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The finite-volume spectrum in the presence of a long-range force

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We derive modified Lüscher equation that enables one to extract scattering phase shifts from the lattice energy levels even in the presence of a long-range force. The method avoids large partial-wave mixing which is *a priori* expected in the presence of such force, and is applicable in the region of the left-hand cut. More details can be found in the recent publication [1].

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1. Introduction

Recently, the two-body quantization condition on the lattice in the presence of the long-range forces came under scrutiny. The interest in this problem is justified by the fact that the standard Lüscher approach [2] is not applicable for certain physical systems which are studied at present by using lattice QCD. For example, in case of the NN scattering, some low-lying energy levels may occur in the so-called left-hand cut region, where the K-matrix is complex and, hence, the use of the Lüscher equation does not make sense [3]. Furthermore, at the physical quark masses, D^* meson is unstable, and looking for the resonances on the D^*D system amounts to solving a three-body problem. However, at present, lattice calculations are carried out at larger-than-physical quark masses, for which the D^* does not decay. The $T_{cc}^+(3875)$ state, emerging in the D^*D elastic scattering, is strongly affected by a nearby left-hand cut and, again, a straightforward use of the standard Lüscher approach is not justified (see, e.g., [4]). In both physical systems, an existence of a left-hand cut close to the physical threshold can be traced back to a t- and u-channel exchange of the light particle (the pion). Finally, it is clear from the beginning that the Lüscher approach has to be modified in the presence of massless particles (photons). A possible modification has been proposed in Ref. [5] which, however, leaves some questions still open. Namely, it is based on the QED$_L$ implementation of the photon field on the lattice with periodic boundary conditions. This theory is known to be non-local and violating the Appelquist-Carazzone decoupling theorem needed for the matching to the non-relativistic effective field theory [6], which is then used for the derivation of the quantization condition. Furthermore, only the Coulomb photon exchange has been considered in Ref. [5], and it has been implemented perturbatively. Albeit the arguments were given, justifying this choice, the approximation is set to fail, obviously, at least in the close vicinity of the free-particle poles. In the present work, we do not consider massless particles and concentrate on the two physical systems, mentioned in the beginning.

It is important to realize that, in the presence of light particle exchange in the t- and u-channels that give rise to a long-range static potential between particles in the non-relativistic scattering, several basic assumptions that provide a foundation of the standard Lüscher approach, are invalidated. Namely,

- Standard Lüscher approach is based on the assumption $RL \gg 1$, where L is the spatial extent of the cubic lattice, and R denotes the range of the potential, which is inversely proportional to the mass of the exchanged particle. If this mass becomes light, one needs a very large box to fit the two-particle system in.

- As one knows, the Lüscher equation holds up to the exponentially suppressed terms that contain $\exp(-nLR)$, with $n \geq 1$. If the condition $RL \gg 1$ is not fulfilled, the exponentially suppressed corrections become sizable and introduce an uncontrollable systematic error.

- Consider, for example, the NN scattering. It is well known that the partial-wave projection of the one-pion exchange diagram has the left-hand cut, whose upper rim is located at $s_\ell = 4m^2 - M^2$ (here, m and M denote the nucleon and the pion masses, respectively). For physical masses, s_ℓ lies very close to the elastic threshold, $s_{th} = 4m^2$. Namely, $\sqrt{s_{th}} \simeq 1880\,\text{MeV}$ and $\sqrt{s_\ell} \simeq 1875\,\text{MeV}$. The NN scattering K-matrix in a given partial wave is complex below

s_ℓ and so is the scattering phase. The use of the Lüscher equation for the energy levels, measured below $\sqrt{s_\ell}$, is obviously not justified. It is important to realize that the migration of the energy levels below $\sqrt{s_\ell}$ is caused by large exponential corrections. In order to prove this, let us assume the opposite. Namely, assume that the exponential corrections are negligible. Since below threshold we are dealing with the exponential corrections only, the measured energy level would correspond, to a good accuracy, to a physical bound state on the top of the left-hand cut. Since such a bound state does not exist, our initial assumption turns out to be wrong. Last but not least, note that one expects large exponentially suppressed contributions not only in the left-hand cut region, but slightly above it as well, where the effect of the singularity is still felt. Since s_ℓ and s_{th} are so close, this, in its turn, might introduce a large systematic error in the analysis of lattice data also in the vicinity of the elastic threshold.

- Owing to the presence of the long-range force, the partial-wave expansion converges rather slowly. This leads to a large partial-wave mixing in the Lüscher equation, even above the elastic threshold, and makes it mandatory to retain the contributions of higher partial waves.

In view of the above difficulties, several alternative approaches have been recently proposed for the analysis of scattering data in the systems featuring the long-range force. The most straightforward idea is to solve directly the two-body Lippmann-Schwinger equation in a finite volume, using the plane wave basis [4, 7]. Fitting the resulting energy levels to lattice data allows one to extract the parameters of the effective Hamiltonian, which are devoid of the left-hand singularity due to the light particle exchange (because this process is explicitly included in the framework used). Furthermore, using the plane wave basis allows one to circumvent the problem related with the partial-wave mixing. A different method was proposed in Ref. [8], where the effective Hamiltonian is split in the long- and short-range parts. Last but not least, in Refs. [9, 10] the three-body formalism is used to study the D^*D scattering for different quark masses, even in the case of a stable D^* meson. An advantage of this approach is that it allows to study the extrapolation in quark masses and the impact of the inner structure of the D^* meson on the finite-volume spectrum [10]. It should be pointed out that the solution of the problem of the left-hand cut within this approach is not related with the use of the three-body formalism *per se,* but with the fact that the discretization of the spectator momentum in this formalism is equivalent to the use of the plane-wave basis in the two-body equation. To summarize, all proposed approaches have one thing in common. Namely, in the presence of the long-range force, both the scattering amplitude and the K matrix feature the left-hand cut and thus cannot be directly extracted on the lattice. The solution of the problem consists in identifying appropriate quantities (the parameters of the effective Hamiltonian, an auxiliary scattering amplitude on the short-range potential, etc.) which are devoid of the nearby singularities and thus can be determined from lattice data. The observable quantities like scattering amplitudes can be then determined by solving integral equations with an input from the lattice.

Recently, we have proposed yet another approach to the problem. The formalism derived here is almost identical to the original Lüscher equation, up to a modification of the Lüscher zeta-function, caused by the presence of the long-range force. Below, we shall present a brief description of this formalism. More details can be found in Ref. [1].

2. Modified effective-range expansion

Our formalism is based on the use of the modified effective range expansion, derived by van Haeringen and Kok in Ref. [11]. In that paper, the authors consider a non-relativistic scattering problem on a sum of long- and short-range local potentials $V(r) = V_L(r) + V_S(r)$. The long-range part is taken to be superregular, i.e., $r^{-2\ell_{\sf max}}V_L(r)$ is assumed to be finite in the limit $r \to 0$ (here, $\ell_{\sf max}$ denotes the maximally allowed value of the orbital angular momentum). Next, a modified effective-range function has been defined

$$K_\ell^M(q^2) = M_\ell(q) + \frac{q^{2\ell+1}}{|f_\ell(q)|^2}\,(\cot(\delta_\ell(q) - \sigma_\ell(q)) - i)\,. \tag{1}$$

Here, $\ell \le \ell_{\sf max}$ is the angular momentum and $\delta_\ell(q), \sigma_\ell(q)$ denote, respectively, the full phase shift and the phase shift in the problem with the long-range potential $V_L(r)$ only. Furthermore,

$$M_\ell(q) = \frac{1}{\ell!}\left(-\frac{iq}{2}\right)^\ell \lim_{r\to 0}\frac{d^{2\ell+1}}{dr^{2\ell+1}}\,r^\ell\,\frac{f_\ell(q,r)}{f_\ell(q)}\,, \tag{2}$$

where $f_\ell(q,r)$ is the Jost solution for the long-range potential, and

$$f_\ell(q) = \frac{q^\ell e^{-i\ell\pi/2}(2\ell+1)}{(2\ell+1)!!}\,\lim_{r\to 0} r^\ell f_\ell(q,r)\,. \tag{3}$$

The main result of Ref. [11] consists in demonstrating that the quantity $K_\ell^M(q^2)$, in difference to the full K-matrix $K_\ell(q^2)$, does not feature the left-hand singularity and is thus an appropriate quantity to be determined from the lattice spectrum. On the other hand, after fitting $K_\ell^M(q^2)$ to lattice data, it is straightforward to determine the full phase shift $\delta_\ell(q)$ from Eq. (1). This step does not imply solving an integral equation, since Eq. (1) is an algebraic relation between $K_\ell^M(q^2)$ and $\delta_\ell(q)$.

3. NREFT framework

In order to set up the finite-volume formalism, it is convenient to reformulate the two-body scattering problem within non-relativistic effective field theory (NREFT) framework. In this framework, the short-range potential is given by a string of terms containing three-dimensional δ-function and the derivatives thereof. In the momentum space, one may write

$$\langle \boldsymbol{p}|V_S|\boldsymbol{q}\rangle = C_0^{00} + 3C_1^{00}\boldsymbol{p}\boldsymbol{q} + C_0^{10}(\boldsymbol{p}^2 + \boldsymbol{q}^2) + \cdots \tag{4}$$

where C_0^{00}, C_1^{00}, $C_0^{10}, \ldots$ stand for the effective couplings. Using the two-potential scattering theory, the full T-matrix can be written in the form

$$T = T_L + (1 + T_L G_0)T_S(G_0 T_L + 1)\,, \tag{5}$$

where G_0 denotes the free two-particle Green function, and T_L, T_S are defined through the equations

$$\begin{aligned} T_L &= V_L + V_L G_0 T_L\,,\\ T_S &= V_S + V_S G_L T_S\,,\\ G_L &= G_0 + G_0 V_L G_L\,. \end{aligned} \tag{6}$$

Considering, for simplicity, the case, when there are no bound states in the potential V_L, the following spectral representation for the Green function can be written down:

$$\langle \boldsymbol{p}|G_L(q_0^2)|\boldsymbol{q}\rangle = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{p})\left(\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{q})\right)^*}{\boldsymbol{k}^2-q_0^2-i\varepsilon}\,. \tag{7}$$

Here q_0 is the magnitude of the on-shell three-momentum, and $\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{p})$ denotes the scattering wave function.

In order to explain the essence of the method, let us restrict ourselves to the lowest order approximation $C_0^{00}\neq 0,\ C_1^{00}=C_0^{10}=\cdots=0$. An exact solution for the matrix T_S is then given by

$$\langle \boldsymbol{p}|T_S(q_0^2)|\boldsymbol{q}\rangle = \frac{1}{(C_0^{00})^{-1}-\langle G_L(q_0^2)\rangle}\,,\qquad \langle G_L(q_0^2)\rangle = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{d^3\boldsymbol{q}}{(2\pi)^3}\,\langle \boldsymbol{p}|G_L(q_0^2)|\boldsymbol{q}\rangle\,. \tag{8}$$

Note that the scattering occurs only in the S-wave.

In order to arrive at the result given in Ref. [11], one needs two more relations:

$$\langle G_L(q_0^2)\rangle = \frac{1}{4\pi}\,M_0(q_0) + \text{real polynomial in } q_0^2\,,\qquad \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\left|\psi_{\boldsymbol{k}}^{(+)}(\boldsymbol{p})\right| = \frac{1}{|f_0(k)|}\,, \tag{9}$$

where the quantities $M_\ell(q),\ f_\ell(q)$ were defined in Eqs. (2) and (3). The polynomial in Eq. (9) can be safely dropped – it amounts to a choice of a particular renormalization prescription.

Using now the explicit expression (8), it is straightforward to obtain that

$$\frac{4\pi}{C_0^{00}} = M_0(q_0) + \frac{q_0}{|f_0(q_0)|^2}\,\left(\cot(\delta_0(q_0)-\sigma_0(q_0))-i\right)\,. \tag{10}$$

The physical meaning of the modified effective range function $K_\ell^M(q_0^2)$ becomes crystal clear now. Namely, at lowest order, it is given by the effective range expansion in case of *only* the short-range potential V_S. In higher orders, and higher partial waves, the coefficients of this expansion are modified by long-range interactions. Still, the main message is the same: the scale that determines the radius of convergence of $K_\ell^M(q_0^2)$, is defined solely by the short-range part of the potential.

4. Modified Lüscher equation

In order to obtain the quantization condition, it suffices to discretize the three-momentum integrals in the two-body scattering equation. We skip the intermediate steps and display the final result for the modified Lüscher equation in the presence of the long-range force. It has the form $\det\mathscr{A}=0$, where

$$\mathscr{A}_{\ell m,\ell'm'}(q_0) = \delta_{\ell\ell'}\delta_{mm'}K_\ell^M(q_0^2) - H_{\ell m,\ell'm'}(q_0)\,. \tag{11}$$

Here, $H_{\ell m,\ell'm'}(q_0)$ can be loosely termed as the modified Lüscher zeta-function:

$$H_{\ell m,\ell'm'}(q_0) = \frac{4\pi}{L^6}\sum_{\boldsymbol{p},\boldsymbol{q}} \mathscr{Y}^*_{\ell m}(\boldsymbol{p})\langle \boldsymbol{p}|G_L(q_0^2)|\boldsymbol{q}\rangle\mathscr{Y}_{\ell'm'}(\boldsymbol{q})\,, \tag{12}$$

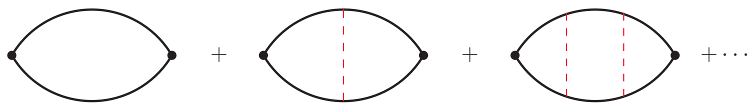

Figure 1: Modified Lüscher zeta-function: a loop with any number of insertions of the long-range potential. Standard zeta-function is given by the first term only

where $\mathscr{Y}_{\ell m}(\boldsymbol{p}) = p^\ell Y_{\ell m}(\hat{p})$ and $Y_{\ell m}(\hat{p})$ denotes the spherical function. Note that the expression for the standard zeta-function contains G_0 instead of G_L. Thus, the modified zeta function is given by a loop with any number of insertions of the long-range potential, see Fig. 1.

The analysis of lattice data proceeds pretty much along the same pattern as in case of an ordinary Lüscher equation. The key to this conclusion is the assumption that the form of the long-range potential is exactly known and its parameters can be determined independently and at a high precision. Then, $H_{\ell m,\ell' m'}(q_0)$ can be calculated prior to the analysis. The fit to the lattice data allows one to extract $K_\ell^M(q_0^2)$, which is algebraically related to the full phase shift, see Eq. (1). Note also that $K_\ell^M(q_0^2)$ is devoid of the left-hand. Furthermore, the modified quantization condition is obtained from the Lippmann-Schwinger equation, containing short-range potential only and, therefore, does not suffer from large partial-wave mixing.

5. Conclusions

We propose a formulation of the two-body quantization condition in the presence of the long-range force. The analysis of lattice data within this approach closely follows the standard pattern based on the use of the ordinary Lüscher equation and thus should be straightforward to implement in practice.

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References

[1] R. Bubna, H. W. Hammer, F. Müller, J. Y. Pang, A. Rusetsky and J. J. Wu, *Lüscher equation with long-range forces*, *JHEP* **05** (2024) 168 [2402.12985]

[2] M. Lüscher, *Two particle states on a torus and their relation to the scattering matrix*, *Nucl. Phys. B* **354** (1991) 531.

[3] J. R. Green, A. D. Hanlon, P. M. Junnarkar and H. Wittig, *Weakly bound H dibaryon from $SU(3)$-flavor-symmetric QCD*, *Phys. Rev. Lett.* **127** (2021) 242003 [2103.01054].

[4] L. Meng, V. Baru, E. Epelbaum, A. A. Filin and A. M. Gasparyan, *Solving the left-hand cut problem in lattice QCD: $T_{cc}^{+}(3875)$ from finite volume energy levels*, *Phys. Rev. D* **109** (2024) L071506 [2312.01930].

[5] S. R. Beane and M. J. Savage, *Two-Particle Elastic Scattering in a Finite Volume Including QED*, *Phys. Rev. D* **90** (2014) 074511 [1407.4846].

[6] Z. Fodor, C. Hoelbling, S. D. Katz, L. Lellouch, A. Portelli, K. K. Szabo and B. C. Toth, *Quantum electrodynamics in finite volume and nonrelativistic effective field theories*, *Phys. Lett. B* **755** (2016) 245 [1502.06921].

[7] L. Meng and E. Epelbaum, *Two-particle scattering from finite-volume quantization conditions using the plane wave basis*, *JHEP* **10** (2021) 051 [2108.02709].

[8] A. B. Raposo and M. T. Hansen, *Finite-volume scattering on the left-hand cut*, *JHEP* **08** (2024) 075 [2311.18793].

[9] M. T. Hansen, F. Romero-López and S. R. Sharpe, *Incorporating $DD\pi$ effects and left-hand cuts in lattice QCD studies of the $T_{cc}^{+}(3875)$*, *JHEP* **06** (2024) 051 [2401.06609].

[10] S. M. Dawid, F. Romero-López and S. R. Sharpe, *Finite- and infinite-volume study of $DD\pi$ scattering*, *JHEP* **01** (2025) 060 [2409.17059].

[11] H. van Haeringen and L. P. Kok, *Modified effective range function*, *Phys. Rev. A* **26** (1982) 1218.