PROCEEDINGS OF SCIENCE

Eigenspectra of Minimally Doubled Fermions

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In this work, we explored the eigenspectra of minimally doubled fermions, in both Karsten-Wilczek and Borici-Creutz realizations. We generated 4-dim $SU(3)$ gauge fields with a definite topological charge and calculated the chiralities of the eigenmodes for KW and BC fermions. We used the spectral flow of the eigenvalues for this purpose and demonstrated the Index theorem.

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1. Introduction

Minimally doubled fermions (MDF) are proposed as the local actions with chiral symmetry on the lattice. In the presence of a background gauge field with an integer-valued topological charge, they should satisfy the Atiyah-Singer index theorem. Two popular MDF formulations, namely Karsten-Wilczek (KW) [1, 2] and Borici-Creutz (BC) [3, 4] fermions in 2-dim are shown to satisfy the index theorem [5–9]. To make a case for MDFs in full QCD simulations, it is important to investigate the same in 4-dim. In this work, we investigate the index theorem for both KW and BC fermions in 4-dim with $SU(3)$ gauge field, having a specific topological charge in the background by employing the spectral flow of the eigenvalues of the corresponding Hamiltonian. We observe that γ_5 is not the appropriate operator to identify the chirality of the zero modes. The modified chirality operators with the flavored mass term have been defined for both KW and BC fermions, which are found to reproduce correct chiral states of the doublers or taste as it is often called.

2. KW and BC fermions

According to Nielsen-Ninomiya theorem, chiral fermion on lattice comes with an extra number of fermion species or doublers. The minimum number of doubler fermions on the lattice one can have is two. The two most popular variants of minimally doubled fermions (MDF) are Karsten-Wilczek and Borici-Creutz fermions [1–4]. Both actions add terms to the naive fermion action that anticommutes with γ_5, and hence, the changed actions remain chiral, each with two chiral zero modes. In KW fermions, a Wilson-like term with only spatial derivatives multiplied by $i\gamma_4$ is added to the free Dirac operator. In 4-dim discrete spacetime lattice, the KW action in the presence of a background gauge field is

$$\begin{aligned} S_{\rm KW} = \sum_x \Big[\frac{1}{2} \sum_{\mu=1}^{4} \bar{\psi}(x)\gamma_\mu \Big\{ U_\mu(x)\psi(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu}) \Big\} \\ -\frac{i}{2} \sum_{\mu=1}^{3} \bar{\psi}(x)\gamma_4 \Big\{ U_\mu(x)\psi(x+\hat{\mu}) - 2\psi(x) + U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu}) \Big\} + m\bar{\psi}(x)\psi(x) \Big] \end{aligned} \tag{1}$$

The appearance of zero modes in two different places in the Brillouin zone becomes apparent in momentum space. In free massless theory ($U = 1$), the KW operator in momentum space is,

$$D_{\rm KW}(p) = \sum_{\mu=1}^{4} i\gamma_\mu \sin p_\mu + 2i\gamma_4 \sum_{\mu=1}^{3} [\sin(p_\mu/2)]^2 \tag{2}$$

Inverting the momentum space Dirac KW operator, the zeros show up at $(0, 0, 0, 0)$ and $(0, 0, 0, \pi)$ which corresponds to the two doublers. Having a preferred time direction, it breaks the hypercubic symmetry of the naive lattice action. Alternatively, the MDF action proposed by Borici and Creutz on 4-dim orthogonal lattice is written as,

$$\begin{aligned} S_{\rm BC} = \sum_x \Big[\frac{1}{2} \sum_{\mu=1}^{4} \bar{\psi}(x)\gamma_\mu \Big\{ U_\mu(x)\psi(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu}) \Big\} \\ +\frac{i}{2} \sum_{\mu=1}^{4} \bar{\psi}(x)\gamma'_\mu \Big\{ U_\mu(x)\psi(x+\hat{\mu}) - 2\psi(x) + U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu}) \Big\} + m\bar{\psi}(x)\psi(x) \Big] \end{aligned} \tag{3}$$

where $\gamma'_\mu = \Gamma - \gamma_\mu,\ \ \Gamma = (1/2)\sum_\mu \gamma_\mu$. The two zeros of BC operator in momentum space are at $(0,0,0,0)$ and $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ as is evident from the poles of the inverse of Dirac BC operator in the momentum space,

$$D_{\text{BC}}(p) \;=\; \sum_\mu i\gamma_\mu \,\sin p_\mu - 2i\sum_\mu \gamma'_\mu\,[\sin(p_\mu/2)]^2 \tag{4}$$

The BC operator satisfies the γ_5-hermiticity, and like KW, is diagonal in momentum space.

Some exploratory studies of minimally doubled fermions, both KW and BC, have been carried out, like mixed action spectroscopy [10–14], eigenspectra [15], taste structures [16–18], chiral symmetry breaking [19], and index theorem [6, 7]. Some studies of formal aspects include renormalization properties [20–28], phase structure [29–31], anomaly structure [9], and construction of chiral perturbation theory [32]. However, for MDF to become a serious contender for QCD simulation, we feel that more analytical and numerical studies are needed, particularly in the topological sector. In the present work, we have studied the spectral flow of the eigenvalues with respect to the flavored mass [6, 7] and index theorem of KW and BC action in 4-dim with a definite topological charge for background $SU(3)$ gauge fields.

3. Spectral flow and the index theorem

Atiyah-Singer Index Theorem relates the difference of numbers of left and right-handed (*i.e.,* of opposite chiralities) zero eigenmodes of massless Dirac operator D to topological charge Q_{top},

$$\text{index}(D) \;=\; n_+ - n_- = (-1)^{d/2} Q_{\text{top}} \tag{5}$$

$$\text{index}(D_{\text{mdf}}) \;=\; 2\times(-1)^{d/2} Q_{\text{top}} \tag{6}$$

The Q_{top} is a property of the gauge fields. In eqn. (6), an extra factor of 2 arises from two fermion species [6] of MDF. We obtained the index(D) by spectral flow [6, 33], where the flow of eigenvalues of the Dirac operator is considered as a function of both the bare and flavored mass parameters. The index(D) is calculated by counting the net number of times the eigenvalues change sign in the flow near $m = 0$, counted with sign $\pm$ depending on the slope of crossings, and equals $n_+ - n_-$. For the computation of the eigenspectrum, we use a combination of Kalkreuter-Simma algorithm [34] for eigenvectors and LAPACK for eigenvalues. We implemented the algorithm by suitably modifying appropriate subroutines in the publicly available MILC code [35]. We computed eigenspectra for $D_{\text{mdf}} + M$ and its hermitian counterpart, where mdf stands for either KW or BC and M for either bare mass (m) or *flavored mass*. The hermitian version of MDF with flavored mass is defined as [6, 7],

$$H_{\text{KW}}(m) \;=\; \gamma_5(D_{\text{KW}} + m[C_{\text{sym}}\otimes 1]) \tag{7}$$

$$H_{\text{BC}}(m) \;=\; \gamma_5(D_{\text{BC}} + m[(2C_{\text{sym}} - 1)\otimes 1]) \tag{8}$$

where

$$C_{\text{sym}} \;=\; \frac{1}{4!}\sum_{\text{perm}} C_1C_2C_3C_4, \tag{9}$$

$$\text{and}\ \ C_\mu(x,y)\,\psi(y) \;=\; \frac{1}{2}\big[U_\mu(x)\delta_{x+\hat{\mu},y} + U^\dagger_\mu(x-\hat{\mu})\delta_{x-\hat{\mu},y}\big]\psi(y). \tag{10}$$

We calculated numerically the eigenvalue flows of $H_{\rm KW}(m)$ and $H_{\rm BC}(m)$ on a background $SU(3)$ gauge field configuration with a fixed topological charge $Q_{\rm top}$. For this, we followed the proposal given in reference [36, 37]. The simplest gauge field with a non-zero $Q_{\rm top}$ is when $F_{\mu\nu}(x)$ is a constant. On a 4-dim torus with $x_\mu \in [0, L_\mu]$, the link variables, $U_\mu(x)$ are

$$U_1(x) = exp(-i\omega_1 a x_2 \tau_j), \text{ and } U_2(x) = \begin{cases} 1, & \text{for } x_2 = 0, a, ..., (N-2)a \\ exp(i\omega_1 L x_1 \tau_j), & \text{for } x_2 = (N-1)a \end{cases} \quad (11)$$

$$U_3(x) = exp(-i\omega_2 a x_4 \tau_j), \text{ and } U_4(x) = \begin{cases} 1, & \text{for } x_4 = 0, a, ..., (N-2)a \\ exp(i\omega_2 L x_3 \tau_j), & \text{for } x_4 = (N-1)a \end{cases} \quad (12)$$

$$Q_{\rm top} = 2n_1 n_2 \quad \text{where, } L = Na, \quad \omega_i a^2 = 2\pi n_i / L^2 \quad (13)$$

The topological charge $Q_{\rm top}$ for a smooth $SU(3)$ gauge field is,

$$Q_{\rm top} = -\frac{1}{4\pi^2} \sum_x \mathrm{Tr}\big[F_{12}(x)F_{34}(x) - F_{13}(x)F_{24}(x) + F_{23}(x)F_{14}(x)\big] \quad (14)$$

where $F_{\mu\nu}(x)$ is the field strength tensor. The generated field corresponding to different $Q_{\rm top}$ are considered smooth in the sense they are not elements of Markov chains. A smooth configuration does not always give well-separated sign flips for the eigenvalues, and hence we roughen the $U_\mu(x)$ by keeping $Q_{\rm top}$ approximately invariant,

$$U_\mu(x)_{(\delta)} = exp(i \sum_j \theta^{(j)}_\mu \tau_j) \implies U_\mu(x)_{\rm rough} = U_\mu(x)_{(\delta)} U_\mu(x)_{old} \quad (15)$$

where $U_\mu(x)_{old}$ are smooth links (11, 12) and $U_\mu(x)_{(\delta)}$ are $SU(3)$ elements in the vicinity of identity. τ_j are Gell-Mann matrices and $\theta^{(j)}_\mu$ $j = 1, 2, \ldots, 8$ are small random numbers uniformly distributed in $(-\delta\pi, \delta\pi)$. In reference [37], the roughening is done to mimic Monte Carlo configurations to increase statistics. In Fig. 1 the flow of eigenvalues of $H_{\rm KW}$ and $H_{\rm BC}$ fermions with flavored mass

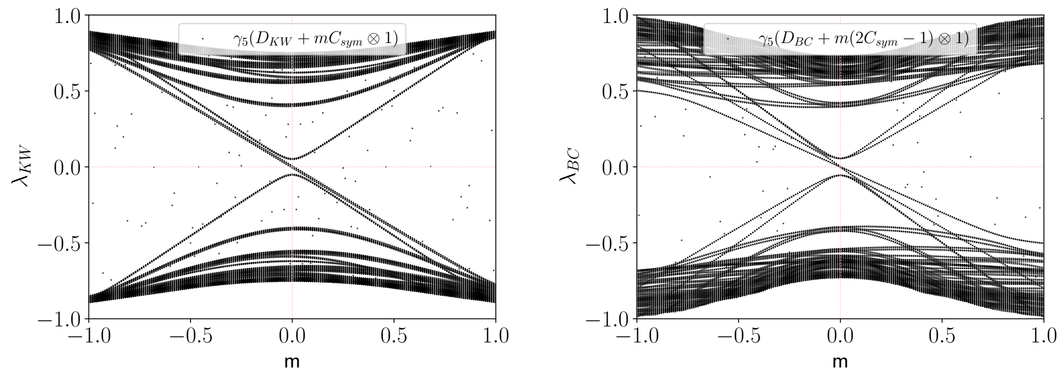

Figure 1: Spectral flow with respect to flavored masses of KW and BC fermion on 8^4 lattice with $Q_{\rm top} = -2$ and $\delta = 0.05$. Eigenvalues in left and right panels correspond to $H_{\rm KW}$ (eqn. 7) and $H_{\rm BC}$ (eqn. 8), respectively.

are shown. When eigenvalues are computed with bare quark mass instead of flavored mass, *i.e.,* with $\gamma_5(D_{\rm mdf} + m)$, it shows the net crossing to be zero as the would-be zero modes of either KW or BC Dirac operator cancel between species doublers. This gets resolved with the use of flavored mass, which is similar to what has been observed with staggered fermions [38, 39]. We see two doubled crossings for each of BC and KW around $m = 0$ with negative slopes corresponding to zero modes implying $\text{index}(D_{\text{mdf}}) = 2Q_{\text{top}} = -4$. The Q_{top} calculated from the eqn. (14) using the gauge field(s) generated is $Q_{\text{calc}} = -1.742$.

4. Chiralities

In this section, we present the details of the eigenvalues and eigenvectors along with measured γ_5 and modified γ_5-chiralities X,

$$X_{\text{KW}} = [C_{\text{sym}} \otimes \gamma_5] \quad \text{and} \quad X_{\text{BC}} = [(2C_{\text{sym}} - 1) \otimes \gamma_5] \tag{16}$$

Following [40], in the first step, we determined N_{max} eigenvectors $|\psi_k\rangle$, corresponding to that many lowest eigenvalues of D^2_{mdf}. For this, we used the Kalkreuter-Simma algorithm. In the next step, a reduced $N_{\text{max}} \times N_{\text{max}}$ MDF Dirac operator is created by $\langle\psi_j|D_{\text{mdf}}|\psi_k\rangle$. Subsequently, LAPACK is used on the reduced D_{mdf} matrix to obtain our desired eigenvalues λ_{mdf}. The same eigenvectors are used to obtain $\langle\psi_i, \gamma_5\psi_i\rangle$ and $\langle\psi_i, X_{\text{mdf}}\psi_i\rangle$.

S.No	λ_{KW}	$\langle\psi_i, \gamma_5\psi_i\rangle$	$\langle\psi_i, X_{KW}\psi_i\rangle$	λ_{BC}	$\langle\psi_i, \gamma_5\psi_i\rangle$	$\langle\psi_i, X_{BC}\psi_i\rangle$
1	0.004962i	0.00	-0.80	0.000636i	0.00	-0.80
2	-0.004962i	0.00	-0.80	-0.000636i	0.00	-0.80
3	0.006934i	0.00	-0.80	0.004267i	0.00	-0.78
4	-0.006934i	0.00	-0.80	-0.004267i	0.00	-0.78
5	0.049193i	0.00	0.05	0.054355i	0.00	0.06
6	-0.049193i	0.00	0.05	-0.054355i	0.00	0.06
7	0.049675i	0.00	0.05	0.055118i	0.00	0.06

Table 1: Eigenvalues and computed chiralities of massless D_{KW} and D_{BC} for $Q_{\text{top}} = -2$ and $\delta = 0.05$ background 8^4 lattice.

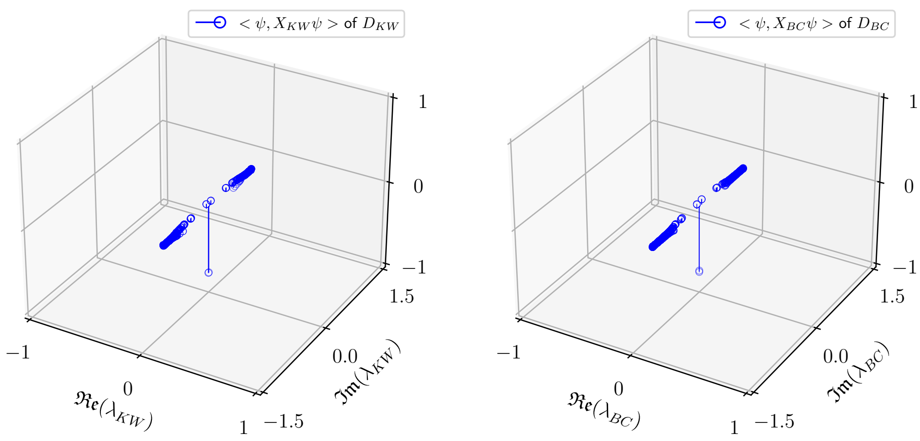

Figure 2: Needle plots of the modified γ_5−chiralities for massless D_{KW} and D_{BC} for $Q_{\text{top}} = -2$ and $\delta = 0.05$ background 8^4 lattice. The λ_{mdf} are concentrated on the imaginary axis as tabulated in the Table. 1, where the vertical axis corresponds to $\langle X_{\text{mdf}}\rangle$ with the stand-alone points reaching out to -1.

In Table 1, we tabulated the first few lowest eigenvalues and their corresponding chiralities. We found the zero eigenvalues λ_{mdf} are of $O(10^{-3})$, whereas those of the first excited states are $O(10^{-2})$. We also found that $\langle\gamma_5\rangle$ is always zero regardless of whether it is a zero eigenmode or not, implying that γ_5 is not an appropriate measure to capture the ± 1 chirality corresponding to the zero eigenvalues. On the other hand, the modified γ_5-chirality operator X_{mdf} (16) correctly reproduced the chirality -0.80 (≈ -1) for four zero eigenmodes, implying index$(D_{\text{mdf}}) = -4$ which equals $2Q_{\text{top}}$, validating the index theorem.

The corresponding needle plots for KW and BC for modified chirality operators are given in Fig. 2, which, as is expected from the table, show $\langle X_{\text{mdf}}\rangle \rightarrow\approx -1$ for four zero eigenmodes and $\langle X_{\text{mdf}}\rangle \rightarrow\approx 0$ for non-zero eigenmodes. The plots are almost identical to those obtained in [7], showing no new physics for KW or BC emerging in 4-dim.

5. Eigenspectra with flavored mass

We have seen above the operator X_{mdf} can identify the chiralities of the zero modes of the eigenstates of the massless KW and BC Dirac operators in 4-dim. It now remains to see how the tastes of the operators $(D_{\text{KW}} + mC_{\text{sym}} \otimes 1)$ and $(D_{\text{BC}} + m(2C_{\text{sym}} - 1) \otimes 1)$ are separated based on their chiralities. A previous study in 2-dim [7] showed that a flavored mass term successfully removes the degeneracy in tastes. Here, too, we generated reduced matrices for $(D_{\text{KW}} + mC_{\text{sym}} \otimes 1)$ and $(D_{\text{BC}} + m(2C_{\text{sym}} - 1) \otimes 1)$ for $m = 1$ using corresponding eigenvectors $|\psi_k\rangle$ of $(D_{\text{KW}} + mC_{\text{sym}} \otimes 1)^2$ and $(D_{\text{BC}} + m(2C_{\text{sym}} - 1) \otimes 1)^2$, respectively, and determined the eigenvalues of these reduced matrices.

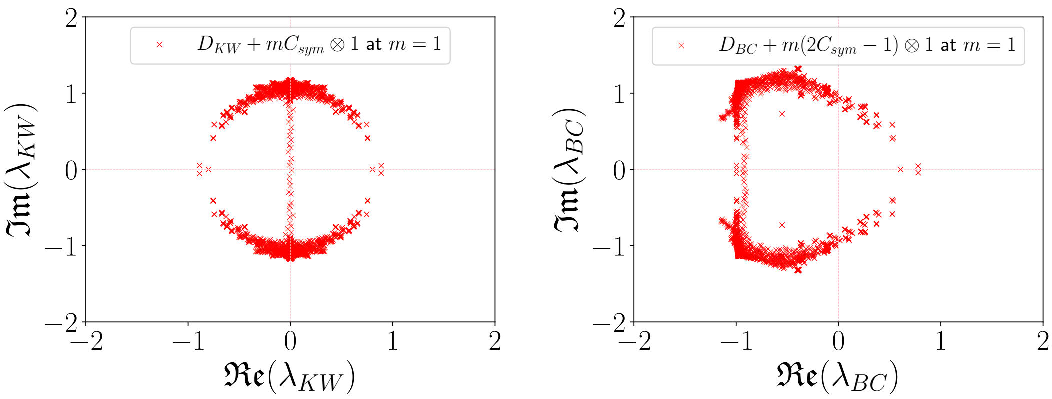

Figure 3: Complex eigenvalues of KW and BC Dirac operators with flavored mass terms: the eigenspectra split in two branches crossing the real axis at $m = \pm 1$.

Fig. 3, shows that the modified mass term separated the doublers (or tastes) according to their ± 1 chirality appearing at $|m|$ on the real axis. However, for KW fermion, some concentrations of eigenvalues are still at $m = 0$, which possibly result from using a small lattice volume and are expected to vanish for larger lattices.

6. Conclusions

The above set of observations indicates that the eigenspectra of the two variants of minimally doubled fermions, namely Karsten-Wilczek and Borici-Creutz, in 4-dim spacetime, is consistent with the index theorem. For this, we generated gauge field configuration, subjected to appropriate roughening, with a fixed topological charge $Q_{\text{top}} = -2$. The charge is cross-checked by recalculating it using eqn. (14). Using flavored mass terms, we obtained the separation of doubler tastes of both KW and BC according to their chirality.

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