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Structure-dependent electromagnetic finite-volume effects to the hadronic vacuum polarisation

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In this talk we present some preliminary results and discuss the prospects of determining the leading structure-dependent finite-volume effects in the hadronic vacuum polarisation associated to order e^2 electromagnetic corrections. In the quantum electrodynamics prescription $\mathrm{QED_L}$ these arise at order $1/L^3$ in the large-volume expansion, which is also the leading order because of the neutrality of the currents defining the underlying correlation function. Knowing the size of the finite-volume effects in question is relevant for determinations of the leading isospin-breaking corrections to the muon anomalous magnetic moment coming from the hadronic vacuum polarisation.

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1. Introduction

The muon anomalous magnetic moment, $a_\mu = (g-2)_\mu/2$ where g is the gyromagnetic ratio, has historically attracted much attention as a potential place to discover new physics [1]. The apparent tension between theory and experiment that for long had persisted is now questioned, given recent years' progress in e.g. lattice quantum chromodynamics (QCD) [2, 3]. It is important to understand the origin of the issue, and also for additional lattice QCD calculations to predict the hadronic vacuum polarisation (HVP) contribution to a_μ including isospin-breaking effects[1].

Isospin-breaking effects arising from non-degenerate light-quark masses and electromagnetism typically enter as per cent level corrections, meaning that they have to be included for precision goals beyond that. The long-range nature of the electromagnetic effects forbids charged states in finite-volume spacetimes with periodic boundary conditions [4]. This underlying problem is related to Gauss' law, the absence of a mass gap in quantum electrodynamics (QED) and photon zero-momentum modes [4]. However, the issue can be circumvented by defining finite-volume prescriptions for QED, such as $\mathrm{QED_L}$ and infrared-improvement schemes [4–7], $\mathrm{QED_C}$ [8], $\mathrm{QED_M}$ [9] and $\mathrm{QED_\infty}$ [10, 11].

For $\mathrm{QED_L}$ and $\mathrm{QED_C}$, there generally are finite-volume effects (FVEs) scaling as inverse powers of the spatial volume extent, $1/L$. To extract physical predictions from lattice data, it is often useful to analytically subtract FVEs determined using effective field theory techniques [5–7, 12–15] and fit the residual volume-dependence from the numerical data. For the HVP, it was in Ref. [14] shown using pointlike scalar $\mathrm{QED_L}$ that the leading effects start at order $1/L^3$, as expected from the neutrality of the current [16]. Moreover, Ref. [14] argued from the analytical properties of the hadronic light-by-light tensor [17, 18] that the internal structure of the pion does not alter the cancellation at order $1/L^2$. In this talk, we take the first steps to determine the leading structure-dependent FVEs for the HVP in $\mathrm{QED_L}$, arising at order $1/L^3$ in the large-volume expansion.

2. Structure dependence in finite-volume effects

Let us consider an observable $O(L)$ in lattice QCD+QED. We will neglect finite-time effects and only consider continuous Euclidean spacetime geometries $\mathbb{R}\times L^3$. The FVEs are given by $\Delta O(L) = O(L) - O(\infty)$. The volume dependence can be obtained from a skeleton expansion of the underlying correlation function, which will generate a set of Feynman diagrams with one-particle irreducible vertex functions that depend on physical particle properties such as masses, charges and structure in terms of form factors [15].[2] At leading order in QED, i.e. order e^2, diagrams with virtual QED corrections will contain one photon, and consequently $\Delta O(L)$ can be written

$$\Delta O(L) = \left\{ \frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right\} \int \frac{dk_0}{2\pi} \frac{f_O(k_0,\mathbf{k})}{k_0^2+\mathbf{k}^2}\,, \tag{1}$$

where the photon momentum $k = (k_0, \mathbf{k})$ has been introduced. Our choice of $\mathrm{QED_L}$ here is manifested in terms of the absence of $\mathbf{k} = \mathbf{0}$ in the sum. The function $f_O(k_0, \mathbf{k})$ depends on the observable O and in particular the physical properties of the particles in the process. One should further note that $f_O(k_0, \mathbf{k})$ can contain analytical structure in k_0 as well, in particular poles from intermediate particles propagating and branch-cuts from form factors. Assuming that there are no

[1] There were several talks about this at the conference.

[2] For an alternative but equivalent procedure, see the talk [19] at this conference.

infrared divergences in O or external spatial momenta, in a large-L expansion the quantity $\Delta O(L)$ takes the form [6, 7, 15]

$$\Delta O(L) = \frac{c_2\, A_2}{m_\pi L} + \frac{c_1\, A_1}{(m_\pi L)^2} + \frac{c_0\, A_0}{(m_\pi L)^3} + \ldots . \tag{2}$$

Here exponentially suppressed terms $e^{-m_\pi L}$ as well as power-suppressed effects of order $1/(m_\pi L)^4$ have been neglected. The c_j in the numerators are dimensionless finite-volume coefficients defined e.g. in Refs. [5, 15]. The A_j contain the physics, in particular structure. It was observed in Ref. [15] that A_0 contains structure-dependent contributions associated to branch-cuts in the underlying correlation function. These cuts are difficult to estimate, which means that it is challenging to subtract $\Delta O(L)$ in analyses of lattice data beyond order $1/(m_\pi L)^2$, see e.g. Refs. [6, 7, 15, 20]. As will be discussed below, for the HVP the expansion starts at order $1/(m_\pi L)^3$, meaning that unless one pins down the structure-dependence and cuts, the full leading correction cannot be subtracted.

3. Hadronic vacuum polarisation

The HVP is defined as the vector-vector 2-point function

$$\Pi_{\mu\nu}(q) = \int d^4x\, e^{iq\cdot x} \langle 0|\, T\big[J_\mu(x) J_\nu^\dagger(0)\big]\, |0\rangle , \tag{3}$$

where $J_\mu(x)$ is the electromagnetic current and $q = (q_0, \vec{q})$ is an external momentum. In the following, we will consider the kinematical setting $q = (q_0, \mathbf{0})$ with $q^2 > 0$ in Euclidean space. From the Ward-Takahashi identity $q_\mu \Pi_{\mu\nu} = 0$ it follows that $\Pi_{\mu\nu}(q^2)$ is transverse, namely $\Pi_{\mu\nu}(q^2) = \big(q_\mu q_\nu - q^2\delta_{\mu\nu}\big)\, \Pi(q^2)$. We are interested in the subtracted and hence ultraviolet finite quantity

$$\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0) = \frac{1}{3q_0^2}\sum_{j=1}^{3}\Big(\Pi_{jj}(q^2) - \Pi_{jj}(0)\Big) . \tag{4}$$

The corresponding FVEs are then given by[3] $\Delta\hat{\Pi}(q^2, L) = \hat{\Pi}(q^2, L) - \hat{\Pi}(q^2, \infty)$. At order e^2, there are 12 diagrammatic topologies contributing, here shown in Fig. 1, but in practice only 7 are independent (A, B, E, C, T, S and X). Separated into diagrams, we have

$$\begin{aligned}\Delta\hat{\Pi}(q^2, L) = {} & \Delta\hat{\Pi}_A(q^2, L) + \Delta\hat{\Pi}_B(q^2, L) + 2\,\Delta\hat{\Pi}_E(q^2, L) + 4\,\Delta\hat{\Pi}_C(q^2, L) + 2\,\Delta\hat{\Pi}_T(q^2, L) \\ & + \Delta\hat{\Pi}_S(q^2, L) + \Delta\hat{\Pi}_X(q^2, L) .\end{aligned} \tag{5}$$

The vertices in the Feynman diagrams of Fig. 1 correspond to the structure-dependent irreducible vertex functions obtained from a skeleton expansion of the correlation function in Eq. (3). These all contain two pions and in addition one or two photons, which we respectively denote $\Gamma_\mu(p, k)$ and $\Gamma_{\mu\nu}(p, k, q)$ for incoming pion momentum p, incoming photon momentum k and outgoing photon momentum q. The form-factor decompositions of the vertex functions are known from virtual Compton scattering [21], and given by

$$\begin{aligned}\Gamma_{\mu\nu}(p, k, q) &= 2\delta_{\mu\nu}[1 - F(k^2) - F(q^2)] - 2\,k_\mu k_\nu\,\frac{1 - F(k^2)}{k^2} - 2\,q_\mu q_\nu\,\frac{1 - F(q^2)}{q^2} + \Gamma^{\mathrm{T}}_{\mu\nu}(p, k, q) . \\ \Gamma_\mu(p, k) &= (2p + k)_\mu\, F(k^2) + k_\mu\,\frac{(p + k)^2 - p^2}{k^2}\,[1 - F(k^2)],\end{aligned} \tag{6}$$

[3] Although this choice differs from the time-momentum representation approach typically employed in lattice calculations of the HVP [1], the FVEs can be used when integrating $\Delta\hat{\Pi}(q^2, L)$ with the appropriate kernel to get the contribution to a_μ.

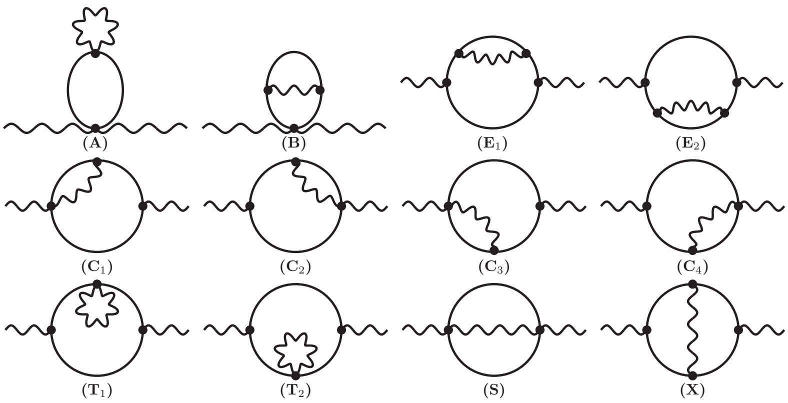

Figure 1: The 12 diagrams contributing at order e^2.

Here $F(k^2)$ is the electromagnetic form factor of the pion, with $F(0) = 1$ being the charge and $F'(0) = \langle r_\pi^2 \rangle/6$ proportional to the charge radius. The function $\Gamma^{\mathrm{T}}_{\mu\nu}(p,k,q)$ is transverse with respect to the photon momenta, i.e. $k_\mu \Gamma^{\mathrm{T}}_{\mu\nu}(p,k,q) = -q_\nu \Gamma^{\mathrm{T}}_{\mu\nu}(p,k,q) = 0$, and is purely structure dependent. There are 5 form factors in this transverse quantity [21] $G_1, G_2, ..., G_5$, each being a function of $k \cdot q$, k^2+q^2, k^2-q^2, $(k+q)\cdot(2p+k-q)$. Physically these are e.g. related to the pion electromagnetic polarisabilities $\bar{\alpha}$ and $\bar{\beta}$. For brevity, we refrain from writing down the whole expression which can be obtained from section IV of Ref. [21].[4] The pointlike scalar QED calculation in Ref. [14] can be obtained from these vertex functions through the limit $F(k^2) = F(q^2) = 1$ and $\Gamma^{\mathrm{T}}_{\mu\nu}(p,k,q) = 0$.

Having thus defined the structure-dependent vertex functions, we may proceed to evaluate the respective contributions $\Delta\hat{\Pi}_U(q^2, L)$. Since these diagrams have two loops and the pions are also in finite-volume, $\Delta\hat{\Pi}_U(q^2, L)$ takes the form

$$\Delta\hat{\Pi}_U = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq \mathbf{0}} \frac{1}{L^3} \sum_{\boldsymbol{\ell}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi} \frac{d\ell_0}{2\pi} \, \hat{\pi}_U \left(q_0^2, k, \ell \right) . \tag{7}$$

Here $\ell = (\ell_0, \boldsymbol{\ell})$ is the pion momentum, and $\hat{\pi}_U \left(q_0^2, k, \ell \right)$ is the integrand of Feynman diagram U in Fig. 1. From the kinematical choice $q^2 > 0$, there are no kinematical singularities in the integrand associated to the pions, and we may therefore replace the sum over $\boldsymbol{\ell}$ with an integral, which is valid up to corrections exponentially suppressed with the volume [14]. We thus have the simple sum-integral difference over the photon momentum as in Eq. (1), i.e.

$$\Delta\hat{\Pi}_U(q^2, L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq \mathbf{0}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} \int \frac{dk_0}{2\pi} \frac{d\ell_0}{2\pi} \, \hat{\pi}_U \left(q_0^2, k, \ell \right) + \dots . \tag{8}$$

As examples of integrands, we have diagrams E and X,

$$\hat{\pi}_E \left(q_0^2, k, \ell \right) = \frac{1}{3q_0^2} \left\{ \frac{\Gamma_j(\ell - q, q)\, \Gamma_\mu(\ell, k) \Gamma_\mu(k+\ell, -k)\, \Gamma_j(\ell, -q)}{k^2 \, [\ell^2 + m_\pi^2]^2 \, [(k+\ell)^2 + m_\pi^2] \, [(q-\ell)^2 + m_\pi^2]} \right\}_q , \tag{9}$$

$$\hat{\pi}_X \left(q_0^2, k, \ell \right) = \frac{1}{3q_0^2} \left\{ \frac{\Gamma_j(\ell - q, q)\, \Gamma_\mu(\ell, k) \Gamma_\mu(k+\ell, -q)\, \Gamma_j(\ell + k - q, -k)}{k^2 \, [\ell^2 + m_\pi^2] \, [(k+\ell)^2 + m_\pi^2] \, [(q-\ell)^2 + m_\pi^2] \, [(q-k-\ell)^2 + m_\pi^2]} \right\}_q . \tag{10}$$

[4]Eq. (6) only depends on on-shell information, which is a choice since $\Delta\hat{\Pi}(q^2, L)$ only can depend on physical quantities. Consequently, it would be independent of any off-shellness in the form factors [12, 13, 15], as can be understood from the equivalence between the skeleton expansion and on-shell approaches such as in the talk [19].

Here the subscript q indicates that whatever is in the curly brackets has to have a subtraction at $q^2 = 0$ in accordance with Eq. (4).

In the pointlike scalar QED limit, one should find [14]

$$\Delta\hat{\Pi}(q^2) \overset{\text{point}}{=} \frac{c_0}{(m_\pi L)^3}\left(\frac{16}{3}\,\Omega_{0,3} + \frac{5}{3}\,\Omega_{2,2} - \frac{40}{9}\,\Omega_{2,3} + \frac{3}{8}\,\Omega_{4,1} - \frac{7}{6}\,\Omega_{4,2} - \frac{8}{9}\,\Omega_{4,3}\right). \tag{11}$$

where the integrals $\Omega_{i,j} = \Omega_{i,j}(q_0^2/m_\pi^2)$ are defined in Ref. [14]. Diagram by diagram there are also $1/(m_\pi L)^2$ terms, but these cancel in the full sum due to the neutrality of the currents in the HVP [14, 16]. As was also argued in Refs. [14, 16], even in the structure-dependent case we should see that $\Delta\hat{\Pi}(q^2)$ starts at order $1/(m_\pi L)^3$. We briefly note that in $\mathrm{QED_r}$ [6, 7] and $\mathrm{QED_C}$ [8] the leading effects start at order $1/(m_\pi L)^4$, since the equivalent of c_0 there is zero.

4. Towards an evaluation of the finite-volume effects

Next we discuss the prospects of evaluating the leading FVEs including structure dependence in $\Delta\hat{\Pi}(q^2, L)$, and report on some preliminary findings. It should be noted that the k_0 and ℓ_0 integrals in Eq. (8) pick up all the analytical structure in the integrand $\tilde{\pi}_U\left(q_0^2, k, \ell\right)$, i.e. both pole and branch-cuts. We may then separate $\Delta\hat{\Pi}(q^2, L)$ into pure pole contributions and a remainder,

$$\Delta\hat{\Pi}(q^2, L) = \Delta_{\text{poles}}\hat{\Pi}(q^2, L) + \Delta_{\text{rem}}\hat{\Pi}(q^2, L)\,. \tag{12}$$

The pole contributions can be directly evaluated from the singularities in k_0 and ℓ_0 from the propagators in the integrands $\tilde{\pi}_U\left(q_0^2, k, \ell\right)$. Doing this and a large-volume expansion one then obtains e.g., for the sum of diagrams E and X,

$$\begin{aligned}\Delta_{\text{poles}}\hat{\Pi}_{E+X}(q^2, L) = \frac{c_1}{24\pi z\,(m_\pi L)^2}&\Bigg\{4\Big[4z^2\,\Omega_{1,3} - 7z^2\,\Omega_{3,3} + 3z^2\,\Omega_{5,3} - 96z\,\Omega_{1,3}\\ &+ 40z\,\Omega_{3,3} + 8(7z-66)\,\Omega_{-1,3} + 288\Omega_{-3,3} + 240\,\Omega_{1,3}\Big]F(q_0^2)^2\\ &- 3\Big[6z^3\,\Omega_{3,3} - 11z^3\,\Omega_{5,3} + 5z^3\,\Omega_{7,3} + 72z^2\,\Omega_{1,3} - 132z^2\,\Omega_{3,3} + 60z^2\,\Omega_{5,3}\\ &- 528z\,\Omega_{1,3} + 240z\,\Omega_{3,3} + 32(9z-22)\,\Omega_{-1,3} + 384\,\Omega_{-3,3} + 320\,\Omega_{1,3}\Big]\Bigg\} + \frac{c_0\,C_{E+X}^{\text{poles}}}{(m_\pi L)^3}\,.\end{aligned} \tag{13}$$

Here $z = q_0^2/m_\pi^2$. We have left out an explicit expression of the structure dependent C_{E+X}^{poles} due to its length. By setting $F(q_0^2) = 1$ and $F'(0) = 0$ for the $1/(m_\pi L)^3$ contribution we regain the pointlike scalar QED result from Ref. [14]. One may proceed in this way to evaluate the full $\Delta_{\text{poles}}\hat{\Pi}(q^2, L)$ through order $1/(m_\pi L)^3$, which in the pointlike limit should give back Eq. (11).

The issue to evaluate $\Delta_{\text{rem}}\hat{\Pi}(q^2, L)$ remains, and is crucial since it is structure-dependent as well and can give cancellations with $\Delta_{\text{poles}}\hat{\Pi}(q^2, L)$, thus altering the size of $\Delta\hat{\Pi}(q^2, L)$. To obtain $\Delta_{\text{rem}}\hat{\Pi}(q^2, L)$ we propose to exploit the connection between the HVP and the hadronic light-by-light tensor $\Pi_{\mu\nu\rho\sigma}(q_1, q_2, q_3, q_4)$ in forward kinematics,

$$\hat{\Pi}(q^2) = \frac{1}{3q_0^2}\int \frac{d^4k}{(2\pi)^4}\,\frac{\hat{\Pi}_{jj\mu\mu}(q, -q, k, -k)}{k^2}\,. \tag{14}$$

In Ref. [22] the above relation is rewritten in terms of a dispersive sum rule for $\gamma^*\gamma^*$-fusion cross sections. Moreover, in Refs. [17, 18] it was proven that the two-pion contribution in the dispersive representation of the hadronic light-by-light is in one-to-one correspondence with the Feynman diagram approach involving two pions (equivalent to the approach here). The proposed way forward is therefore to evaluate $\Delta_{\text{poles}}\hat{\Pi}(q^2, L)$ using the form-factor decompositions in Eq. (6), and $\Delta_{\text{rem}}\hat{\Pi}(q^2, L)$ by connecting it to dispersion theory for the hadronic light-by-light.

5. Conclusions

We have discussed the prospects of evaluating the leading FVEs to the HVP in QED_L. This is an extension of Ref. [14] to include also structure dependence through form factors as in Ref. [15]. The work is relevant for evaulation of the leading isospin-breaking corrections to the HVP contribution to the muon anomalous magnetic moment.

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References

[1] T. Aoyama et al., *The anomalous magnetic moment of the muon in the Standard Model*, *Phys. Rept.* **887** (2020) 1 [2006.04822].

[2] S. Borsanyi et al., *Leading hadronic contribution to the muon magnetic moment from lattice QCD*, *Nature* **593** (2021) 51 [2002.12347].

[3] A. Boccaletti et al., *High precision calculation of the hadronic vacuum polarisation contribution to the muon anomaly*, 2407.10913.

[4] M. Hayakawa and S. Uno, *QED in finite volume and finite size scaling effect on electromagnetic properties of hadrons*, *Prog. Theor. Phys.* **120** (2008) 413 [0804.2044].

[5] Z. Davoudi, J. Harrison, A. Jüttner, A. Portelli and M.J. Savage, *Theoretical aspects of quantum electrodynamics in a finite volume with periodic boundary conditions*, *Phys. Rev. D* **99** (2019) 034510 [1810.05923].

[6] N. Hermansson-Truedsson, M. Di Carlo, M.T. Hansen and A. Portelli, *Structure-dependent electromagnetic finite-volume effects through order* $1/L^3$, *PoS* **LATTICE2023** (2024) 265 [2310.13358].

[7] M. Di Carlo, *Isospin-breaking corrections to weak decays: the current status and a new infrared improvement*, *PoS* **LATTICE2023** (2024) 120 [2401.07666].

[8] B. Lucini, A. Patella, A. Ramos and N. Tantalo, *Charged hadrons in local finite-volume QED+QCD with C boundary conditions*, *JHEP* **02** (2016) 076 [1509.01636].

[9] M.G. Endres, A. Shindler, B.C. Tiburzi and A. Walker-Loud, *Massive photons: an infrared regularization scheme for lattice QCD+QED*, *Phys. Rev. Lett.* **117** (2016) 072002 [1507.08916].

[10] N. Asmussen, J. Green, H.B. Meyer and A. Nyffeler, *Position-space approach to hadronic light-by-light scattering in the muon $g-2$ on the lattice*, *PoS* **LATTICE2016** (2016) 164 [1609.08454].

[11] X. Feng and L. Jin, *QED self energies from lattice QCD without power-law finite-volume errors*, *Phys. Rev. D* **100** (2019) 094509 [1812.09817].

[12] BMW collaboration, *Ab initio calculation of the neutron-proton mass difference*, *Science* **347** (2015) 1452 [1406.4088].

[13] V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo, S. Simula and N. Tantalo, *Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD*, *Phys. Rev. D* **95** (2017) 034504 [1611.08497].

[14] J. Bijnens, J. Harrison, N. Hermansson-Truedsson, T. Janowski, A. Jüttner and A. Portelli, *Electromagnetic finite-size effects to the hadronic vacuum polarization*, *Phys. Rev. D* **100** (2019) 014508 [https://arxiv.org/abs/1903.10591].

[15] M. Di Carlo, M.T. Hansen, A. Portelli and N. Hermansson-Truedsson, *Relativistic, model-independent determination of electromagnetic finite-size effects beyond the pointlike approximation*, *Phys. Rev. D* **105** (2022) 074509 [2109.05002].

[16] D. Giusti, V. Lubicz, G. Martinelli, F. Sanfilippo and S. Simula, *Strange and charm HVP contributions to the muon $(g-2)$ including QED corrections with twisted-mass fermions*, *JHEP* **10** (2017) 157 [1707.03019].

[17] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, *Dispersion relation for hadronic light-by-light scattering: theoretical foundations*, *JHEP* **09** (2015) 074 [1506.01386].

[18] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, *Dispersion relation for hadronic light-by-light scattering: two-pion contributions*, *JHEP* **04** (2017) 161 [1702.07347].

[19] Di Carlo, M., *On-shell derivation of QED finite-volume effects*, *PoS* **LATTICE2024** (2024) 273.

[20] P. Boyle et al., *Isospin-breaking corrections to light-meson leptonic decays from lattice simulations at physical quark masses*, *JHEP* **02** (2023) 242 [2211.12865].

[21] H.W. Fearing and S. Scherer, *Virtual Compton scattering off spin zero particles at low-energies*, *Few Body Syst.* **23** (1998) 111 [nucl-th/9607056].

[22] V. Biloshytskyi, E.-H. Chao, A. Gérardin, J.R. Green, F. Hagelstein, H.B. Meyer et al., *Forward light-by-light scattering and electromagnetic correction to hadronic vacuum polarization*, *JHEP* **03** (2023) 194 [2209.02149].