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The Electron Self-Energy at 2 and 3 Loops

Christoph Nega[a,*]

[a] *Technical University of Munich, TUM School of Natural Sciences, Physics Department, James-Franck-Straße 1, D-85748 Garching, Germany*

E-mail: c.nega@tum.de

In these contributions, we report on the computation of the bare electron self-energy up to three loops in terms of iterated integrals over kernels of elliptic type. To systematically compute the master integrals, we use differential equations that are brought into canonical form such that the ϵ-expansion can be easily calculated in terms of iterated integrals. Up to three loops, only sixteen different kernels are necessary to describe all master integrals, whereas at two loops, only seven are necessary. For a numerical evaluation of the iterated integrals, we use local series expansions, which can be analytically continued to any region of the relevant parameter space. Moreover, we show that all elliptic contributions can be traced back to the sunset elliptic curve, already appearing at two loops.

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1. Introduction

In these proceedings, we tackle the computation of the electron self-energy, which is one of the most basic objects in quantum electrodynamics, up to three loops. This continues the long history of calculating the electron self-energy, starting with the one-loop computation by Schwinger [1] which was then carried on with the two-loop treatment by A. Sabry [2] that was just recently completed and rephrased into the modern mathematical language in [3]. Such calculations give us insides of the perturbative structure, not just of quantum electrodynamics itself, but any quantum field theory. To handle the perturbative treatment we use the modern approach of computing scattering amplitudes [4–9] by decomposing the self-energy in terms of a basis of independent scalar Feynman integrals, which are also known as master integrals.

The success of current amplitude computations lies especially in the understanding of the special functions to which the master integrals evaluate. This means the analytic properties like functional relations and identities, as well as the numerical evaluation at given points, are well under control. This is particularly true for the function space of so-called multiple polylogarithms [10–15], where it was shown that many scattering amplitudes can be expressed through these [16]. With these proceedings, we want to go one step further and show that more general functions involving elliptic objects can appear in quantum field theory computations and present techniques to handle these new special functions.

These proceedings consist mainly of three parts. In the first part, we discuss the decomposition of the electron self-energy into scalar Feynman integrals. Afterwards, in section three, we describe how the master integrals necessary for the self-energy up to three loops can be computed using an ϵ-factorized differential equation. Section four deals with the iterated integrals showing up in the self-energy. Finally, in section five we give our conclusions.

2. Decomposition into Feynman integrals

In our discussion, we want to compute the electron self-energy in quantum electrodynamics up to three loops. We denote by m and p the mass and momenta of the electron, and we work in $d = 4 - 2\epsilon$ dimensions. Moreover, we define $x = p^2/m^2$ as our convenient variable, in particular, when we later want to expand the self-energy for small momenta. We work in a general R_ξ gauge for the photon propagator. To simplify the subsequent expressions, we set from here on the electron mass to unity, i.e. $m = 1$. We leave it as an exercise for the reader to reintroduce the mass dependence in our formulas if wanted.

We can write down the electron self-energy in terms of two different form factors

$$\hat{\Sigma}(p) = \Sigma_V(p)\not{p} + \Sigma_S(p)\mathbb{1}\,, \tag{1}$$

where both form factors can be computed in a perturbative series

$$\Sigma_{V,S}(p) = \sum_{\ell=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^\ell \Sigma_{V,S}^{(\ell)}(p)\,, \tag{2}$$

where each coefficient $\Sigma_{V,S}^{(\ell)}(p)$ is expressed through ℓ-loop scalar Feynman integrals.

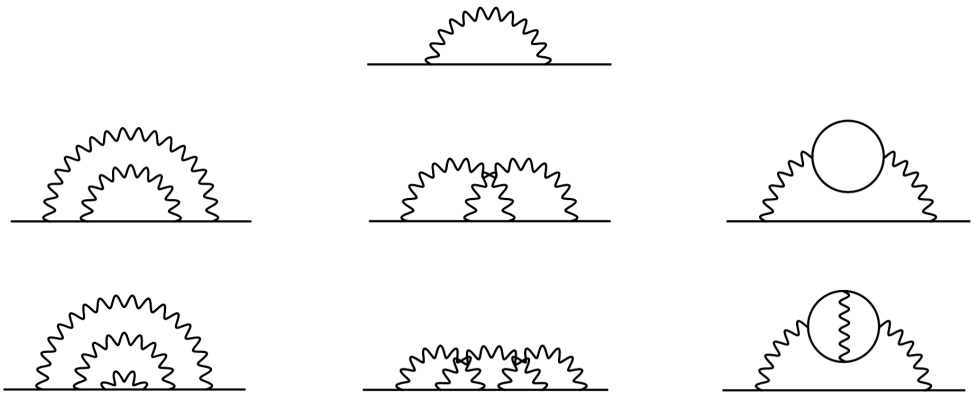

Figure 1: Feynman graphs necessary at one, two and three loops for the electron self-energy. At three loops we only show three out of the 20 different diagrams.

Our workflow for computing $\Sigma^{(\ell)}_{V,S}(p)$ is as follows: We first generate all relevant diagrams using the program `QGRAF` [17]. We handle all tensor structures with `FORM` such that we can decompose $\Sigma^{(\ell)}_{V,S}(p)$ just through scalar integrals [18–20]. Using the program `Reduze 2` [21, 22] we can map all integrals to families of scalar integrals for which we determine a set of master integrals. Besides `Reduze 2` we also apply `Kira 2` [23–25] to compute a set of differential equations our master integrals satisfy. This set of differential equations, which we also call the Gauss-Manin system, is then transformed into an ϵ-form [26] such that we can write down systematically the ϵ-expansion of our master integrals in terms of iterated integrals. Afterwards, these iterated integrals are plugged into our form factors, such that necessary simplifications and cancellations are happening analytically. Finally, the series expansions of the iterated integrals are used to obtain numerical values for the form factors which we visualize in a plot. Notice that we do not renormalize the electron self-energy in these proceedings and leave this open for our upcoming work [27], where also additional material and explicit results will be presented.

Let us start with the relevant Feynman diagrams in quantum electrodynamics up to three loops. At one loop, there is only one possible Feynman diagram one can draw, and at two loops, there are three different ones (see figure 1). For three loops, the combinatorics gets a little bit more complicated, such that `QGRAF` turned out to be very useful. We found 20 different diagrams, some of which are also shown in figure 1.

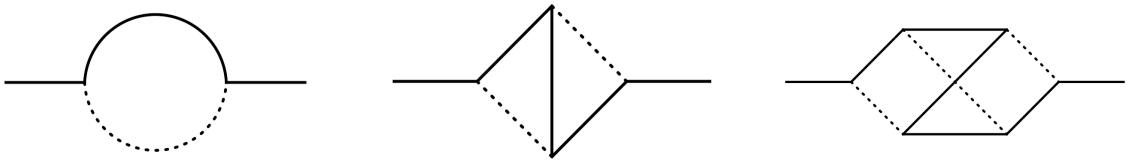

Figure 2: Top sector Feynman graphs defining the three different families for the electron self-energy at one, two and three loops. We call them the bubble, kite and paramecia family.

Now, in our second step, we map for each loop order these diagrams to a family of scalar

Feynman integrals. For one and two loops, the corresponding family is given by the bubble, with one massless propagator having two master integrals, and the massive kite containing eight master integrals (see figure 2). We are particularly interested in the three-loop so-called paramecia family [28] (shown on the right in figure 2) which is defined by the following propagators[1]

$$\begin{aligned}
&D_1 = k_1^2\,, && D_2 = k_2^2\,, && D_3 = k_3^2\,,\\
&D_4 = (k_1-p)^2-m^2\,, && D_5 = (k_2-p)^2-m^2\,, && D_6 = (k_3-p)^2-m^2\,,\\
&D_7 = (k_1+k_3-p)^2-m^2\,, && D_8 = (k_2+k_3-p)^2-m^2\,, && D_9 = (k_1+k_2+k_3-p)^2-m^2\,,
\end{aligned} \tag{3}$$

such that an integral in this family is given by

$$I_{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8,n_9} = e^{3\epsilon\gamma}\int\prod_{\ell=1}^{3}\frac{d^dk_\ell}{i\pi^{d/2}}\frac{1}{D_1^{n_1}\ldots D_9^{n_9}}\,. \tag{4}$$

Then by using integration-by-parts relations we find the following set of 51 master integrals

$$\begin{aligned}
&I_{0,0,0,1,1,1,0,0,0}\,, && I_{1,0,0,1,1,1,0,0,0}\,, && I_{1,0,1,0,1,0,1,0,0}\,, && I_{1,0,1,-1,1,0,1,0,0}\,,\\
&I_{0,0,0,1,1,1,1,0,0}\,, && I_{0,0,0,2,1,1,1,0,0}\,, && I_{1,1,0,0,0,0,1,1,0}\,, && I_{0,1,0,1,0,0,1,1,0}\,,\\
&I_{0,1,0,2,0,0,1,1,0}\,, && I_{-1,1,0,1,0,0,1,1,0}\,, && I_{0,0,0,1,1,0,1,1,0}\,, && I_{1,1,1,0,0,0,0,0,1}\,,\\
&I_{1,1,1,-1,0,0,0,0,1}\,, && I_{1,1,0,1,1,1,0,0,0}\,, && I_{1,1,1,0,1,0,1,0,0}\,, && I_{1,1,1,-1,1,0,1,0,0}\,,\\
&I_{0,1,0,1,1,1,1,0,0}\,, && I_{0,1,0,1,1,1,2,0,0}\,, && I_{1,1,1,0,0,-1,1,1,0}\,, && I_{1,1,1,0,0,-2,1,1,0}\,,\\
&I_{1,0,0,1,1,0,1,1,0}\,, && I_{0,0,0,1,1,1,1,1,0}\,, && I_{0,0,-1,1,1,1,1,1,0}\,, && I_{0,0,-2,1,1,1,1,1,0}\,,\\
&I_{1,1,1,1,0,0,0,0,1}\,, && I_{1,0,1,0,1,0,1,0,1}\,, && I_{1,0,1,0,1,0,2,0,1}\,, && I_{1,1,1,0,1,0,1,0,1}\,,\\
&I_{1,1,1,1,1,1,0,0,0}\,, && I_{1,0,1,1,1,1,1,0,0}\,, && I_{1,1,1,1,0,0,1,2,0}\,, && I_{1,2,1,1,0,0,1,1,0}\,,\\
&I_{1,1,0,1,1,0,2,1,0}\,, && I_{1,1,0,2,1,0,2,1,0}\,, && I_{1,0,0,1,1,1,1,2,0}\,, && I_{1,1,1,1,1,0,0,0,2}\,,\\
&I_{1,1,2,1,1,0,0,0,1}\,, && I_{1,1,1,2,1,0,0,0,2}\,, && I_{1,0,1,1,1,0,1,0,2}\,, && I_{1,1,2,0,0,0,1,1,1}\,,\\
&I_{1,1,1,0,0,1,1,1,1}\,, && I_{1,1,1,1,2,1,1,0,0}\,, && I_{1,1,0,1,1,1,1,1,0}\,, && I_{1,1,1,1,1,0,1,0,1}\,,\\
&I_{1,1,1,2,1,0,1,0,1}\,, && I_{1,1,1,1,1,0,0,1,1,1}\,, && I_{0,1,1,1,1,0,1,1,1}\,, && I_{1,1,1,1,1,0,1,1,1,1}\,,\\
&I_{1,1,1,1,1,1,1,1,1,0}\,, && I_{1,1,1,1,1,1,0,1,1,1}\,, && I_{1,1,1,1,1,-1,1,1,1}\,. &&
\end{aligned} \tag{5}$$

For the later considerations, the sectors

$$\begin{aligned}
&\mathrm{ES}_1: && I_{0,0,0,1,1,1,1,0,0}\,, I_{0,0,0,2,1,1,1,0,0}\\
&\mathrm{ES}_2: && I_{0,1,0,1,0,0,1,1,0}\,, I_{0,1,0,2,0,0,1,1,0}\,, I_{-1,1,0,1,0,0,1,1,0}\\
&\mathrm{ES}_3: && I_{0,1,0,1,1,1,1,0,0}\,, I_{0,1,0,1,1,1,2,0,0}\\
&\mathrm{ES}_4: && I_{1,0,1,0,1,0,1,0,1}\,, I_{1,0,1,0,1,0,2,0,1}\,, I_{1,1,1,0,1,0,1,0,1}
\end{aligned} \tag{6}$$

are particularly interesting, because their maximal cuts can be related, as we show in the next section, to the maximal cuts of the equal-mass sunset integral. This will then show that these integrals are of elliptic type and we need special care for deriving an ϵ-factorized differential equation.

[1] Actually, to capture all scalar integrals, we have to introduce two scalar families at three loops. Nevertheless, we do not discuss here the second family because, after integration-by-parts reduction, one does not find any new additional master integrals that are not already included in the paramecia family.

3. Canonical differential equations for elliptic Feynman integrals

In this section, we want to briefly discuss how one can construct an ϵ-factorized basis, starting with the Laporta basis in (5). We put special effort into the four elliptic sectors in (6). But let us start by briefly recalling the concept of an ϵ-factorized or canonical basis.

From integration-by-parts identities, one can show that a generic set of master integrals satisfies a first-order system of differential equations of the following form

$$\frac{d}{dx}\vec{I} = B(x;\epsilon)\vec{I}\,, \tag{7}$$

where the Gauss-Manin connection $B(x;\epsilon)$ consists of rational functions in x and ϵ, and is naturally in block-triangular form if one collects the master integrals by sectors. Our choice (5) satisfies exactly such a Gauss-Manin system. It is convenient to transform this system into a so-called ϵ-factorized form

$$\vec{J} = T(x;\epsilon)\vec{I} \quad \text{with} \quad \frac{d}{dx}\vec{J} = \epsilon\, A(x)\vec{J} \quad \text{and} \quad \epsilon A(x) = T\left(B(x;\epsilon)T^{-1} - \frac{dT^{-1}}{dx}\right), \tag{8}$$

such that in the new basis of master integrals, the connection form $A(x)$ is not any more dependent on the dimensional regulator ϵ. When such a basis is found, the solution of this Gauss-Manin system can easily be written down through the path-ordered exponential

$$\vec{J}(x) = \mathbb{P}\exp\left[\epsilon\int_\gamma A(x)\right]\vec{J}_0\,, \tag{9}$$

where $\vec{J}_0$ contains the boundary condition at x_0 and the integration path connects the boundary point to the generic point x. So we see that formally, in each order in ϵ the solutions can be expressed through iterated integrals of the kernels appearing in the connection form $A(x)$. If $A(x)$ just contains d log-forms of rational functions in x the function space can be identified with the space of multiple polylogarithms. In our case, we will see that certain entries in $A(x)$ contain elliptic integrals, and therefore a broader function space is required. Subsequently, we describe techniques for how the rotation $T(x;\epsilon)$ can be constructed.

Most of the sectors in the paramecia family, or at least their maximal cuts, can be related to integrals over d log-forms [29] and therefore one can apply standard methods [30–32] to determine good initial integrals from which one can derive an ϵ-basis. For instance, an integrand analysis in Baikov representation [33–35] turned out to yield most of the canonical integrals in our case. Here the first 29 integrals in (5) are built out of bubble-type graphs such that it is more convenient to look for candidates in two dimensions and then relate them to four dimensions using dimensional shift relations [36]. With this approach, one easily finds suitable candidates for an ϵ-factorized basis, except for the elliptic sectors (6).

For the elliptic sectors, we follow the approach introduced in[37] which is based on splitting the matrix of fundamental solutions of the corresponding elliptic differential equation into a unipotent and semi-simple part, which can be seen as a generalization of the leading singularities in the d log-case. To start this approach we have to identify suitable initial candidates that have, in particular, no infrared or ultraviolet singularities. Different from the d log-case, where one looks for integrals

such that their integrands can also be brought into d log-form, we look for integrals such that their integrands can be associated with standard differential forms[2] of an elliptic curve given by

$$\lambda_1 = \frac{X dX}{Y}\,, \quad \lambda_2 = \frac{d}{dz}\left(\frac{dX}{Y}\right), \quad \lambda_3 = \frac{X dX}{Y}\,, \tag{10}$$

where the elliptic curve is defined by a quartic polynomial equation $Y^2 = P_4(X)$ and z is the modulus of the curve. Notice here that the first differential λ_1 is holomorphic and therefore has no poles at all, whereas λ_2 has a double pole at infinity, which is different than what one would consider in the d log-situation. The differential form of the third kind λ_3 has a single pole and is therefore more similar to a d log-form than the other two forms.

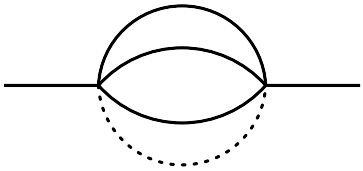

Figure 3: Three-loop banana graph with one massless propagator.

Let us look in detail into the elliptic sector ES_2 which describes the three-loop one massless propagator banana graph shown in figure 3. As already noticed it is more convenient to discuss this sector in $d = 2 - 2\epsilon$ dimensions which is why we use the dimension shift operator $\mathbf{D}^-$ to transform our integrals into the right dimension. Using a loop-by-loop Baikov representation [38] one can show that the maximal cut of the first master integral in this sector is related to λ_1, i.e.

$$\begin{aligned} &\mathbf{D}^- I^{\text{max cut}}_{0,1,0,1,0,0,1,1,0} \sim \int \frac{dX}{Y} \quad \text{with} \\ &Y^2 = P_{\text{sunset}}(X) = X(X-4)(X-(1-\sqrt{x})^2)(X-(1+\sqrt{x})^2)\,, \end{aligned} \tag{11}$$

where the elliptic curve P_{sunset} is exactly the one already appearing in the two-loop equal-mass sunset integral [6]. Very similarly, one can show that the maximal cuts of the other two master integrals in this sector can be related to the other differential forms λ_2 and λ_3. Up to a simple rotation of the master integrals in ES_2 we find that their maximal cuts satisfy the Gauss-Manin system

$$B_{\text{ban}}(x;\epsilon) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3-x}{(9-x)(1-x)x} & -\frac{9-20x+3x^2}{(9-x)(1-x)x} & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\epsilon)\,, \tag{12}$$

where the second row is exactly describing the Picard-Fuchs equation

$$\mathcal{L}_{\text{ban}} = (9-x)(1-x)x\partial_x^2 + (9-20x+3x^2)\partial_x - (3-x)\,, \quad \mathcal{L}_{\text{ban}}\varpi_i(x) = 0 \tag{13}$$

[2]Here we use for convenience as differential of the second kind the derivative of the first one which is in cohomology equivalent to a linear combination of the standard forms $\frac{XdX}{Y}, \frac{X^2dX}{Y}$.

governing the periods ϖ_0, ϖ_1 of the sunset elliptic curve P_{sunset}. The two solutions can be computed using the Frobenius method yielding

$$\begin{aligned}\varpi_0(x) &= 1 + \frac{x}{3} + \frac{5x^2}{27} + \frac{31x^3}{243} + \frac{71x^4}{729} + O(x^5)\,,\\ \varpi_1(x) &= \varpi_0(x)\log(x) + \frac{4x}{9} + \frac{26x^2}{81} + \frac{526x^3}{2187} + \frac{1253x^4}{6561} + O(x^5)\,,\end{aligned} \tag{14}$$

where the solution ϖ_0 is locally holomorphic and ϖ_1 contains also contributions of $\log(x)$. We also call both solutions the holomorphic and logarithmic periods, respectively. With these solutions, we can build the full matrix of fundamental solutions, which we also call the Wronskian matrix

$$W = \begin{pmatrix} \varpi_0(x) & \varpi_1(x) \\ \varpi_0'(x) & \varpi_1'(x) \end{pmatrix} = \begin{pmatrix} \varpi_0(x) & 0 \\ \varpi_0'(x) & \frac{1}{(9-x)(1-x)x\,\varpi_0(x)} \end{pmatrix} \begin{pmatrix} 1 & \tau = \frac{\varpi_1(x)}{\varpi_0(x)} \\ 0 & 1 \end{pmatrix} =: W_{\text{ss}} W_{\text{u}}\,, \tag{15}$$

where we define the unipotent part such that it satisfies the following nilpotent differential equation

$$\frac{d}{d\tau}\mathbf{W}_{\text{u}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{W}_{\text{u}}\,. \tag{16}$$

Here, τ is the τ-parameter of the sunset elliptic curve. We can interpret the semi-simple part of the Wronskian matrix as being the generalized leading singularities of our master integrals. Therefore, the main step to constructing an ϵ-factorized basis is to remove these leading singularities so that we just end up with the pure part of our integrals. Since this analysis was done in exactly two dimensions, we have to take into account a further rotation, removing the last non-ϵ-factorized entries of the differential equations. This rotation can easily be determined, and in total, we have the following result

$$\begin{aligned} T_{\text{ban}}(x;\epsilon) &= \begin{pmatrix} \epsilon & 0 & 0 \\ -\frac{1}{2}\left(9+30x-7x^2\right)\epsilon\varpi_0^2 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} W_{\text{ss}}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1+3\epsilon}{x} & -\frac{3}{x} & 0 \\ -\frac{1}{3}(3+x) & 0 & 1 \end{pmatrix} \quad \text{with} \\ A_{\text{ban}}(x) &= \begin{pmatrix} \frac{9+30x-7x^2}{2(9-x)(1-x)x} & \frac{1}{(9-x)(1-x)x\,\varpi_0^2} & 0 \\ \frac{\left(81+1188x-594x^2+372x^3-23x^4\right)\varpi_0^2}{4(9-x)(1-x)x} & \frac{9+30x-7x^2}{2(9-x)(1-x)x} & 36\varpi_0 \\ -\frac{2}{3}\varpi_0 & 0 & \frac{1}{x} \end{pmatrix}, \end{aligned} \tag{17}$$

which rotates the initial integrals in sector ES_2 to an ϵ-basis. Observe that here we have embedded naturally the 2×2 Wronskian matrix into a 3×3 matrix by adding an identity on the last entry.

Proceeding in a similar fashion with the other three elliptic sectors, we can also construct ϵ-factorized bases for them. Interestingly, in all four elliptic sectors, the elliptic curve can be traced back to the sunset one P_{sunset} defined in (11). To obtain a full ϵ-form for all 51 master integrals, a further rotation, including mixing to the subsectors, has to be considered. Fortunately, there are no couplings between the different elliptic sectors, and thus it is not hard to find the final rotation, which can be found in the ancillary file of [27]. Let us remark that the only new transcendental functions we have to introduce are ϖ_0 and its derivative ϖ_0', whereas ϖ_0' only appears in the rotation and not in the final ϵ-form. Moreover, we find that the top sector integrals, i.e. the kite at two loops

and the paramecia integral at three loops, do not couple explicitly in the Gauss-Manin system to the differential forms of second kind which contain double poles. In the next section, we want to continue discussing the forms appearing in the final ϵ-factorized connection form at two and three loops and how the iterated integrals constructed out of them look like.

4. Two and three loop self-energy in terms of iterated integrals

After we have computed the ϵ-factorized differential equations relevant for the three-loop electron self-energy, we can look in detail into the appearing integration kernels. Collecting the independent kernels, we find the following 16 ones:

$$f_i = \left\{ \frac{1}{3+x}, \frac{1}{1+x}, \frac{1}{x}, \frac{1}{x-1}, \frac{1}{2-x}, \frac{1}{x-9}, \right. \\ \left. \frac{1}{\sqrt{(3+x)(x-1)}}, \frac{1}{\sqrt{(x-1)(x-9)}}, \frac{1}{\sqrt{(x-1)(x-9)}}\frac{1}{x} \right\} \quad \text{for } i = 1, \ldots, 9\,, \tag{18}$$

$$f_i = \left\{ \frac{1}{(x-9)(x-1)x\varpi_0(x)^2}, \varpi_0(x), \frac{\varpi_0(x)}{x-1}, \frac{(x-3)\varpi_0(x)}{\sqrt{(x-1)(x-9)}}, \frac{(x+3)^4\varpi_0(x)^2}{x(x-1)(x-9)}, \right. \\ \frac{(3+x)\left(81+729x-117x^2+11x^3\right)\varpi_0(x)^2}{x(x-1)(x-9)}, \\ \left. \frac{(81+1188x-594x^2+372x^3-23x^4)\varpi_0(x)^2}{x(x-1)(x-9)} \right\} \quad \text{for } i = 10, \ldots, 16\,, \tag{19}$$

in which the first 9 are familiar d log-kernels and the additional 7 kernels are of elliptic type. At one loop, we only find f_3, f_4 as independent entries in the differential equation, giving rise to harmonic polylogarithms[11]. In the two-loop case, we can identify three d log-kernels and four elliptic kernels given by f_3, f_4, f_6 and $f_{10}, f_{11}, f_{12}, f_{14}$, respectively. The iterated integrals constructed out of them can be identified with the space of iterated integrals of modular forms of $\Gamma_1(6)$[39] another well studied function space. These are all the integration kernels necessary to express all master integrals appearing in the self-energy up to three loops.

In the next step, we can analyze the iterated integrals and the associated kernels within the bare electron self-energy $\Sigma^{(3)}_{V,S}$, extending the analysis up to the finite part in the regulator ϵ. To do this, we also have to fix the boundary conditions for our 51 master integrals. We do this numerically around $x = 1$ by using `AMFlow` [40]. With a subsequent PSLQ algorithm, we can reconstruct the transcendental numbers, which are built from suitable products, through the following set of numbers

$$\{1, \pi, \log(2), \zeta(3), \mathrm{Li}_4(1/2), \zeta(5), \mathrm{Li}_5(1/2)\}\,. \tag{20}$$

Notice that these are also the transcendental numbers appearing in the $g-2$ of the electron except $\mathrm{Li}_5(1/2)$ which also drops out in the finite part of the self-energy but is necessary to fix all 51 master integrals up to weight five. For further details about the reconstruction, we again refer to [27]. With the boundary conditions at hand, it turns out that not all of the 19 kernels show up in the finite part of $\Sigma^{(3)}_{V,S}$, and similarly at lower loops an even smaller number as discussed above is required. In more detail, we only find the following iterated integrals in the different ϵ-orders in the self-energy

	$1/\epsilon^3$	$1/\epsilon^2$	$1/\epsilon$	ϵ^0
1L			1	$1, I(f_4) = \log(1-x)$
2L		1	$1, I(f_4)$	$\mathbb{MF}$
3L	1	$1, I(f_4)$	$\mathbb{MF}$	183 different iterated integrals of max. length 5

.

Here we have included also the empty iterated integral $I() \coloneqq 1$ and the set $\mathbb{MF}$ is defined by

$$\begin{aligned}\mathbb{MF} =\{&1, I(f_4), I(f_{11}), I(f_{12}), I(f_3, f_4), I(f_4, f_4), I(f_{10}, f_{11}),\\ &I(f_3, f_4, f_4), I(f_4, f_3, f_4), I(f_{11}, f_{10}, f_{11}), I(f_{12}, f_{10}, f_{11})\}\end{aligned} \tag{21}$$

and can be identified with a subset of iterated integrals of modular forms of $\Gamma_1(6)$. We can now make three important observations: Foremost, the maximal length of iterated integrals is given by 1, 3 and 5 for 1, 2 and 3 loops, respectively. Moreover, we see that poles at two and three loops contain exactly the same iterated integrals as the ones appearing already at lower loops. This shows directly that cancellations between the iterated integrals are happening analytically, such that the iterated integrals group in the right way as expected from renormalization. Thirdly, in the 183 iterated integrals in the finite part at three loops, the d log-kernel f_1 and all elliptic kernels proportional to ϖ_0^2, which can be traced back to the differential of the second kind, drop out and contribute only in higher orders in ϵ to the self-energy. This is an interesting observation since it suggests that these kernels are of too high transcendental weight to be allowed in the finite part of the self-energy.

To end our discussion about the self-energy, we also mention how one can evaluate the above introduced iterated integrals. For this, we consider a local series expansion of the kernels in (19) such that the multiple integrations can easily be performed on the level of generalized power series, including also $\log(x)$ terms. For this purpose, we have to use the local holomorphic series of the period ϖ_0 given in (14). By plugging all series expansions of all iterated integrals into the self-energy, we obtain a local series expansion of $\Sigma_{V,S}^{(\ell)}$ for $\ell \leq 3$. To extend this to a global function, we have to perform analytic continuation, which can be done using the differential equation. Notice that if we want to continue, e.g., to values $x > 1$, we have to use a new local holomorphic period ϖ_0^1 which is given by the holomorphic solution of (13) now around $x = 1$. As an example, we show in figure 4 the finite part of the bare self energy $\Sigma_V^{(3)}$. From figure 4 we can see immediately the threshold singularity at $x = 1$ and the vanishing imaginary part for $x < 1$ as expected.

5. Conclusions

In these proceedings, we have discussed the bare electron self-energy up to three loops. This is one fundamental object in quantum electrodynamics, which also exhibits elliptic contributions starting from two loops. We have given a strategy of tackling espacially these elliptic contributions by constructing an ϵ-factorized basis such that the master integrals and even the self-energy can be expressed through independent elliptic iterated integrals. These can be used to analyze the analytic structure of the electron self-energy, and, moreover, these functions can be evaluated in the whole parameter space by using generalized series expansions together with analytic continuation.

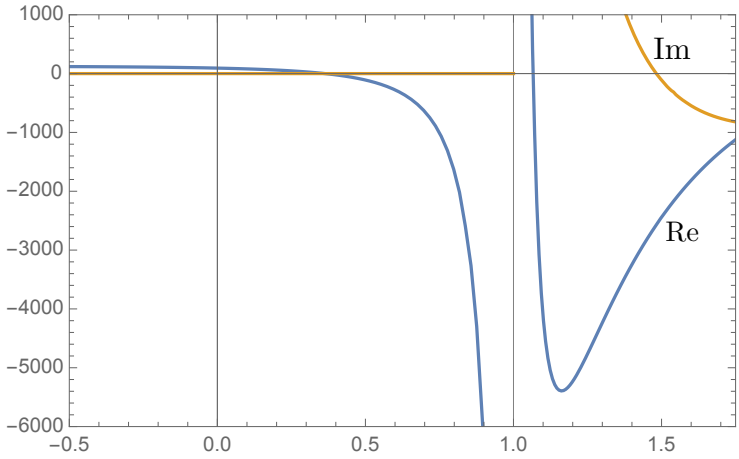

Figure 4: Plot of the finite part of the self-energy $\Sigma_V^{(3)}$ for $\xi = 1$.

We leave it to future work appearing in our upcoming publication [27] to discuss the renormalization of the electron self-energy. Moreover, further details and explicit results can be found in [27] and the thereby attached ancillary file.

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