J}} &= F + Q \triangleright \mathcal{J} + P' \diamond \mathcal{J}^2 + H' \odot \mathcal{J}^3 \\
f_- \cdot e^{\mathcal{J}} &= H + P \triangleright \mathcal{J} + Q' \diamond \mathcal{J}^2 + F' \odot \mathcal{J}^3 \,.
\end{aligned}
\tag{32}
$$

Here, the flux-actions on the $\mathcal{J}_{ijkl}$ four-form polynomial pieces resulting in three-forms are [5, 10, 14, 15, 79]

$$
\begin{aligned}
(Q \triangleright \mathcal{J})_{a_1 a_2 a_3} &= \frac{3}{2} Q^{b_1 b_2}_{[\underline{a_1}} \mathcal{J}_{\underline{a_2}\,\underline{a_3}]b_1 b_2} \,, \tag{33} \\[4pt]
(P \triangleright \mathcal{J})_{a_1 a_2 a_3} &= \frac{3}{2} P^{b_1 b_2}_{[\underline{a_1}} \mathcal{J}_{\underline{a_2}\,\underline{a_3}]b_1 b_2} \,, \\[6pt]
\left(P' \diamond \mathcal{J}^2 \right)_{a_1 a_2 a_3} &= \frac{1}{4} P'^{c, b_1 b_2 b_3 b_4} \mathcal{J}_{[\underline{a_1 a_2}|cb_1|} \mathcal{J}_{\underline{a_3}]b_2 b_3 b_4} \,, \\[4pt]
\left(Q' \diamond \mathcal{J}^2 \right)_{a_1 a_2 a_3} &= \frac{1}{4} Q'^{c, b_1 b_2 b_3 b_4} \mathcal{J}_{[\underline{a_1 a_2}|cb_1|} \mathcal{J}_{\underline{a_3}]b_2 b_3 b_4} \,, \\[6pt]
\left(H' \odot \mathcal{J}^3 \right)_{a_1 a_2 a_3} &= \frac{1}{192} H'^{c_1 c_2 c_3, b_1 b_2 b_3 b_4 b_5 b_6} \mathcal{J}_{[\underline{a_1 a_2}|c_1 c_2|} \mathcal{J}_{\underline{a_3}]c_3 b_1 b_2} \mathcal{J}_{b_3 b_4 b_5 b_6} \,, \\[4pt]
\left(F' \odot \mathcal{J}^3 \right)_{a_1 a_2 a_3} &= \frac{1}{192} F'^{c_1 c_2 c_3, b_1 b_2 b_3 b_4 b_5 b_6} \mathcal{J}_{[\underline{a_1 a_2}|c_1 c_2|} \mathcal{J}_{\underline{a_3}]c_3 b_1 b_2} \mathcal{J}_{b_3 b_4 b_5 b_6} \,.
\end{aligned}
$$

Subsequently one finds an explicit and expanded version of the generalized flux superpotential W with 128 terms, each having one of the 128 flux parameters such that they are coupled with the complexified moduli resulting in cubic polynomial in T_α as well as U^i moduli while being linear in the axio-dilaton S.

2.3 Axionic fluxes and the scalar potential taxonomy

From the iterative models studied/revisited so far, one has the educated guess to invoke the following set of so-called axionic-flux combinations which will turn out to be extremely useful for

rearranging the scalar potential pieces into a compact form [17],

$$
\begin{aligned}
\mathbb{F}_{ijk} = \Bigg(&F_{ijk} + \frac{3}{2}\, Q_{[\underline{i}}^{b_1 b_2}\, \rho_{\underline{j}\,\underline{k}]b_1 b_2} + \frac{1}{4}\, P'^{c,b_1 b_2 b_3 b_4}\, \rho_{[\underline{i}\,\underline{j}|cb_1|}\, \rho_{\underline{k}]b_2 b_3 b_4} \\
&+ \frac{1}{192}\, H'^{c_1 c_2 c_3, b_1 b_2 b_3 b_4 b_5 b_6}\, \rho_{[\underline{i}\,\underline{j}|c_1 c_2|}\, \rho_{\underline{k}]c_3 b_1 b_2}\, \rho_{b_3 b_4 b_5 b_6}\Bigg) - C_0\, \mathbb{H}_{ijk},
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\mathbb{H}_{ijk} = \Bigg(&H_{ijk} + \frac{3}{2}\, P_{[\underline{i}}^{b_1 b_2}\, \rho_{\underline{j}\,\underline{k}]b_1 b_2} + \frac{1}{4}\, Q'^{c,b_1 b_2 b_3 b_4}\, \rho_{[\underline{i}\,\underline{j}|cb_1|}\, \rho_{\underline{k}]b_2 b_3 b_4} \\
&+ \frac{1}{192}\, F'^{c_1 c_2 c_3, b_1 b_2 b_3 b_4 b_5 b_6}\, \rho_{[\underline{i}\,\underline{j}|c_1 c_2|}\, \rho_{\underline{k}]c_3 b_1 b_2}\, \rho_{b_3 b_4 b_5 b_6}\Bigg),
\end{aligned}
$$

$$
\mathbb{Q}_i{}^{jk} = \Bigg(Q_i{}^{jk} - \frac{1}{2}\, P'^{c,jkb_1 b_2}\, \rho_{i\,c\,b_1 b_2} + \frac{1}{48} H'^{jkc,b_1 b_2 b_3 b_4 b_5 b_6}\, \rho_{i\,c\,b_1 b_2}\, \rho_{b_3 b_4 b_5 b_6}\Bigg) - C_0\, \mathbb{P}_i{}^{jk},
$$

$$
\mathbb{P}_i{}^{jk} = \Bigg(P_i{}^{jk} - \frac{1}{2}\, Q'^{c,jkb_1 b_2}\, \rho_{i\,c\,b_1 b_2} + \frac{1}{48} F'^{jkc,b_1 b_2 b_3 b_4 b_5 b_6}\, \rho_{i\,c\,b_1 b_2}\, \rho_{b_3 b_4 b_5 b_6}\Bigg),
$$

$$
\mathbb{P}'^{i,jklm} = \Bigg(P'^{i,jklm} + \frac{1}{4}\, H'^{ij'k',l'm'jklm}\, \rho_{j'k'l'm'}\Bigg) - C_0\, \mathbb{Q}'^{i,jklm},
$$

$$
\mathbb{Q}'^{i,jklm} = \Bigg(Q'^{i,jklm} + \frac{1}{4}\, F'^{ij'k',l'm'jklm}\, \rho_{j'k'l'm'}\Bigg),
$$

$$
\mathbb{H}'^{ijk,lmnpqr} = H'^{ijk,lmnpqr} - C_0\, \mathbb{F}'^{ijk,lmnpqr},
$$

$$
\mathbb{F}'^{ijk,lmnpqr} = F'^{ijk,lmnpqr}.
$$

Using Eq. (34) for our toroidal construction, one finds that there are 128 axionic fluxes corresponding to 128 standard fluxes, and one can solve this set of linear relations to determine one set of fluxes from the other,

$$
\{F, H, Q, P, P', Q', H', F'\} \Longleftrightarrow \{\mathbb{F}, \mathbb{H}, \mathbb{Q}, \mathbb{P}, \mathbb{P}', \mathbb{Q}', \mathbb{H}', \mathbb{F}'\}.
\tag{35}
$$

Subsequently using the $N = 1$ formula (2) for this toroidal model, one can compute the scalar potential induced by the generalized flux superpotential (31) having 128 terms. The final result consists of 76276 terms in the scalar potential, in a so-called bilinear formulation presented in [42, 43, 93–96]. All these 76276 terms can be distributed into 36 types of pieces, in the form of diagonal and off-diagonal flux bilinears arising from $\{F, H, Q, P, P', Q', H', F'\}$. As an immediate illustration of the significant use of the axionic fluxes (34), let us mention that the scalar potential can be equivalently expressed by 10888 terms if we use the axionic flux components, distributed into the new 36 pieces arising as bilinears from $\{\mathbb{F}, \mathbb{H}, \mathbb{Q}, \mathbb{P}, \mathbb{P}', \mathbb{Q}', \mathbb{H}', \mathbb{F}'\}$.

It is not very illuminating to display all the 36 pieces of the scalar potential which are detailed in [17], however to appreciate the importance of the axionic-flux polynomials we present the taxonomy of various pieces from the perspective of counting terms in scalar potential written with usual fluxes or the axionic fluxes in Table 1.

Another point worth mentioning here is the fact that all the RR axionic dependencies are encoded in the axionic fluxes (34) and the scalar potential does not have an explicit dependence on any of the RR axions. In our detailed analysis of rewriting the scalar potential pieces we find that using the set

	Fluxes	#(terms) in V using standard fluxes	Axionic-fluxes	#(terms) in V using axionic-fluxes in (60)
1	F	76	$\mathbb{F}$	76
2	F, H	361	$\mathbb{F}, \mathbb{H}$	160
3	F, H, Q	2422	$\mathbb{F}, \mathbb{H}, \mathbb{Q}$	772
4	F, H, Q, P	9661	$\mathbb{F}, \mathbb{H}, \mathbb{Q}, \mathbb{P}$	2356
5	F, H, Q, P, P'	23314	$\mathbb{F}, \mathbb{H}, \mathbb{Q}, \mathbb{P}, \mathbb{P}'$	4855
6	F, H, Q, P P', Q'	50185	$\mathbb{F}, \mathbb{H}, \mathbb{Q}, \mathbb{P},$ $\mathbb{P}', \mathbb{Q}'$	8326
7	$F, H, Q, P,$ P', Q', H'	60750	$\mathbb{F}, \mathbb{H}, \mathbb{Q}, \mathbb{P},$ $\mathbb{P}', \mathbb{Q}', \mathbb{H}'$	9603
8	$F, H, Q, P,$ P', Q', H', F'	76276	$\mathbb{F}, \mathbb{H}, \mathbb{Q}, \mathbb{P},$ $\mathbb{P}', \mathbb{Q}', \mathbb{H}', \mathbb{F}'$	10888

Table 1: Counting of scalar potential terms for a set of fluxes being turned-on at a time. [17, 51]

of axionic fluxes reduces the number of scalar potential terms quite significantly. This subsequently helps us in understanding the insights within each terms towards seeking a compact and concise formulation of the full scalar potential and we can take an educated route to arrive at generic setups beyond the torodial orientifold. This is what is the motivation for the symplectic formulation we present next.

3. Symplectic formulation

To begin with, let us also recollect some relevant ingredients for rewriting the F-term scalar potential in a symplectic formulation. We start by resetting our notations and conventions for symplectic formulation which would be valid for setups beyond the toroidal orientifolds.

3.1 Necessary cohomological and symplectic ingredients

The massless states in the four dimensional (4D) effective theory are in one-to-one correspondence with harmonic forms which are either even or odd under the action of an isometric, holomorphic involution (σ) acting on the internal compactifying sixfold X, and these do generate the equivariant cohomology groups $H_\pm^{p,q}(X)$. For that purpose, let us fix our conventions, and denote the bases of even/odd two-forms as (μ_α, ν_a) and four-forms as $(\tilde{\mu}_\alpha, \tilde{\nu}_a)$ where $\alpha \in h_+^{1,1}(X)$, $a \in h_-^{1,1}(X)$. However for our current objectives, we will be interested in orientifold setups resulting in $h_-^{1,1}(X) = 0$ which means that the so-called odd-moduli G^a counted by $h_-^{1,1}(X) \neq 0$ will be absent, and explicit construction of such CY orientifolds with odd two-cycles can be found in [82, 97–101]. In addition, we consider the orientifold involutions resulting in O3/O7 system with a trivial $H_+^3(X)$, and basis of 3-forms in the odd (2,1)-cohomology sector being denoted as $(\mathcal{A}_\Lambda, \mathcal{B}^\Delta)$ where $\{\Lambda, \Delta\} \in \{0, 1, .., h_-^{2,1}(X)\}$. We fix the normalization in the various cohomology bases as,

$$\int_X \mu_\alpha \wedge \tilde{\mu}^\beta = \delta_\alpha{}^\beta, \quad \int_X \mu_\alpha \wedge \mu_\beta \wedge \mu_\gamma = \kappa_{\alpha\beta\gamma}, \quad \int_X \mathcal{A}_\Lambda \wedge \mathcal{B}^\Delta = \delta_\Lambda{}^\Delta. \tag{36}$$

Subsequently the Kähler form J, and the RR four-form C_4 can be expanded as [102]: $J = t^\alpha \mu_\alpha$, $C_4 \simeq \rho_\alpha \tilde{\mu}^\alpha$ where t^α, and ρ_α denote the Einstein-frame two-cycle volume moduli, and axions descending from the 4-form potential C_4. For the choice of involution σ we have $\sigma^* \Omega_3 = -\Omega_3$, where Ω_3 denotes the nowhere vanishing holomorphic three-form depending on the complex structure moduli U^i counted in the $h^{2,1}_-(X)$ cohomology, which can be generically given as below,

$$\Omega_3 \equiv \mathcal{X}^\Lambda \, \mathcal{A}_\Lambda - \mathcal{F}_\Lambda \, \mathcal{B}^\Lambda \,. \tag{37}$$

Here, the period vectors $(\mathcal{X}^\Lambda, \mathcal{F}_\Lambda)$ are encoded in a pre-potential $(\mathcal{F})$ of the following form,

$$\mathcal{F} = (\mathcal{X}^0)^2 \, f(U^i) \,, \qquad f(U^i) = -\frac{1}{6} \ell_{ijk} \, U^i \, U^j \, U^k + \frac{1}{2} a_{ij} \, U^i \, U^j + b_i \, U^i + \frac{i}{2} \gamma \,. \tag{38}$$

In fact, the function $f(U^i)$ can generically have an infinite series of non-perturbative contributions, which we ignore for the current work assuming to be working in the large complex structure limit. The quantities a_{ij}, b_i and γ are real numbers where γ is related to the perturbative $(\alpha')^3$-corrections on the mirror side [103–105]. Furthermore, the chiral coordinates U^i's are defined as $U^i = \frac{\delta^i_\Lambda \mathcal{X}^\Lambda}{\mathcal{X}^0}$ where ℓ_{ijk}'s are triple intersection numbers on the mirror (CY) threefold.

In addition to the normalizations defined in (36) we define the following Hodge star operations acting on the various (odd) 3-forms via introducing a set of so-called $\mathcal{M}$ matrices [45],

$$\star \, \mathcal{A}_\Lambda = \mathcal{M}_\Lambda{}^\Sigma \, \mathcal{A}_\Sigma + \mathcal{M}_{\Lambda\Sigma} \, \mathcal{B}^\Sigma, \quad \text{and} \quad \star \, \mathcal{B}^\Lambda = \mathcal{M}^{\Lambda\Sigma} \, \mathcal{A}_\Sigma + \mathcal{M}^\Lambda{}_\Sigma \, \mathcal{B}^\Sigma, \tag{39}$$

where

$$\mathcal{M}^{\Lambda\Delta} = \text{Im}\mathcal{N}^{\Lambda\Delta}, \qquad\qquad \mathcal{M}_\Lambda{}^\Delta = \text{Re}\mathcal{N}_{\Lambda\Gamma} \, \text{Im}\mathcal{N}^{\Gamma\Delta}, \tag{40}$$

$$\mathcal{M}^\Lambda{}_\Delta = -\left(\mathcal{M}_\Lambda{}^\Delta\right)^T, \qquad\qquad \mathcal{M}_{\Lambda\Delta} = -\text{Im}\mathcal{N}_{\Lambda\Delta} - \text{Re}\mathcal{N}_{\Lambda\Sigma} \, \text{Im}\mathcal{N}^{\Sigma\Gamma} \, \text{Re}\mathcal{N}_{\Gamma\Delta}.$$

Here the period matrix $\mathcal{N}$ for the involutively odd (2,1)-cohomology sector can be given by the derivatives of the prepotential (38) as below,

$$\mathcal{N}_{\Lambda\Delta} = \overline{\mathcal{F}}_{\Lambda\Delta} + 2\,i \, \frac{Im(\mathcal{F}_{\Lambda\Gamma}) \, \mathcal{X}^\Gamma X^\Sigma \, (Im\mathcal{F}_{\Sigma\Delta})}{Im(\mathcal{F}_{\Gamma\Sigma}) \mathcal{X}^\Gamma X^\Sigma}. \tag{41}$$

Moreover, it was observed in [49] that an interesting relation analogous to the period matrix expression (41) holds which is given as below:

$$\mathcal{F}_{\Lambda\Delta} = \overline{\mathcal{N}}_{\Lambda\Delta} + 2\,i \, \frac{Im(\mathcal{N}_{\Lambda\Gamma}) \, \mathcal{X}^\Gamma X^\Sigma \, (Im\mathcal{N}_{\Sigma\Delta})}{Im(\mathcal{N}_{\Gamma\Sigma}) \mathcal{X}^\Gamma X^\Sigma}, \tag{42}$$

and, similar to the period matrices given in (40), one can define another set of symplectic quantities as below,

$$\mathcal{L}^{\Lambda\Delta} = \text{Im}\mathcal{F}^{\Lambda\Delta}, \qquad\qquad \mathcal{L}_\Lambda{}^\Delta = \text{Re}\mathcal{F}_{\Lambda\Gamma} \, \text{Im}\mathcal{F}^{\Gamma\Delta}, \tag{43}$$

$$\mathcal{L}^\Lambda{}_\Delta = -\left(\mathcal{L}_\Lambda{}^\Delta\right)^T, \qquad\qquad \mathcal{L}_{\Lambda\Delta} = -\text{Im}\mathcal{F}_{\Lambda\Delta} - \text{Re}\mathcal{F}_{\Lambda\Sigma} \, \text{Im}\mathcal{F}^{\Sigma\Gamma} \, \text{Re}\mathcal{F}_{\Gamma\Delta}.$$

Let us mention here at the outset that in addition to having $h^{2,1}_+(X) = 0 = h^{1,1}_-(X)$ in order to avoid the D-term effects and the presence of odd moduli (G^a), in our current analysis we will make another assumption in our superpotential by considering the fluxes to adopt appropriate rational shifts [95, 105] in order to absorb the appearance of a_{ij} and b_i sourced from the prepotential (38). Also, we set $\gamma = 0$ assuming the large complex structure limit.

3.2 The Kähler potential and the flux superpotential

Using these pieces of information one defines a set of chiral coordinates, namely S, T_α and U^i, as below [3],

$$S \equiv C_0 + i\, e^{-\phi} = C_0 + i\, s, \quad T_\alpha = \rho_\alpha - i\, \tau_\alpha, \quad U^i = v^i - i\, u^i, \tag{44}$$

where $\{C_0, \rho_\alpha, v^i\}$ are axionic moduli while $\{s, \tau_\alpha, u^i\}$ are saxionic moduli. The Einstein-frame overall volume ($\mathcal{V}$) of the internal background is connected with the 2-cycle volume moduli t^α and the 4-cycle volume moduli as below,

$$\mathcal{V} = \frac{1}{6}\kappa_{\alpha\beta\gamma}\, t^\alpha\, t^\beta\, t^\gamma, \qquad \tau_\alpha = \partial_{t^\alpha}\mathcal{V} = \frac{1}{2}\kappa_{\alpha\beta\gamma}t^\beta t^\gamma. \tag{45}$$

Using appropriate chiral variables (S, T_α, U^i) as defined in (44), a generic form of the tree-level Kähler potential can be written as below,

$$K = -\ln\left(-i\int_X \Omega_3 \wedge \bar{\Omega}_3\right) - \ln\left(-i(S - \overline{S})\right) - 2\ln\mathcal{V}. \tag{46}$$

The tree-level Kähler potential (46) leads to a decoupled sector for U^i moduli and the $\{S, T_\alpha\}$ moduli sectors, thus implying the following derivatives along with the block diagonals matrices,

$$K_S = \frac{i}{2\,s} = -K_{\overline{S}}, \quad K_{T_\alpha} = -\frac{i\,t^\alpha}{2\mathcal{V}} = -K_{\overline{T}_\alpha}, \tag{47}$$

$$K_{S\overline{S}} = \frac{1}{4\,s^2}, \quad K_{S\overline{T}_\alpha} = 0 = K_{T_\alpha\overline{S}}, \quad K_{T_\alpha\overline{T}_\beta} = 4\left(\tau_\alpha\tau_\beta - \mathcal{V}\kappa_{\alpha\beta}\right) \equiv 4\,\mathcal{G}_{\alpha\beta},$$

$$K^{S\overline{S}} = 4\,s^2, \quad K^{S\overline{T}_\alpha} = 0 = K^{T_\alpha\overline{S}}, \quad K^{T_\alpha\overline{T}_\beta} = \frac{1}{16\mathcal{V}^2}\left(2\,t^\alpha t^\beta - 4\mathcal{V}\kappa^{\alpha\beta}\right) \equiv \frac{1}{4}\mathcal{G}^{\alpha\beta},$$

where we have introduced two new quantities $\mathcal{G}_{\alpha\beta}$ and $\mathcal{G}^{\alpha\beta}$ to avoid any confusion with lower/upper indices in the Kähler metric and its inverse metric. We also note that the metrics defined in (47) result in the following useful identities [67],

$$\mathcal{G}_{\alpha\beta}t^\alpha = \mathcal{V}\,\tau_\beta, \quad \mathcal{G}_{\alpha\beta}t^\alpha t^\beta = 3\mathcal{V}^2, \quad \mathcal{G}^{\alpha\beta}\tau_\alpha = \frac{t^\alpha}{\mathcal{V}}, \quad \mathcal{G}^{\alpha\beta}\tau_\alpha\tau_\beta = 3. \tag{48}$$

Now, it turns out that making successive applications of T/S-dualities results in the need of introducing more and more fluxes. In fact, one needs a total of four S-dual pairs of fluxes, commonly denotes as: (F, H), (Q, P), (P', Q') and (H', F') [5, 5, 10, 10, 14, 15, 17, 71, 79]. This leads to the following generalized flux superpotential,

$$W = \int_X\left[(F - S\,H) + (Q - S\,P) \triangleright \mathcal{J} + (P' - S\,Q') \diamond \mathcal{J}^2 + (H' - S\,F') \odot \mathcal{J}^3\right] \wedge \Omega_3, \tag{49}$$

where $\mathcal{J} = T_\alpha\,\mu^\alpha$ the dual complexified four-form dual to the Kähler form, and the symplectic form of the various flux actions are defined as below [51],

$$(Q \triangleright \mathcal{J}) = T_\alpha Q^\alpha, \qquad\qquad (P \triangleright \mathcal{J}) = T_\alpha P^\alpha, \tag{50}$$

$$\left(P' \diamond \mathcal{J}^2\right) = \frac{1}{2} P'^{\beta\gamma} T_\beta T_\gamma, \qquad\qquad \left(Q' \diamond \mathcal{J}^2\right) = \frac{1}{2} Q'^{\beta\gamma} T_\beta T_\gamma,$$

$$\left(H' \odot \mathcal{J}^3\right) = \frac{1}{3!} H'^{\alpha\beta\gamma} T_\alpha T_\beta T_\gamma, \qquad \left(F' \odot \mathcal{J}^3\right) = \frac{1}{3!} F'^{\alpha\beta\gamma} T_\alpha T_\beta T_\gamma.$$

Here $Q^\alpha, P^\alpha, P'^{\beta\gamma}, Q'^{\beta\gamma}, H'^{\alpha\beta\gamma}$ and $F'^{\alpha\beta\gamma}$ denote the 3-forms similar to the standard F_3/H_3 fluxes and can be expanded in the symplectic basis $\{\mathcal{A}_\Lambda, \mathcal{B}^\Delta\}$ in the following manner,

$$
\begin{aligned}
F &= F^\Lambda \mathcal{A}_\Lambda - F_\Lambda \mathcal{B}^\Lambda, & H &= H_\Lambda \mathcal{A}_\Lambda - H_\Lambda \mathcal{B}^\Lambda, & (51)\\
Q^\alpha &= Q^{\alpha\Lambda} \mathcal{A}_\Lambda - Q^\alpha{}_\Lambda \mathcal{B}^\Lambda, & P^\alpha &= P^{\alpha\Lambda} \mathcal{A}_\Lambda - P^\alpha{}_\Lambda \mathcal{B}^\Lambda, \\
P'^{\beta\gamma} &= P'^{\beta\gamma\Lambda} \mathcal{A}_\Lambda - P'^{\beta\gamma}{}_\Lambda \mathcal{B}^\Lambda, & Q'^{\beta\gamma} &= Q'^{\beta\gamma\Lambda} \mathcal{A}_\Lambda - Q'^{\beta\gamma}{}_\Lambda \mathcal{B}^\Lambda, \\
H'^{\alpha\beta\gamma} &= H'^{\alpha\beta\gamma\Lambda} \mathcal{A}_\Lambda - H'^{\alpha\beta\gamma}{}_\Lambda \mathcal{B}^\Lambda, & F'^{\alpha\beta\gamma} &= F'^{\alpha\beta\gamma\Lambda} \mathcal{A}_\Lambda - F'^{\alpha\beta\gamma}{}_\Lambda \mathcal{B}^\Lambda.
\end{aligned}
$$

Note that the flux components Q^α and P^α having a single index $\alpha \in h^{1,1}_+$ (and $\Lambda \in h^{2,1}_-$ indices) are counted accordingly. However, counting of the prime flux components is a bit tricky because their appearance as $P'^{\beta\gamma}, Q'^{\beta\gamma}, H'^{\alpha\beta\gamma}$ and $F'^{\alpha\beta\gamma}$ in (50) does not seem to have any constraint except for being symmetric in indices $\{\alpha, \beta, \gamma\} \in h^{1,1}_+$. For that purpose, motivated by the previous toroidal studies [14, 15, 17, 79], one can equivalently define [51],

$$
P'_{\alpha\Lambda}, P'_\alpha{}^\Lambda, \qquad Q'_{\alpha\Lambda}, Q'_\alpha{}^\Lambda, \qquad H'_\Lambda, H'^\Lambda, \qquad F'_\Lambda, F'^\Lambda, \tag{52}
$$

which are related to the earlier set of prime fluxes as below

$$
\begin{aligned}
P'^{\beta\gamma\Lambda} &= P'_\alpha{}^\Lambda \kappa^{\alpha\beta\gamma}, & P'^{\beta\gamma}{}_\Lambda &= P'_{\alpha\Lambda} \kappa^{\alpha\beta\gamma}, & (53)\\
Q'^{\beta\gamma\Lambda} &= Q'_\alpha{}^\Lambda \kappa^{\alpha\beta\gamma}, & Q'^{\beta\gamma}{}_\Lambda &= Q'_{\alpha\Lambda} \kappa^{\alpha\beta\gamma}, \\
H'^{\alpha\beta\gamma\Lambda} &= H'^\Lambda \kappa^{\alpha\beta\gamma}, & H'^{\alpha\beta\gamma}{}_\Lambda &= H'_\Lambda \kappa^{\alpha\beta\gamma}, \\
F'^{\alpha\beta\gamma\Lambda} &= F'^\Lambda \kappa^{\alpha\beta\gamma}, & F'^{\alpha\beta\gamma}{}_\Lambda &= F'_\Lambda \kappa^{\alpha\beta\gamma}.
\end{aligned}
$$

Here $\kappa^{\alpha\beta\gamma}$ is defined as $\kappa^{\alpha\beta\gamma} = \mathcal{V}^3 \kappa_{\alpha'\beta'\gamma'} \mathcal{G}^{\alpha\alpha'} \mathcal{G}^{\beta\beta'} \mathcal{G}^{\gamma\gamma'}$ where the tree-level metric $\mathcal{G}^{\alpha\beta}$ and its inverse $\mathcal{G}_{\alpha\beta}$ are introduced in (47). This leads to the following interesting relation [51]

$$
t^\alpha = \frac{1}{2} \kappa^{\alpha\beta\gamma} \tau_\beta \tau_\gamma, \quad \mathcal{V} = \frac{1}{3!} \kappa^{\alpha\beta\gamma} \tau_\alpha \tau_\beta \tau_\gamma,
$$

where τ_α corresponds to the volume of the 4-cycle and can be written in terms of the 2-cycle volumes as: $\tau_\alpha = \frac{1}{2} \kappa_{\alpha\beta\gamma} t^\beta t^\gamma = \frac{1}{2} \kappa_\alpha$. Here we used the shorthand notations $\kappa_\alpha = \kappa_{\alpha\beta} t^\beta = \kappa_{\alpha\beta\gamma} t^\beta t^\gamma$ etc.

From this approach it is clear that counting of prime fluxes is also similar to those of the non-prime fluxes, i.e. $\{H', F'\}$ are counted by the $h^{2,1}_-(\text{CY})$ indices similar to the standard H_3/F_3 fluxes while $\{P', Q'\}$ are counted via one α index and one Λ index. However, the crucial property we must take into account is the holomorphicity of the flux superpotential which shows that fluxes used in actions (50) and (51) are the ones to be considered in the superpotential along with the chiral variables S, T_α and U^i. Subsequently one has the following generalized flux superpotential,

$$
\begin{aligned}
W = \int_X &\left[\left(F + Q^\alpha T_\alpha + \frac{1}{2} P'^{\alpha\beta} T_\alpha T_\beta + \frac{1}{6} H'^{\alpha\beta\gamma} T_\alpha T_\beta T_\gamma \right) \right. & (54)\\
&\left. - S \left(H + P^\alpha T_\alpha + \frac{1}{2} Q^{\alpha\beta} T_\alpha T_\beta + \frac{1}{6} F'^{\alpha\beta\gamma} T_\alpha T_\beta T_\gamma \right) \right]_3 \wedge \Omega_3,
\end{aligned}
$$

where all the terms appearing inside the bracket $[\ldots]_3$ denote a collection of three-forms as expanded in (51). Let us also make a side remark that for the holomorphicity of the flux superpotential all

these 3-form flux ingredients in (54), or equivalently in (51), are independent of the volume moduli and therefore such fluxes are used for expressing the superpotential for the purpose of computing the scalar potential using the $N = 1$ formula (2). After the scalar potential is computed one can always reshuffle and rewrite the scalar potential pieces in order to achieve a more compact way of expressing the full potential, which a priori is a complicated task.

Using the flux actions in Eq. (51), the generalized flux superpotential W in (54) can be equivalently given as,

$$W \equiv \int_X [\text{Flux}]_3 \wedge \Omega_3 = e_\Lambda \mathcal{X}^\Lambda - m^\Lambda \mathcal{F}_\Lambda, \tag{55}$$

where the symplectic electromagnetic vectors (e_Λ, m^Λ) are given as,

$$e_\Lambda = (F_\Lambda - S H_\Lambda) + T_\alpha (Q^\alpha{}_\Lambda - S P^\alpha{}_\Lambda) + \frac{1}{2} T_\alpha T_\beta \left(P'^{\alpha\beta}{}_\Lambda - S Q'^{\alpha\beta}{}_\Lambda \right) \tag{56}$$

$$+ \frac{1}{6} T_\alpha T_\beta T_\gamma \left(H'^{\alpha\beta\gamma}{}_\Lambda - S F'^{\alpha\beta\gamma}{}_\Lambda \right),$$

$$m^\Lambda = \left(F^\Lambda - S H^\Lambda \right) + T_\alpha \left(Q^{\alpha\Lambda} - S P^{\alpha\Lambda} \right) + \frac{1}{2} T_\alpha T_\beta \left(P'^{\alpha\beta\Lambda} - S Q'^{\alpha\beta\Lambda} \right)$$

$$+ \frac{1}{6} T_\alpha T_\beta T_\gamma \left(H'^{\alpha\beta\gamma\Lambda} - S F'^{\alpha\beta\gamma\Lambda} \right).$$

The generic superpotential (55) is linear in the axio-dilaton modulus S and has cubic dependence in moduli U^i and T_α both. Using the symplectic vectors (e_Λ, m^Λ) of (56) in the flux superpotential (55), one can compute the derivatives with respect to chiral variables, S and T_α which are given by

$$W_S = (e_1)_\Lambda \mathcal{X}^\Lambda - (m_1)^\Lambda \mathcal{F}_\Lambda, \qquad W_{T_\alpha} = (e_2)_\Lambda^\alpha \mathcal{X}^\Lambda - (m_2)^{\alpha\Lambda} \mathcal{F}_\Lambda, \tag{57}$$

where the two new pairs of symplectic vectors (e_1, m_1) and (e_2, m_2) are given as:

$$(e_1)_\Lambda = -\left[H_\Lambda + T_\alpha P^\alpha{}_\Lambda + \frac{1}{2} T_\alpha T_\beta Q'^{\alpha\beta}{}_\Lambda + \frac{1}{6} T_\alpha T_\beta T_\gamma F'^{\alpha\beta\gamma}{}_\Lambda \right], \tag{58}$$

$$(m_1)^\Lambda = -\left[H^\Lambda + T_\alpha P^{\alpha\Lambda} + \frac{1}{2} T_\alpha T_\beta Q'^{\alpha\beta\Lambda} + \frac{1}{6} T_\alpha T_\beta T_\gamma F'^{\alpha\beta\gamma\Lambda} \right],$$

and

$$(e_2)_\Lambda^\alpha = (Q^\alpha{}_\Lambda - S P^\alpha{}_\Lambda) + T_\beta \left(P'^{\alpha\beta}{}_\Lambda - S Q'^{\alpha\beta}{}_\Lambda \right) + \frac{1}{2} T_\beta T_\gamma \left(H'^{\alpha\beta\gamma}{}_\Lambda - S F'^{\alpha\beta\gamma}{}_\Lambda \right), \tag{59}$$

$$(m_2)^{\alpha\Lambda} = \left(Q^{\alpha\Lambda} - S P^{\alpha\Lambda} \right) + T_\beta \left(P'^{\alpha\beta\Lambda} - S Q'^{\alpha\beta\Lambda} \right) + \frac{1}{2} T_\beta T_\gamma \left(H'^{\alpha\beta\gamma\Lambda} - S F'^{\alpha\beta\gamma\Lambda} \right).$$

3.3 Invoking the axionic fluxes

With the concrete form of the holomorphic superpotential at hand, now, we define the following set of the so-called axionic-flux combinations which will turn out to be extremely useful for rearranging

the scalar potential pieces into a compact form,

$$\mathbb{F}_\Lambda = F_\Lambda + \rho_\alpha \, Q^\alpha{}_\Lambda + \frac{1}{2} \rho_\alpha \, \rho_\beta \, P'^{\alpha\beta}{}_\Lambda + \frac{1}{6} \rho_\alpha \, \rho_\beta \, \rho_\gamma \, H'^{\alpha\beta\gamma}{}_\Lambda - C_0 \, \mathbb{H}_\Lambda \,, \tag{60}$$

$$\mathbb{H}_\Lambda = H_\Lambda + \rho_\alpha \, P^\alpha{}_\Lambda + \frac{1}{2} \rho_\alpha \, \rho_\beta \, Q'^{\alpha\beta}{}_\Lambda + \frac{1}{6} \rho_\alpha \, \rho_\beta \, \rho_\gamma \, F'^{\alpha\beta\gamma}{}_\Lambda,$$

$$\mathbb{Q}^\alpha{}_\Lambda = Q^\alpha{}_\Lambda + \rho_\beta \, P'^{\alpha\beta}{}_\Lambda + \frac{1}{2} \rho_\beta \, \rho_\gamma \, H'^{\alpha\beta\gamma}{}_\Lambda - C_0 \, \mathbb{P}^\alpha{}_\Lambda,$$

$$\mathbb{P}^\alpha{}_\Lambda = P^\alpha{}_\Lambda + \rho_\beta \, Q'^{\alpha\beta}{}_\Lambda + \frac{1}{2} \rho_\beta \, \rho_\gamma \, F'^{\alpha\beta\gamma}{}_\Lambda \,,$$

$$\mathbb{P}'^{\alpha\beta}{}_\Lambda = P'^{\alpha\beta}{}_\Lambda + \rho_\gamma \, H'^{\alpha\beta\gamma}{}_\Lambda - C_0 \, \mathbb{Q}'^{\alpha\beta}{}_\Lambda \,,$$

$$\mathbb{Q}'^{\alpha\beta}{}_\Lambda = Q'^{\alpha\beta}{}_\Lambda + \rho_\gamma \, F'^{\alpha\beta\gamma}{}_\Lambda,$$

$$\mathbb{H}'^{\alpha\beta\gamma}{}_\Lambda = H'^{\alpha\beta\gamma}{}_\Lambda - C_0 \, \mathbb{F}'^{\alpha\beta\gamma}{}_\Lambda,$$

$$\mathbb{F}'^{\alpha\beta\gamma}{}_\Lambda = F'^{\alpha\beta\gamma}{}_\Lambda \,,$$

where the analogous axionic fluxes with upper Λ index are expressed in a similar fashion. Using these axionic-flux combinations (60) along with the definitions of chiral variables in Eq. (44), the three pairs of symplectic vectors, namely (e, m), (e_1, m_1) and (e_2, m_2) which are respectively given in Eqs. (56), (58) and (59), can be expressed in the following compact form,

$$e_\Lambda = (\mathbb{F}_\Lambda - s \, \mathbb{P}_\Lambda - \mathbb{P}'_\Lambda + s \, \mathbb{F}'_\Lambda) + i \, (-s \, \mathbb{H}_\Lambda - \mathbb{Q}_\Lambda + s \, \mathbb{Q}'_\Lambda + \mathbb{H}'_\Lambda) \,, \tag{61}$$

$$m^\Lambda = \left(\mathbb{F}^\Lambda - s \, \mathbb{P}^\Lambda - \mathbb{P}'^\Lambda + s \, \mathbb{F}'^\Lambda \right) + i \left(-s \, \mathbb{H}^\Lambda - \mathbb{Q}^\Lambda + s \, \mathbb{Q}'^\Lambda + \mathbb{H}'^\Lambda \right),$$

$$(e_1)_\Lambda = (- \mathbb{H}_\Lambda + \mathbb{Q}'_\Lambda) + i \, (\mathbb{P}_\Lambda - \mathbb{F}'_\Lambda) \,, \quad (m_1)^\Lambda = \left(- \mathbb{H}^\Lambda + \mathbb{Q}'^\Lambda \right) + i \left(\mathbb{P}^\Lambda - \mathbb{F}'^\Lambda \right) \,, \tag{62}$$

$$(e_2)^\alpha{}_\Lambda = (\mathbb{Q}^\alpha{}_\Lambda - s \, \mathbb{Q}'^\alpha{}_\Lambda - \mathbb{H}'^\alpha{}_\Lambda) + i \, (-s \, \mathbb{P}^\alpha{}_\Lambda - \mathbb{P}'^\alpha{}_\Lambda + s \, \mathbb{F}'^\alpha{}_\Lambda) \,, \tag{63}$$

$$(m_2)^{\alpha\Lambda} = \left(\mathbb{Q}^{\alpha\Lambda} - s \, \mathbb{Q}'^{\alpha\Lambda} - \mathbb{H}'^{\alpha\Lambda} \right) + i \left(-s \, \mathbb{P}^{\alpha\Lambda} - \mathbb{Q}'^{\alpha\Lambda} + s \, \mathbb{F}'^{\alpha\Lambda} \right) \,,$$

where we have used the shorthand notations such as $\mathbb{Q}_\Lambda = \tau_\alpha \, \mathbb{Q}^\alpha{}_\Lambda$, $\mathbb{Q}'^\alpha_\Lambda = \tau_\beta \mathbb{Q}'^{\alpha\beta}{}_\Lambda$, $\mathbb{Q}'_\Lambda = \frac{1}{2} \tau_\alpha \, \tau_\beta \mathbb{Q}'^{\alpha\beta}{}_\Lambda$, $\mathbb{H}'_\Lambda = \frac{1}{6} \tau_\alpha \tau_\beta \tau_\gamma \mathbb{H}'^{\alpha\beta\gamma}{}_\Lambda$ etc. In addition, we mention that such shorthand notations are applicable only with τ_α contractions, and not to be (conf)used with axionic (ρ_α) contractions. This convention will be used wherever the (Q, P), (P', Q') and (H', F') fluxes are seen with/without a free index $\alpha \in h^{1,1}_+(X)$. Here we recall that $\tau_\alpha = \frac{1}{2} \kappa_{\alpha\beta\gamma} t^\beta t^\gamma$.

Now the main task is to rewrite the scalar potential (2) in a generic symplectic formulation. The blocks in the Kähler metric and inverse metric corresponding to the Kähler moduli and axio-dilaton dependent sector are determined by (47) while for simplifying the complex structure dependent pieces one needs to use the period matrix components defined in (40)-(43). In this regard, one of the most important identities for simplifying the scalar potential is the following one [45],

$$K^{i\bar{j}} \, (D_i \mathcal{X}^\Lambda) \, (\overline{D}_{\bar{j}} \overline{\mathcal{X}^\Delta}) = - \overline{\mathcal{X}^\Lambda} \, \mathcal{X}^\Delta - \frac{1}{2} \, e^{-K_{cs}} \, \mathrm{Im} \mathcal{N}^{\Lambda\Delta}. \tag{64}$$

Subsequently after some tedious computations, one arrives at a generic scalar potential promoted for the beyond toroidal case. This scalar potential can be expressed using the following 36 pieces,

$$V = V_{\mathrm{FF}} + V_{\mathrm{HH}} + V_{\mathrm{QQ}} + V_{\mathrm{PP}} + V_{\mathrm{P'P'}} + V_{\mathrm{Q'Q'}} + V_{\mathrm{H'H'}} + V_{\mathrm{F'F'}}, \tag{65}$$

$$+ V_{\mathrm{FH}} + V_{\mathrm{FQ}} + V_{\mathrm{FP}} + V_{\mathrm{FP'}} + V_{\mathrm{FQ'}} + V_{\mathrm{FH'}} + V_{\mathrm{FF'}} + V_{\mathrm{HQ}} + V_{\mathrm{HP}} + V_{\mathrm{HP'}} + V_{\mathrm{HQ'}} + V_{\mathrm{HH'}} + V_{\mathrm{HF'}}$$

$$+ V_{\mathrm{QP}} + V_{\mathrm{QP'}} + V_{\mathrm{QQ'}} + V_{\mathrm{QH'}} + V_{\mathrm{QF'}} + V_{\mathrm{PP'}} + V_{\mathrm{PQ'}} + V_{\mathrm{PH'}} + V_{\mathrm{PF'}} + V_{\mathrm{P'Q'}} + V_{\mathrm{P'H'}} + V_{\mathrm{P'F'}}$$

$$+ V_{\mathrm{Q'H'}} + V_{\mathrm{Q'F'}} + V_{\mathrm{H'F'}}\,.$$

The explicit forms of the various pieces are elaborated in the appendix of [51].

3.4 Master formula for the scalar potential

The attempts so far have just been to elaborate on the insights of various terms and how they could appear from the flux superpotential in connection with the standard U-dual flux parameters, and it is desirable that we club these 36 terms in a more concise symplectic formulation. With this goal in mind, we investigated the 36 pieces in some more detail and found that using the axionic fluxes (60), the following triplet $\{\chi, \psi, \Psi^\alpha\}$ of quantities turn out to be useful,

$$\psi_\Lambda = s\,(-\mathbb{H}_\Lambda + \mathbb{Q}'_\Lambda) + i\,s\,(\mathbb{P}_\Lambda - \mathbb{F}'_\Lambda)\,, \tag{66}$$
$$\chi_\Lambda = (\mathbb{F}_\Lambda - \mathbb{P}'_\Lambda) + i\,(-\mathbb{Q}_\Lambda + \mathbb{H}'_\Lambda) + i\,\psi_\Lambda\,,$$
$$\Psi^\alpha{}_\Lambda = (\mathbb{Q}^\alpha{}_\Lambda - s\,\mathbb{Q}'^\alpha{}_\Lambda - \mathbb{H}'^\alpha{}_\Lambda) + i\,(-s\,\mathbb{P}^\alpha{}_\Lambda - \mathbb{P}'^\alpha{}_\Lambda + s\,\mathbb{F}'^\alpha{}_\Lambda)\,,$$

where the analogous quantities with upper Λ index are expressed in a similar fashion. Also we used the shorthand notations such as $\mathbb{Q}_\Lambda = \tau_\alpha \mathbb{Q}^\alpha{}_\Lambda$, $\mathbb{Q}'^\alpha_\Lambda = \tau_\beta \mathbb{Q}'^{\alpha\beta}{}_\Lambda$, $\mathbb{Q}'_\Lambda = \frac{1}{2}\tau_\alpha \tau_\beta \mathbb{Q}'^{\alpha\beta}{}_\Lambda$, $\mathbb{H}'_\Lambda = \frac{1}{6}\tau_\alpha\tau_\beta\tau_\gamma \mathbb{H}'^{\alpha\beta\gamma}{}_\Lambda$ etc. In the similar way we write $\Psi_\Lambda = \tau_\alpha \Psi^\alpha{}_\Lambda$ and $\Psi^\Lambda = \tau_\alpha \Psi^{\alpha\Lambda}$ wherever Ψ appears without an $h^{1,1}_+$ index α. Subsequently, we will have the following relations consistent with out shorthand notations,

$$\Psi_\Lambda = (\mathbb{Q}_\Lambda - 2\,s\,\mathbb{Q}'_\Lambda - 3\,\mathbb{H}'_\Lambda) + i\,(-s\,\mathbb{P}_\Lambda - 2\,\mathbb{P}'_\Lambda + 3\,s\,\mathbb{F}'_\Lambda)\,, \tag{67}$$
$$\Psi^\Lambda = \left(\mathbb{Q}^\Lambda - 2\,s\,\mathbb{Q}'^\Lambda - 3\,\mathbb{H}'^\Lambda\right) + i\,\left(-s\,\mathbb{P}^\Lambda - 2\,\mathbb{Q}'^\Lambda + 3\,s\,\mathbb{F}'^\Lambda\right).$$

Also, let us recall that the electric/magnetic components of the fluxes defined in (66), are expanded in the three-form basis as $\mathrm{Flux} = \mathrm{Flux}^\Lambda \mathcal{A}_\Lambda - \mathrm{Flux}_\Lambda \mathcal{B}^\Lambda$, where $\mathrm{Flux} = \{\chi, \psi, \Psi\}$.

Subsequently, the full scalar potential can be expressed in just a few lines, and can be clubbed into two types of terms, namely $(O_1 \wedge *\overline{O}_2)$ and $(O_1 \wedge \overline{O}_2)$, which we present below,

$$V = V_{(O_1 \wedge *\overline{O}_2)} + V_{(O_1 \wedge \overline{O}_2)}, \tag{68}$$

where

$$V_{(O_1 \wedge *\overline{O}_2)} = -\frac{1}{4\,s\,\mathcal{V}^2} \int_{X_6} \left[\chi \wedge *\overline{\chi} + \widetilde{\psi} \wedge *\overline{\widetilde{\psi}} + \mathcal{G}_{\alpha\beta}\, \widetilde{\Psi}^\alpha \wedge *\overline{\widetilde{\Psi}^\beta} \right. \tag{69}$$
$$\left. + \frac{i}{2}\left(\widetilde{\chi} \wedge *\overline{\widetilde{\psi}} - \overline{\widetilde{\chi}} \wedge *\widetilde{\psi} \right) + \frac{i}{2}\left(\widetilde{\Psi} \wedge *\overline{\widetilde{\chi}} - \overline{\widetilde{\Psi}} \wedge *\widetilde{\chi} \right) \right],$$

$$V_{(O_1 \wedge \overline{O}_2)} = -\frac{1}{4\,s\,\mathcal{V}^2} \int_{X_6} \left[(-i)\left(\chi \wedge \overline{\chi} + \chi \wedge \overline{\widetilde{\chi}} + 2\,\widetilde{\psi} \wedge \overline{\widetilde{\psi}} + 2\,\mathcal{G}_{\alpha\beta}\, \Psi^\alpha \wedge \overline{\widetilde{\Psi}^\beta} \right) \right.$$
$$\left. + \left(\widetilde{\chi} \wedge \overline{\widetilde{\psi}} + \overline{\widetilde{\chi}} \wedge \widetilde{\psi} \right) + \left(\widetilde{\Psi} \wedge \overline{\widetilde{\chi}} + \overline{\widetilde{\Psi}} \wedge \widetilde{\chi} \right) \right],$$

17

where the so-called tilde fluxes for χ, ψ and Ψ^α are defined as below,

$$
\widetilde{\chi} = -\left(\mathcal{S}^{\Sigma\Delta}\chi_\Delta + \mathcal{S}^\Sigma{}_\Delta\chi^\Delta\right)\mathcal{A}_\Sigma + \left(\mathcal{S}_\Sigma{}^\Delta\chi_\Delta + \mathcal{S}_{\Sigma\Delta}\chi^\Delta\right)\mathcal{B}^\Sigma, \tag{70}
$$

$$
\widetilde{\psi} = -\left(\mathcal{S}^{\Sigma\Delta}\psi_\Delta + \mathcal{S}^\Sigma{}_\Delta\psi^\Delta\right)\mathcal{A}_\Sigma + \left(\mathcal{S}_\Sigma{}^\Delta\psi_\Delta + \mathcal{S}_{\Sigma\Delta}\psi^\Delta\right)\mathcal{B}^\Sigma,
$$

$$
\widetilde{\Psi}^\alpha = -\left(\mathcal{S}^{\Sigma\Delta}\Psi^\alpha{}_\Delta + \mathcal{S}^\Sigma{}_\Delta\Psi^{\alpha\Delta}\right)\mathcal{A}_\Sigma + \left(\mathcal{S}_\Sigma{}^\Delta\Psi^\alpha{}_\Delta + \mathcal{S}_{\Sigma\Delta}\Psi^{\alpha\Delta}\right)\mathcal{B}^\Sigma
$$

and using (40) and (43) the $\mathcal{S}$ matrices are defined as

$$
\mathcal{S}^{\Lambda\Delta} = \left(\mathcal{M}^\Lambda{}_\Sigma\,\mathcal{L}^{\Sigma\Delta} + \mathcal{M}^{\Lambda\Sigma}\,\mathcal{L}_\Sigma{}^\Delta\right), \tag{71}
$$

$$
\mathcal{S}^\Lambda{}_\Delta = -\left(\mathcal{M}^\Lambda{}_\Sigma\,\mathcal{L}^\Sigma{}_\Delta + \mathcal{M}^{\Lambda\Sigma}\,\mathcal{L}_{\Sigma\Delta}\right) + \delta^\Lambda{}_\Delta,
$$

$$
\mathcal{S}_\Lambda{}^\Delta = \left(\mathcal{M}_{\Lambda\Sigma}\,\mathcal{L}^{\Sigma\Delta} + \mathcal{M}_\Lambda{}^\Sigma\,\mathcal{L}_\Sigma{}^\Delta\right) - \delta_\Lambda{}^\Delta,
$$

$$
\mathcal{S}_{\Lambda\Delta} = -\left(\mathcal{M}_{\Lambda\Sigma}\,\mathcal{L}^\Sigma{}_\Delta + \mathcal{M}_\Lambda{}^\Sigma\,\mathcal{L}_{\Sigma\Delta}\right).
$$

In order to demonstrate the use of the mater formula in the symplectic proposal (68)-(69), we get back to the toroidal type IIB model model based on $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orientifold. As we have already observed for the previous formulation using the metric of the toroidal orbifold, the symplectic formulation (65) also leads to a total of 76276 terms while being expressed in terms of the usual fluxes, however this number reduces to 10888 terms when the total scalar potential is expressed in terms of the axionic-fluxes (60), subsequently leading to the numerics about the number of terms in each of the 36 pieces as presented in Table 1. As a particular case, several scenarios can be considered by switching-off certain fluxes at a time. Moreover, in light of recovering the results from our master formula (68)-(69), we have a clear splitting of 10888 terms in the following manner,

$$
\#\left(V_{(O_1 \wedge * \overline{O}_2)}\right) = 5576, \qquad \#\left(V_{(O_1 \wedge \overline{O}_2)}\right) = 5312. \tag{72}
$$

4. Summary and conclusions

In this work we presented a brief review about the various available formulations of non-geometric scalar potential arising from the generalized flux superpotential. The basic starting point is the four-dimensional type IIB flux superpotential which is induced by the standard three-form RR and NS-NS fluxes (F, H) resulting in a cubic polynomial in complex structure moduli along with a linear dependence on the axio-dilaton modulus. A set of successive applications of S/T-dualities lead to the U-dual completion of the flux superpotential which has a cubic dependence for the complex structure moduli as well as the Kähler moduli, in addition to having a linear dependence on the axio-dilaton. We discussed multiple methods of arriving at the same $N = 1$ four-dimensional scalar potentials generated by the generalized flux superpotentials. These methods are:

- Method 1: This is the direct method of computing the scalar potential using the $N = 1$ formula (2) for a given generic form of Kähler potential K and the superpotenial W.

- Method 2: In this method the scalar potential is calculated using the metric of the internal sixfold background. For the same reason this method is applicable for the cases where the metric of the compactified space is known, e.g. the case of toroidal models.

18

- Method 3: The notion of non-geometric fluxes has been motivated by the toroidal setup, and in order to understand/promote the idea to cases like Calabi Yau compactifications, one has to understand the insights of the effective scalar potential without the need of using the internal metric. For this purpose, the role of symplectic geometries gets very crucial [45, 48] and it turns out that the scalar potential can be equivalently expressed in terms of symplectic ingredients. We presented the master formula (68) which is generically applicable to such cases.

- Method 4: After writing all the relevant quantities in terms of symplectic ingredients (68), one finds that although the formulation is quite generic and compact, making attempts for phenomenology or addressing any model building issues is still hard. This is because of the fact that the required symplectic quantities are implicit in terms of moduli dependence and it is hard to make direct use of symplectic understanding at the level of scalar potential for the search of physical vacua. In this regard, a particular type IIB setting with the absence of prime fluxes has been studied in [13, 62] which have been extended for the most generic case recently in [67]. These formulations are such that one can directly read-off the scalar potentials by using the topological data of the compactifying CY and its mirror threefold, and we hope that it can be useful for search physics vacua, including the de-Sitter ones in the vast landscape of the non-geometric fluxes.

The main aim of formulating the scalar potential in various equivalent forms is to write down the same in a compact and concise way which can be useful and easy-to-handle for the phenomenological purposes such as moduli stabilization and the search for de-Sitter vacua [106].

References

[1] J. Shelton, W. Taylor, and B. Wecht, "Nongeometric flux compactifications," *JHEP* **0510** (2005) 085, `hep-th/0508133`.

[2] J. Shelton, W. Taylor, and B. Wecht, "Generalized Flux Vacua," *JHEP* **02** (2007) 095, `hep-th/0607015`.

[3] I. Benmachiche and T. W. Grimm, "Generalized N=1 orientifold compactifications and the Hitchin functionals," *Nucl.Phys.* **B748** (2006) 200–252, `hep-th/0602241`.

[4] M. Grana, J. Louis, and D. Waldram, "SU(3) x SU(3) compactification and mirror duals of magnetic fluxes," *JHEP* **04** (2007) 101, `hep-th/0612237`.

[5] G. Aldazabal, P. G. Camara, A. Font, and L. Ibanez, "More dual fluxes and moduli fixing," *JHEP* **0605** (2006) 070, `hep-th/0602089`.

[6] D. Robbins and T. Wrase, "D-terms from generalized NS-NS fluxes in type II," *JHEP* **0712** (2007) 058, `0709.2186`.

[7] A. Font, A. Guarino, and J. M. Moreno, "Algebras and non-geometric flux vacua," *JHEP* **0812** (2008) 050, `0809.3748`.

[8] A. Guarino and G. J. Weatherill, "Non-geometric flux vacua, S-duality and algebraic geometry," *JHEP* **0902** (2009) 042, `0811.2190`.

[9] C. Hull, "A Geometry for non-geometric string backgrounds," *JHEP* **0510** (2005) 065, `hep-th/0406102`.

[10] G. Aldazabal, P. G. Camara, and J. Rosabal, "Flux algebra, Bianchi identities and Freed-Witten anomalies in F-theory compactifications," *Nucl.Phys.* **B814** (2009) 21–52, `0811.2900`.

[11] A. Kumar and C. Vafa, "U manifolds," *Phys.Lett.* **B396** (1997) 85–90, `hep-th/9611007`.

[12] C. M. Hull and A. Catal-Ozer, "Compactifications with S duality twists," *JHEP* **0310** (2003) 034, `hep-th/0308133`.

[13] P. Shukla, "Reading off the nongeometric scalar potentials via the topological data of the compactifying Calabi-Yau manifolds," *Phys. Rev.* **D94** (2016), no. 8, 086003, `1603.01290`.

[14] D. M. Lombardo, F. Riccioni, and S. Risoli, "*P* fluxes and exotic branes," *JHEP* **12** (2016) 114, `1610.07975`.

[15] D. M. Lombardo, F. Riccioni, and S. Risoli, "Non-geometric fluxes & tadpole conditions for exotic branes," *JHEP* **10** (2017) 134, `1704.08566`.

[16] E. Plauschinn, "The tadpole conjecture at large complex-structure," *JHEP* **02** (2022) 206, `2109.00029`.

[17] G. K. Leontaris and P. Shukla, "Taxonomy of scalar potential with U-dual fluxes," *Phys. Rev. D* **108** (2023), no. 12, 126020, `2308.15529`.

[18] J.-P. Derendinger, C. Kounnas, P. M. Petropoulos, and F. Zwirner, "Superpotentials in IIA compactifications with general fluxes," *Nucl.Phys.* **B715** (2005) 211–233, `hep-th/0411276`.

[19] M. Graña and D. Marques, "Gauged Double Field Theory," *JHEP* **1204** (2012) 020, `1201.2924`.

[20] G. Dibitetto, J. Fernandez-Melgarejo, D. Marques, and D. Roest, "Duality orbits of non-geometric fluxes," *Fortsch.Phys.* **60** (2012) 1123–1149, `1203.6562`.

[21] U. Danielsson and G. Dibitetto, "On the distribution of stable de Sitter vacua," *JHEP* **1303** (2013) 018, `1212.4984`.

[22] J. Blåbäck, U. Danielsson, and G. Dibitetto, "Fully stable dS vacua from generalised fluxes," *JHEP* **1308** (2013) 054, `1301.7073`.

[23] C. Damian, L. R. Diaz-Barron, O. Loaiza-Brito, and M. Sabido, "Slow-Roll Inflation in Non-geometric Flux Compactification," *JHEP* **1306** (2013) 109, 1302.0529.

[24] C. Damian and O. Loaiza-Brito, "More stable de Sitter vacua from S-dual nongeometric fluxes," *Phys.Rev.* **D88** (2013), no. 4, 046008, 1304.0792.

[25] F. Hassler, D. Lust, and S. Massai, "On Inflation and de Sitter in Non-Geometric String Backgrounds," *Fortsch. Phys.* **65** (2017), no. 10-11, 1700062, 1405.2325.

[26] M. Ihl, D. Robbins, and T. Wrase, "Toroidal orientifolds in IIA with general NS-NS fluxes," *JHEP* **0708** (2007) 043, 0705.3410.

[27] B. de Carlos, A. Guarino, and J. M. Moreno, "Complete classification of Minkowski vacua in generalised flux models," *JHEP* **1002** (2010) 076, 0911.2876.

[28] U. H. Danielsson, S. S. Haque, G. Shiu, and T. Van Riet, "Towards Classical de Sitter Solutions in String Theory," *JHEP* **09** (2009) 114, 0907.2041.

[29] J. Blåbäck, U. H. Danielsson, G. Dibitetto, and S. C. Vargas, "Universal dS vacua in STU-models," *JHEP* **10** (2015) 069, 1505.04283.

[30] G. Dibitetto, A. Guarino, and D. Roest, "Vacua Analysis in Extended Supersymmetry Compactifications," *Fortsch. Phys.* **60** (2012) 987–990, 1112.1306.

[31] C. Damian and O. Loaiza-Brito, "Two-Field Axion Inflation and the Swampland Constraint in the Flux-Scaling Scenario," *Fortsch. Phys.* **67** (2019), no. 1-2, 1800072, 1808.03397.

[32] N. Cabo Bizet, C. Damian, O. Loaiza-Brito, and D. M. Peña, "Leaving the Swampland: Non-geometric fluxes and the Distance Conjecture," *JHEP* **09** (2019) 123, 1904.11091.

[33] E. Plauschinn, "Non-geometric backgrounds in string theory," *Phys. Rept.* **798** (2019) 1–122, 1811.11203.

[34] C. Damian and O. Loaiza-Brito, "Galois groups of uplifted de Sitter vacua," 2307.08468.

[35] R. Blumenhagen, A. Font, M. Fuchs, D. Herschmann, and E. Plauschinn, "Towards Axionic Starobinsky-like Inflation in String Theory," *Phys. Lett.* **B746** (2015) 217–222, 1503.01607.

[36] R. Blumenhagen, A. Font, M. Fuchs, D. Herschmann, E. Plauschinn, Y. Sekiguchi, and F. Wolf, "A Flux-Scaling Scenario for High-Scale Moduli Stabilization in String Theory," *Nucl. Phys.* **B897** (2015) 500–554, 1503.07634.

[37] R. Blumenhagen, A. Font, M. Fuchs, D. Herschmann, and E. Plauschinn, "Large field inflation and string moduli stabilization," *PoS* **PLANCK2015** (2015) 021, 1510.04059.

[38] R. Blumenhagen, C. Damian, A. Font, D. Herschmann, and R. Sun, "The Flux-Scaling Scenario: De Sitter Uplift and Axion Inflation," *Fortsch. Phys.* **64** (2016), no. 6-7, 536–550, 1510.01522.

[39] T. Li, Z. Li, and D. V. Nanopoulos, "Helical Phase Inflation via Non-Geometric Flux Compactifications: from Natural to Starobinsky-like Inflation," *JHEP* **10** (2015) 138, 1507.04687.

[40] P. Shukla, "Revisiting the two formulations of Bianchi identities and their implications on moduli stabilization," *JHEP* **08** (2016) 146, 1603.08545.

[41] P. Shukla, "On stable type IIA de-Sitter vacua with geometric flux," *Eur. Phys. J. C* **83** (2023), no. 3, 196, 2202.12840.

[42] F. Marchesano, D. Prieto, J. Quirant, and P. Shukla, "Systematics of Type IIA moduli stabilisation," *JHEP* **11** (2020) 113, 2007.00672.

[43] F. Marchesano, D. Prieto, and M. Wiesner, "F-theory flux vacua at large complex structure," *JHEP* **08** (2021) 077, 2105.09326.

[44] D. Prieto, J. Quirant, and P. Shukla, "On the limitations of non-geometric fluxes to realize dS vacua," *JHEP* **05** (2024) 008, 2402.13899.

[45] A. Ceresole, R. D'Auria, and S. Ferrara, "The Symplectic structure of N=2 supergravity and its central extension," *Nucl.Phys.Proc.Suppl.* **46** (1996) 67–74, hep-th/9509160.

[46] T. R. Taylor and C. Vafa, "R R flux on Calabi-Yau and partial supersymmetry breaking," *Phys.Lett.* **B474** (2000) 130–137, hep-th/9912152.

[47] R. Blumenhagen, D. Lust, and T. R. Taylor, "Moduli stabilization in chiral type IIB orientifold models with fluxes," *Nucl.Phys.* **B663** (2003) 319–342, hep-th/0303016.

[48] R. D'Auria, S. Ferrara, and M. Trigiante, "On the supergravity formulation of mirror symmetry in generalized Calabi-Yau manifolds," *Nucl. Phys.* **B780** (2007) 28–39, hep-th/0701247.

[49] P. Shukla, "A symplectic rearrangement of the four dimensional non-geometric scalar potential," *JHEP* **11** (2015) 162, 1508.01197.

[50] R. Blumenhagen, A. Font, and E. Plauschinn, "Relating double field theory to the scalar potential of N = 2 gauged supergravity," *JHEP* **12** (2015) 122, 1507.08059.

[51] G. K. Leontaris and P. Shukla, "Symplectic formulation of the type IIB scalar potential with U-dual fluxes," *Phys. Rev. D* **109** (2024), no. 6, 066018, 2309.08664.

[52] R. Blumenhagen, X. Gao, D. Herschmann, and P. Shukla, "Dimensional Oxidation of Non-geometric Fluxes in Type II Orientifolds," *JHEP* **1310** (2013) 201, 1306.2761.

[53] X. Gao, P. Shukla, and R. Sun, "Symplectic formulation of the type IIA nongeometric scalar potential," *Phys. Rev.* **D98** (2018), no. 4, 046009, 1712.07310.

[54] J.-P. Derendinger, C. Kounnas, P. Petropoulos, and F. Zwirner, "Fluxes and gaugings: N=1 effective superpotentials," *Fortsch.Phys.* **53** (2005) 926–935, hep-th/0503229.

[55] G. Dall'Agata, G. Villadoro, and F. Zwirner, "Type-IIA flux compactifications and N=4 gauged supergravities," *JHEP* **0908** (2009) 018, 0906.0370.

[56] G. Aldazabal, D. Marques, C. Nunez, and J. A. Rosabal, "On Type IIB moduli stabilization and N = 4, 8 supergravities," *Nucl. Phys.* **B849** (2011) 80–111, 1101.5954.

[57] G. Aldazabal, W. Baron, D. Marques, and C. Nunez, "The effective action of Double Field Theory," *JHEP* **1111** (2011) 052, 1109.0290.

[58] D. Geissbuhler, "Double Field Theory and N=4 Gauged Supergravity," *JHEP* **1111** (2011) 116, 1109.4280.

[59] G. Villadoro and F. Zwirner, "N=1 effective potential from dual type-IIA D6/O6 orientifolds with general fluxes," *JHEP* **0506** (2005) 047, hep-th/0503169.

[60] M. Cicoli, J. P. Conlon, and F. Quevedo, "Systematics of String Loop Corrections in Type IIB Calabi-Yau Flux Compactifications," *JHEP* **01** (2008) 052, 0708.1873.

[61] M. Cicoli, I. Garcìa-Etxebarria, C. Mayrhofer, F. Quevedo, P. Shukla, and R. Valandro, "Global Orientifolded Quivers with Inflation," *JHEP* **11** (2017) 134, 1706.06128.

[62] P. Shukla, "Dictionary for the type II nongeometric flux compactifications," *Phys. Rev. D* **103** (2021), no. 8, 086009, 1909.07391.

[63] S. AbdusSalam, S. Abel, M. Cicoli, F. Quevedo, and P. Shukla, "A systematic approach to Kähler moduli stabilisation," *JHEP* **08** (2020), no. 08, 047, 2005.11329.

[64] M. Cicoli, I. n. G. Etxebarria, F. Quevedo, A. Schachner, P. Shukla, and R. Valandro, "The Standard Model quiver in de Sitter string compactifications," *JHEP* **08** (2021) 109, 2106.11964.

[65] M. Cicoli, A. Schachner, and P. Shukla, "Systematics of type IIB moduli stabilisation with odd axions," *JHEP* **04** (2022) 003, 2109.14624.

[66] G. K. Leontaris and P. Shukla, "Stabilising all Kähler moduli in perturbative LVS," *JHEP* **07** (2022) 047, 2203.03362.

[67] S. Biswas, G. K. Leontaris, and P. Shukla, "Reading-off the non-geometric scalar potentials with U-dual fluxes," 2407.15822.

[68] X. Gao and P. Shukla, "Dimensional oxidation and modular completion of non-geometric type IIB action," *JHEP* **1505** (2015) 018, 1501.07248.

[69] D. Andriot and A. Betz, "β-supergravity: a ten-dimensional theory with non-geometric fluxes, and its geometric framework," *JHEP* **1312** (2013) 083, 1306.4381.

[70] D. Andriot, M. Larfors, D. Lust, and P. Patalong, "A ten-dimensional action for non-geometric fluxes," *JHEP* **1109** (2011) 134, 1106.4015.

[71] P. Shukla, "On modular completion of generalized flux orbits," *JHEP* **11** (2015) 075, 1505.00544.

[72] P. Shukla, "Implementing odd-axions in dimensional oxidation of 4D non-geometric type IIB scalar potential," *Nucl. Phys.* **B902** (2016) 458–482, 1507.01612.

[73] D. Andriot, O. Hohm, M. Larfors, D. Lust, and P. Patalong, "A geometric action for non-geometric fluxes," *Phys.Rev.Lett.* **108** (2012) 261602, 1202.3060.

[74] D. Andriot, O. Hohm, M. Larfors, D. Lust, and P. Patalong, "Non-Geometric Fluxes in Supergravity and Double Field Theory," *Fortsch.Phys.* **60** (2012) 1150–1186, 1204.1979.

[75] D. Andriot and A. Betz, "Supersymmetry with non-geometric fluxes, or a β-twist in Generalized Geometry and Dirac operator," *JHEP* **04** (2015) 006, 1411.6640.

[76] C. D. A. Blair and E. Malek, "Geometry and fluxes of SL(5) exceptional field theory," *JHEP* **03** (2015) 144, 1412.0635.

[77] S. Gukov, C. Vafa, and E. Witten, "CFT's from Calabi-Yau four folds," *Nucl. Phys.* **B584** (2000) 69–108, hep-th/9906070. [Erratum: Nucl. Phys.B608,477(2001)].

[78] K. Dasgupta, G. Rajesh, and S. Sethi, "M theory, orientifolds and G - flux," *JHEP* **08** (1999) 023, hep-th/9908088.

[79] G. Aldazabal, E. Andres, P. G. Camara, and M. Grana, "U-dual fluxes and Generalized Geometry," *JHEP* **11** (2010) 083, 1007.5509.

[80] P. Shukla, "T-dualizing de Sitter no-go scenarios," *Phys. Rev. D* **102** (2020), no. 2, 026014, 1909.08630.

[81] P. Shukla, "Rigid nongeometric orientifolds and the swampland," *Phys. Rev. D* **103** (2021), no. 8, 086010, 1909.10993.

[82] X. Gao and P. Shukla, "On Classifying the Divisor Involutions in Calabi-Yau Threefolds," *JHEP* **1311** (2013) 170, 1307.1139.

[83] F. Carta, J. Moritz, and A. Westphal, "A landscape of orientifold vacua," *JHEP* **05** (2020) 107, 2003.04902.

[84] R. Altman, J. Carifio, X. Gao, and B. D. Nelson, "Orientifold Calabi-Yau threefolds with divisor involutions and string landscape," *JHEP* **03** (2022) 087, 2111.03078.

[85] F. Carta, A. Mininno, and P. Shukla, "Divisor topologies of CICY 3-folds and their applications to phenomenology," *JHEP* **05** (2022) 101, 2201.02165.

[86] C. Crinò, F. Quevedo, A. Schachner, and R. Valandro, "A database of Calabi-Yau orientifolds and the size of D3-tadpoles," *JHEP* **08** (2022) 050, 2204.13115.

[87] P. Shukla, "Classifying divisor topologies for string phenomenology," *JHEP* **12** (2022) 055, 2205.05215.

[88] T. W. Grimm, "Non-Perturbative Corrections and Modularity in N=1 Type IIB Compactifications," *JHEP* **0710** (2007) 004, 0705.3253.

[89] A. Font, L. E. Ibanez, D. Lust, and F. Quevedo, "Strong - weak coupling duality and nonperturbative effects in string theory," *Phys. Lett.* **B249** (1990) 35–43.

[90] M. Cvetic, A. Font, L. E. Ibanez, D. Lust, and F. Quevedo, "Target space duality, supersymmetry breaking and the stability of classical string vacua," *Nucl. Phys.* **B361** (1991) 194–232.

[91] M. Ihl and T. Wrase, "Towards a Realistic Type IIA T**6/Z(4) Orientifold Model with Background Fluxes. Part 1. Moduli Stabilization," *JHEP* **07** (2006) 027, hep-th/0604087.

[92] X. Gao, P. Shukla, and R. Sun, "On Missing Bianchi Identities in Cohomology Formulation," *Eur. Phys. J.* **C79** (2019), no. 9, 781, 1805.05748.

[93] F. Carta, F. Marchesano, W. Staessens, and G. Zoccarato, "Open string multi-branched and Kähler potentials," *JHEP* **09** (2016) 062, 1606.00508.

[94] A. Herraez, L. E. Ibanez, F. Marchesano, and G. Zoccarato, "The Type IIA Flux Potential, 4-forms and Freed-Witten anomalies," *JHEP* **09** (2018) 018, 1802.05771.

[95] D. Escobar, F. Marchesano, and W. Staessens, "Type IIA flux vacua and α'-corrections," *JHEP* **06** (2019) 129, 1812.08735.

[96] F. Marchesano and J. Quirant, "A Landscape of AdS Flux Vacua," 1908.11386.

[97] D. Lust, S. Reffert, E. Scheidegger, W. Schulgin, and S. Stieberger, "Moduli Stabilization in Type IIB Orientifolds (II)," *Nucl. Phys.* **B766** (2007) 178–231, hep-th/0609013.

[98] D. Lust, S. Reffert, E. Scheidegger, and S. Stieberger, "Resolved Toroidal Orbifolds and their Orientifolds," *Adv. Theor. Math. Phys.* **12** (2008) 67–183, hep-th/0609014.

[99] R. Blumenhagen, V. Braun, T. W. Grimm, and T. Weigand, "GUTs in Type IIB Orientifold Compactifications," *Nucl.Phys.* **B815** (2009) 1–94, 0811.2936.

[100] M. Cicoli, S. Krippendorf, C. Mayrhofer, F. Quevedo, and R. Valandro, "D-Branes at del Pezzo Singularities: Global Embedding and Moduli Stabilisation," *JHEP* **09** (2012) 019, 1206.5237.

[101] X. Gao and P. Shukla, "F-term Stabilization of Odd Axions in LARGE Volume Scenario," *Nucl.Phys.* **B878** (2014) 269–294, 1307.1141.

[102] T. W. Grimm and J. Louis, "The Effective action of N = 1 Calabi-Yau orientifolds," *Nucl.Phys.* **B699** (2004) 387–426, hep-th/0403067.

[103] S. Hosono, A. Klemm, and S. Theisen, "Lectures on mirror symmetry," hep-th/9403096. [Lect. Notes Phys.436,235(1994)].

[104] M. Arends, A. Hebecker, K. Heimpel, S. C. Kraus, D. Lust, *et al.*, "D7-Brane Moduli Space in Axion Monodromy and Fluxbrane Inflation," *Fortsch.Phys.* **62** (2014) 647–702, 1405.0283.

[105] R. Blumenhagen, D. Herschmann, and E. Plauschinn, "The Challenge of Realizing F-term Axion Monodromy Inflation in String Theory," *JHEP* **01** (2015) 007, 1409.7075.

[106] S. AbdusSalam, X. Gao, G. K. Leontaris, and P. Shukla, "On dS vacua with isotropic non-geometric fluxes," 2408.xxxxx.