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SU(6) model revisited

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Toward elucidating infrared dynamics of chiral gauge theories, we reevisit the vacuum structure of an SU(6) model from the viewpoint of the 't Hooft anomaly matching. For this purpose, we identify 't Hooft anomalies in the UV theory using the Stora-Zumino procedure. Subsequently, we construct an effective theory by applying the idea of the Wess-Zumino-Witten action to derive the topological terms that encode the 't Hooft anomalies. We also demonstrate that a low-energy effective theory reproducing a part of the anomalies is described by a $\mathbb{Z}_3$-valued scalar field.

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1. Introduction

Symmetries have always been quite useful tools in the analysis on strongly coupled quantum field theories. The concepts of global symmetries are generalized respecting the topological nature of symmetry defects in [1]. By making use of the 't Hooft anomalies [2] for generalized symmetries such as higher-form symmetries, we can extract non-trivial infrared dynamics of gauge theories [3].

In this paper, we investigate the vacuum structure of the $\mathrm{SU}(6)$ gauge theory with a Weyl fermion in an irreducible self-conjugate representation. As elaborated in the main text, there is a mixed 't Hooft anomaly between a one-form center symmetry and a discrete chiral symmetry, which leads to the following dynamical consequences [4, 5]. Firstly, there must be physical degrees of freedom in order to match the mixed anomaly. Secondly, assuming that the model flows to the confinement phase, the chiral symmetry is expected to be spontaneously broken. Since the fermion bilinear condensate seems to be forbidden due to the Fermi-Dirac statistics, the four-fermion condensate is a minimal candidate for the order parameter of the chiral symmetry breaking.

While, as pointed out in [5], there are not only the mixed anomaly described above but also the pure anomaly of the discrete chiral symmetry, the full structures of the anomalies in the $\mathrm{SU}(6)$ model have not been completely understood yet. We compute these anomalies by Stora-Zumino procedure [6], and check that our results are perfectly consistent with the result based on the analysis of the bordism group. We also find an IR effective action reproducing the mixed anomaly. This action has a potential to match the pure anomaly although we have not succeeded at this stage.

Before delving into the main text, we clarify our motivation why we study the $\mathrm{SU}(6)$ model from the lattice viewpoint. Lattice construction of chiral gauge theories has been a longstanding problem. Contrary to this difficulty, the $\mathrm{SU}(6)$ model is the prominent candidate to be formulated on the lattice. This is because the Weyl fermion in this model belongs to the self-conjugate representation [1].

This paper is structured as follows. In Section 2, we review the properties of the symmetries in the $\mathrm{SU}(6)$ model. In Section 3, we compute the anomalies using the Stora-Zumino procedure and confirm consistency with the results obtained from the analysis of the bordism group. In Section 4, we propose an IR effective action that reproduces the mixed anomaly and discuss the difficulty of matching the pure anomaly in the IR regime.

2. Global symmetries and the chiral symmetry breaking in the $\mathrm{SU}(6)$ model

The $\mathrm{SU}(6)$ model is a chiral gauge theory with a Weyl fermion in the rank-three antisymmetric representation which is a self-conjugate representation. This model does not suffer from a gauge anomaly. Global symmetries of this model are described by

$$G = \frac{\mathrm{SU}(6)/\mathbb{Z}_3^c \times \mathbb{Z}_6^\chi}{\mathbb{Z}_2}\,, \tag{1}$$

where $\mathbb{Z}_3^c$ and $\mathbb{Z}_6^\chi$ stand for a center subgroup of $\mathrm{SU}(6)$ and discrete chiral symmetry, respectively. The discrete symmetry $\mathbb{Z}_2$ is necessary to remove the redundancy of the gauge symmetry and

[1] It was implied that the locality and gauge invariance of the theory can be preserved if the fermion representation of the gauge group is anomaly-free [7–9].

discrete chiral symmetry. As discussed in Appendix A, we find that a background gauge field for $\mathbb{Z}_2$ does not come in the discussion about anomalies. This observation makes analysis on anomalies easier. Therefore, instead of the group G, we only focus on the following symmetries,

$$\widetilde{G} = \frac{SU(6)}{\mathbb{Z}_3^c} \times \mathbb{Z}_6^\chi \; . \tag{2}$$

In what follows, we present more details about the center and chiral symmetries in order.

Center symmetry. The center symmetry crucially depends on the representation of charged matters. In our case, the N-ality of the rank-three aniti-symmetric representation is three, hence the center symmetry is given by $\mathbb{Z}_3^c$.

Chiral symmetry. At the classical level, this model possesses a U(1) chiral symmetry. The Adler–Bell–Jackiw (ABJ) anomaly, however, breaks this chiral symmetry group into the discrete subgroup $\mathbb{Z}_6^\chi$, where the order 6 is the Dynkin index of the rank-three aniti-symmetric representation.

Finally, we give some comments on the chiral symmetry breaking in the SU(6) model. As mentioned in [4, 5], this model exhibits the mixed 't Hooft anomaly between the center symmetry $\mathbb{Z}_3^c$ and chiral symmetry $\mathbb{Z}_6^\chi$. As a result, $\mathbb{Z}_6^\chi$ is spontaneously broken down to the anomaly-free subgroup $\mathbb{Z}_2^\chi$. This leads to three distinct vacua related by the broken symmetry. It is remarkable that a fermion bilinear cannot condense due to the Fermi-Dirac statistics. This fact implies that there is presumably some order parameter for detecting this chiral symmetry breaking.

3. Gauging the total symmetry

This model possesses a mixed anomaly between $\mathbb{Z}_6^\chi$ and $\mathbb{Z}_3^c$ [4]. This anomaly can be observed by performing a discrete $\mathbb{Z}_6^\chi$ chiral transformation in the presence of the background gauge field associated with the $\mathbb{Z}_3^c$ center symmetry. In this section, our goal is to elucidate the full structure of 't Hooft anomalies in the UV regime by gauging the whole global symmetry G described in (1). Although our global symmetry contains the discrete chiral symmetry, we demonstrate that the Stora-Zumino procedure [10] correctly produces 't Hooft anomalies.

Following [11], we construct the $\mathrm{SU}(6)/\mathbb{Z}_3^c$ gauge field by introducing a pair of U(1) two-form and one-form gauge fields $(B_c^{(2)}, B_c^{(1)})$ subject to the constraint,

$$3B_c^{(2)} = \mathrm{d}B_c^{(1)} \; . \tag{3}$$

The background field is coupled to the SU(6) gauge field a locally as $\tilde{a} = a + B_c^{(1)}/3$. In order to gauge $\mathbb{Z}_3^c$ center symmetry, we impose the invariance under the gauge transformations,

$$B_c^{(1)} \to B_c^{(1)} + 3\Lambda^{(1)} \,, \quad B_c^{(2)} \to B_c^{(2)} + \mathrm{d}\Lambda^{(1)} \,, \quad \tilde{a} \to \tilde{a} + \Lambda^{(1)} \,, \tag{4}$$

where $\Lambda^{(1)}$ is a one-form gauge transformation parameter. Subsequently, we have to replace the field strength $\tilde{F}$ constructed by $\tilde{a}$ into $\tilde{F} - B_c^{(2)}$.

The anomaly polynomial in the SU(6) model is given by [2]

$$\mathscr{A}_6 = \int \frac{2\pi}{3!(2\pi)^3} \mathrm{Tr}\left[\left(\tilde{F} - B_c^{(2)}\right) + \mathrm{d}A_\chi^{(1)}\right]^3 , \tag{5}$$

where Tr is taken in the rank-three antisymmetric representation, and $A_\chi^{(1)}$ is the background gauge field associated with $\mathbb{Z}_6^\chi$. This anomaly polynomial (5) leads to the five-dimensional SPT action

$$S_{\mathrm{SPT}} = \int A_\chi^{(1)} \wedge \left[\frac{2\pi}{3!(2\pi)^3}\left(18\,\mathrm{tr}(\tilde{F} - B_c^{(2)})^2 + 20\ (\mathrm{d}A_\chi^{(1)})^2\right)\right] . \tag{6}$$

This action is manifestly invariant under the $\mathbb{Z}_3^c$ transformation. Under the $\mathbb{Z}_6^\chi$ chiral transformation,

$$A_\chi^{(1)} \to A_\chi^{(1)} + \mathrm{d}\lambda^{(0)} , \tag{7}$$

where λ is a gauge transformation parameter, the SPT action yields

$$\delta S_{\mathrm{SPT}} = \int \lambda \left[\frac{2\pi}{3!(2\pi)^3}\left(18\ \ \mathrm{tr}(\tilde{F})^2 - 108(B_c^{(2)})^2 + 20(\mathrm{d}A_\chi^{(1)})^2\right)\right] . \tag{8}$$

Here we specify $\lambda^{(0)}$ as $\frac{2\pi}{6}$. The second term, then, leads to a mixed anomaly between $\mathbb{Z}_3^c$ and $\mathbb{Z}_6^{\chi(0)}$,

$$\mathcal{A}_{[\mathbb{Z}_6^{\chi(0)}]-[\mathbb{Z}_3^{c(1)}]^2} \equiv -\frac{2\pi}{3!(2\pi)^3}\int \frac{2\pi}{6} 108(B_c^{(2)})^2 \in -\frac{2}{3}\cdot 2\pi\mathbb{Z} , \tag{9}$$

which is consistent with [4, 5]. The third term corresponds to the pure anomaly $[\mathbb{Z}_6^{\chi(0)}]^3$,

$$\mathcal{A}_{[\mathbb{Z}_6^{\chi(0)}]^3} \equiv \frac{2\pi}{3!(2\pi)^3}\int \frac{2\pi}{6} 20(dA_\chi^{(1)})^2 \in \frac{1}{9}\cdot 2\pi\mathbb{Z} . \tag{10}$$

Note that the above result (10) coincides with the result from the computation of bordism group [12, 13]. Analysis of the η-invariant provides the result for the pure anomaly $[\mathbb{Z}_n^{(0)}]^3$ under the symmetry transformation of $\mathrm{Spin}(4)\times\mathbb{Z}_N$, as follows [12, 13]:

$$\mathcal{A}_{[\mathbb{Z}_n^{(0)}]^3} \equiv (N^2 + 3N + 2)\left(\sum_L s_L^3 - \sum_R s_R^3\right) \quad \mathrm{mod}\ 6n, \tag{11}$$

where s_L and s_R denote the $\mathbb{Z}_n$ charges of the left- and right-handed fermions, respectively. Applying Eq. (11) to our system, we find that the pure discrete chiral anomaly $[\mathbb{Z}_6^{\chi(0)}]^3$ takes the value 1 mod 9.

It is important to recall that non-perturbative anomalies such as $[\mathbb{Z}_6^{\chi(0)}]^3$ can sometimes take nontrivial values that are not captured by the anomaly polynomial. Therefore, in the next section, we verify our result using a rigorous computation of the η-invariant. As a consequence, we confirm that the non-perturbative anomalies in our system can be precisely evaluated using the Stora-Zumino procedure.

4. IR effective theory

In this section, assuming that SU(6) model is in the confinement phase, we construct an effective IR action incorporating the idea of the Wess-Zumino-Witten(WZW) term[6, 16]. [3]

[2]Gravitational anomalies should also be treated. However, as shown in [12], this model does not exhibit such anomalies. Therefore, we do not consider gravitational contributions in our discussion.

[3]Our procedure is similar to [14, 15] which match the mixed anomalies arising UV region.

4.1 Effective action for mixed anomaly

The WZW term states that the contributions from Numbu-Goldstone (NG) fields to the anomaly is expressed as the difference between the shifted CS term of bare gauge fields (A_h, A) and dressed one $((A^{U^{-1}})_h, A^{U^{-1}})$. In other wards, we can choose the NG fields, so that the shifted CS term of dressed gauge fields $A^{U^{-1}}$ is gauge invariant in one dimensional higher manifold. Of course we know that no NG boson arises since broken chiral symmetry is discrete group. However, some composite scalar fields such as $\psi\psi$ or $\psi\psi\psi\psi$ interpreted as the order parameter of the chiral symmetry breaking should exist in low energy region to reproduce the UV anomalies.

From this view point, let us assume that there exists the composite scalar field $\phi \in \mathbb{Z}_3$ with charge $Q = 2$ or 4 under the $\mathbb{Z}_6^{\chi(0)}$ transformation, which satisfies (13). Then, we can construct the chiral invariant action, which corresponds to the shifted CS term $\tilde{\omega}_5^{(0)}((A^{U^{-1}})_h, A^{U^{-1}})$, with Φ in 5-dimensional manifold N

$$\begin{aligned}\Omega_5 = &\int_N (d\Phi - A_\chi^{(1)}) \wedge \left[\frac{2\pi}{3!(2\pi)^3}\left(-3lN(B_q^{(2)})^2 + \dim R\ (dA_\chi^{(1)})^2\right)\right] \\ &+ \frac{q}{2\pi}\int_N d\phi \wedge db^{(3)}\end{aligned} \tag{12}$$

where $\Phi \equiv \frac{\phi}{Q}$ and

$$\mathbb{Z}_6^{\chi(0)} : \Phi \mapsto \Phi + \frac{2\pi}{6}, \quad \phi \mapsto \phi + Q\frac{2\pi}{l}. \tag{13}$$

The last term in eq.(12) is the Lagrange multiplier and $\oint_M db^{(3)} \in 2\pi\mathbb{Z}$. Therefore the effective IR action is

$$S_{IR} = \int_N \Phi \wedge \left[\frac{2\pi}{3!(2\pi)^3}\left(-3lN(B_q^{(2)})^2 + \dim R\ (dA_\chi^{(1)})^2\right)\right] + \frac{q}{2\pi}\int_M \phi \wedge db^{(3)}. \tag{14}$$

Note that we can construct the effective action by just one scalar field which reproduce the mixed 't Hooft anomaly (9) in the UV energy scale.

4.2 The nontrivial topological term for chiral pure anomaly

On the other hand, in the case of the self-anomaly, it is non-trivial which degrees of freedom reproduce the self anomaly. This is my ongoing work. First, the topological term given by Stora-Zumino procedure is actually, ill-defined mathematically. It can bee seen by deforming to the cochain form. And the well-defined topological term in cochain form is proposed in [17] as follows,

$$\eta_{\text{chiral}} = \beta_9\left(\beta_3 A_3 \cup \beta_3 A_3\right), \tag{15}$$

where A_3 is a background gauge field for $\mathbb{Z}_3^\chi$. Here, we have treated only $\mathbb{Z}_3^\chi$ subgroup of $\mathbb{Z}_6^\chi$ since $\mathbb{Z}_2^\chi$ subgroup is anomaly free. β_3, β_9 are Bockstein homomorphism satisfying

$$\beta_9 : H^n(-, \mathbb{Z}_3) \to H^{n+1}(-, \mathbb{Z}_9) \tag{16}$$

$$\beta_3 : H^n(-, \mathbb{Z}_3) \to H^{n+1}(-, \mathbb{Z}_3). \tag{17}$$

Now it turns out what we have to do is to find the 4-dimensioinal term which compensate with this topological term. This is just problem we are working on now.

In the very naive discussion, this topological term may be similar to the CS term. Indeed, under the assumption that the topological term forms like CS term, [18, 19] have succeeded in predicting even more.

If this is true, the self-anomaly might also be matched by the scalar field $\phi \in Z_3$. Then the vacuum structure is very simple. Another possibility is that some degrees of freedom on the domain wall, which is inserted between degenerate vacua, match the anomaly.

5. Summary

This study identifies two anomalies in the model under consideration. The *mixed anomaly* can be matched by the presence of $\phi \in \mathbb{Z}_3$. On the other hand, the nature of the *discrete chiral anomaly* remains partially unresolved. Although a corresponding topological term is known, the associated four-dimensional degrees of freedom remain unclear. Based on insights from previous studies, it is anticipated that this topological term can be reformulated in a form similar to the Chern-Simons term. If this is achieved, the discrete chiral anomaly can also be matched by $\phi \in \mathbb{Z}_3$. This finding implies that the vacuum structure of the model may be fully constructed using $\phi \in \mathbb{Z}_3$. The above discussions provide a crucial foundation for future efforts to realize this model on the lattice.

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A. Cancellation of $\mathbb{Z}_2$ symmetry

In this appendix, we see that a background gauge field for the $\mathbb{Z}_2$ group in Eq. (1) is nothing to do with the discussion on anomalies. First, we consider the $\mathbb{Z}_6^\chi/\mathbb{Z}_2$ bundle. Its cocycle condition is twisted;

$$\omega^{n_{ij}+n_{jk}+n_{ki}} = \omega^{3b_{ijk}} = \exp\left(\frac{2\pi i}{2} b_{ijk}\right), \tag{18}$$

where $\omega^{n_{ij}} = \exp\left(\frac{2\pi i}{6} n_{ij}\right) \in \mathbb{Z}_6^\chi$ is the transition function on $U_{ij} = U_i \cup U_j$ and $b_{ijk} \in \mathbb{Z}_2$ corresponds to the 2-form background gauge field of $\mathbb{Z}_2$. This twisted cocycle condition is compensated with that of $\frac{SU(6)/\mathbb{Z}_3}{\mathbb{Z}_2}$ bundle. The transition functions $\omega^{n_{ij}}$ is changed under $\mathbb{Z}_2$ gauge transformation as

$$\omega^{n_{ij}} \mapsto \omega^{n'_{ij}} = (-1)^{m_{ij}} \omega^{n_{ij}} = \omega^{n_{ij}+3m_{ij}} \tag{19}$$

It leads

$$\omega^{n'_{ij}+n'_{jk}+n'_{ki}} = \omega^{n_{ij}+n_{jk}+n_{ki}+3m_{ij}+m_{jk}+m_{ki}} = \omega^{3\left(b_{ijk}+m_{ij}+m_{jk}+m_{ki}\right)} \equiv \omega^{3b'_{ijk}}, \tag{20}$$

where $b'_{ijk} = b_{ijk} + m_{ij} + m_{jk} + m_{ki}$. Then, if we chose $m_{ij} = n_{ij}$,

$$\omega^{n'_{ij}+n'_{jk}+n'_{ki}} = \omega^{4\cdot 3b_{ijk}} = 1. \tag{21}$$

Therefore, we don't have to gauge 1-form $\mathbb{Z}_2$ symmetry, so that the gauge symmetry group we want is G described in (2).

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