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Perspective on properties of renormalization schemes at high loops

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We report on recent work on a class of renormalization schemes in QCD, termed the $\widetilde{\mathrm{MOM}}$ schemes. None of the renormalization group functions of the eight QCD $\widetilde{\mathrm{MOM}}$ schemes involve even zetas to five loops. A new scheme is introduced for scalar ϕ^3 theory in six dimensions where the full Laurent series in the regularizing parameter of the Green's function is absorbed into the renormalization constants. Designated as the $\overline{\mathrm{MaxS}}$ scheme it is equivalent to the $\widetilde{\mathrm{MOM}}$ scheme in the critical dimension.

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1. Introduction

Over the last decade or so there have been signficant developments in the high loop order renormalization of gauge theories. For instance the $\overline{\rm MS}$ β-function of Quantum Chromodynamics (QCD) is known to high precision, [1–10]. Results for the renormalization group functions in other schemes such as kinematic ones are not available to as many loops. For instance the QCD β-function in the momentum subtraction (MOM) schemes of Celmaster and Gonsalves, [11], is only available at four loops for the Landau gauge, [12, 13]. Once these core quantities are known the next stage is to determine the perturbative expansion of observables to the same order of precision. In this respect while such quantities are invariably evaluated in the $\overline{\rm MS}$ scheme there is no a priori reason why this scheme should be preferred over any other. They can equally well be determined in a MOM scheme for instance. In either case the observable will be available to a finite order in the coupling constant expansion and therefore would only be an approximation to the true value at some momentum scale. If one had a high enough number of terms then the theory uncertainty for a experimental measurement ought to be insignificant. Then the question arises as to how to arrive at an uncertainty value for the perturbative series truncation. One idea is to use the discrepancy in the value of the perturbative series when determined in several different schemes. Indeed an exploratory study of this idea was provided recently in [14] for the R ratio and the Bjorken sum rule. For example using experimental data for the former quantity in the $\overline{\rm MS}$, the MOM schemes of [11] as well as the mini-MOM scheme of [15], estimates from the three and four loop expressions were extracted for $\alpha_s^{\overline{\rm MS}}(M_Z)$, [14]. These respectively were $0.13281 \pm 0.00197^{+0.01171}_{-0.00986}$ and $0.13185 \pm 0.00053^{+0.01072}_{-0.00999}$. Here the error on the average is the envelope of the scheme values and the average value is the centre of the envelope, [14]. We record that these estimates are for an idealized situation where resonances and quark mass effects have not been taken into account. The exercise was carried out by ignoring these aspects so that the scheme issue would be the sole focus of the study. It is reasonably apparent that the uncertainty reduces with increasing loop order. While this is encouraging what would be interesting is to include additional schemes in the analysis to check whether this improves the uncertainty in the sense of tightening it. We will review some recent activity in this direction for QCD here by discussing a suite of schemes introduced in [16] and extended to five loops in [17].

2. Basics

We begin by recalling the basics behind defining a renormalization scheme. The Lagrangian is presented in terms of bare or classical variables that are not the optimum ones for describing quantum phenomena due to the presence of infinities. Therefore the variables have to be redefined in terms of renormalized ones which will lead to predictions from the field theory that are devoid of divergences. The procedure to enact this is far from unique. There is a requirement that after renormalization previously divergent Green's functions are finite. Two criteria are used to define a scheme. First for renormalizable theories the Green's functions that are divergent are evaluated at a specific momentum configuration using the regularized Lagrangian. Then the combination of renormalization constants associated with that function are defined by a specified method called a scheme. The most basic scheme is the minimal subtraction one where only the poles with respect to the regulator are removed by fixing the unknown terms of the relevant renormalization constants. There are other schemes that render the Green's functions finite.

This can be illustrated with a simple massless cubic theory with Lagrangian

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{g}{6}\phi^3 \tag{1}$$

where the coupling constant is g and the critical dimension is six. In terms of bare variables the divergent Green's functions are

$$\Gamma_2(p) = \langle \phi_0(p)\phi_0(-p)\rangle \ , \ \Gamma_3(p_1,p_2,p_3) = \langle \phi_0(p_1)\phi_0(p_2)\phi_0(p_3)\rangle \ . \tag{2}$$

To illustrate the structure of the Green's functions after renormalization we note that for $\Gamma_2(p)$ it will take the two following forms

$$\Gamma_2(p,-p)|_{p^2=\mu^2} = \begin{cases} \mu^2 \left[1 + \sum_{n=1}^{\infty} a_n g^{2n}\right] & \overline{\text{MS}} \\ \mu^2 & \text{MOM} \end{cases} \tag{3}$$

after renormalization in the $\overline{\text{MS}}$ and MOM schemes respectively where a_n are finite contributions. Here MOM denotes the momentum subtraction scheme of [11] which has the prescription that the Green's function takes its tree value at the subtraction point. For the vertex function there are many more potential schemes given the larger choice of subtraction points. For example one can nullify an external momentum, which is infrared safe in six dimensions despite being an exceptional configuration, which introduces the $\widetilde{\text{MOM}}$ schemes of [16] defined by

$$\Gamma_3(p,-p,0)|_{p^2=\mu^2} = \begin{cases} g + \sum_{n=1}^{\infty} b_n g^{2n+1} & \overline{\text{MS}} \\ g & \widetilde{\text{MOM}} \ . \end{cases} \tag{4}$$

Equally there are schemes for non-exceptional configurations one of which is the symmetric point one considered in [11] and defined by

$$\Gamma_3(p_1,p_2,-p_1-p_2)|_{p_i^2=\mu^2} = \begin{cases} g + \sum_{n=1}^{\infty} c_n g^{2n+1} & \overline{\text{MS}} \\ g & \text{MOM} \end{cases} \tag{5}$$

for $i = 1$ and 2 with $p_3^2 = \mu^2$ and c_n are constants like b_n but different in value since the subtraction points are not equivalent. The MOM scheme given in (5) is not unique since others can be constructed using different momentum configurations. For instance the interpolating momentum (iMOM) subtraction scheme of [18] defined by

$$\Gamma_3(p_1,p_2,p_3)|_{p_1^2=p_2^2=\mu^2, p_3^2=\omega\mu^2} = \begin{cases} g + \sum_{n=1}^{\infty} d_n g^{2n+1} & \overline{\text{MS}} \\ g & \text{iMOM} \end{cases} \tag{6}$$

is one such scheme that depends on the parameter ω defined for instance by $\omega = \frac{p_3^2}{p_1^2} = \frac{p_3^2}{p_2^2}$ where the $\omega \to 1$ limit recovers the symmetric point of [11]. However the most general situation for a 3-point vertex would regard these two dimensionless momenta ratios as independent. Given the wide variety of ways the coupling can be renormalized opens up the possibility of having hybrid schemes. These can be constructed where for example the 2-point function is rendered finite in $\overline{\text{MS}}$ say but the vertex function is made finite in one of the momentum subtraction schemes or its generalization to a 2-variable scheme. Finally it is worth noting that the β-function of each of the schemes such as MOM, $\widetilde{\text{MOM}}$ and iMOM carries information about the subtraction point itself via the finite parts that are related to a_n, b_n, c_n and d_n. This first becomes evident at three loops in a single coupling theory or two loops in a gauge theory for a non-zero covariant gauge parameter which produces a β-function which is gauge parameter dependent.

3. $\widetilde{\text{MOM}}$ examples

To illustrate some of the properties of the $\widetilde{\text{MOM}}$ scheme developed in [16] for QCD we have calculated the renormalization group functions in that scheme for ϕ^3 theory to five loops. Two components are required to achieve this. One is the explicit form of the five loop $\overline{\text{MS}}$ renormalization group functions, which are already available in [19, 20], and the other is the same quantities in the $\widetilde{\text{MOM}}$ scheme as well as their corresponding four loop renormalization constants. The latter were computed recently in [21] using the FORCER package provided in [22] written in FORM, [23]. To use FORCER for six dimensional computations required the determination of the FORCER masters to high order in the ϵ expansion in $d = 6 - 2\epsilon$ dimensions as (1) is dimensionally regularized. The masters were deduced via the Tarasov method, [24, 25], and provided in [21] up to weight 9 to be on an equivalent level to the four dimensional ones of the original package, [22]. The five loop $\widetilde{\text{MOM}}$ renormalization group functions were then deduced via properties of the renormalization group equation. For instance the couplings in the respective schemes are related by

$$g_{\widetilde{\text{MOM}}}(\mu) = \frac{Z_g^{\overline{\text{MS}}}}{Z_g^{\widetilde{\text{MOM}}}} g_{\overline{\text{MS}}}(\mu) \tag{7}$$

where

$$Z_g^{\widetilde{\text{MOM}}} \equiv Z_g^{\widetilde{\text{MOM}}}\left(a_{\widetilde{\text{MOM}}}(a_{\overline{\text{MS}}})\right) \tag{8}$$

and $a = g^2$. Then the β-functions are related by

$$\beta^{\phi^3}_{\widetilde{\text{MOM}}}(a_{\widetilde{\text{MOM}}}) = \left[\beta^{\phi^3}_{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \frac{\partial a_{\widetilde{\text{MOM}}}}{\partial a_{\overline{\text{MS}}}}\right]_{\overline{\text{MS}}\to\widetilde{\text{MOM}}} \tag{9}$$

where the mapping on the right hand side means the $\overline{\text{MS}}$ coupling is replaced by the inverse relation to (7). The field anomalous dimension can be deduced by a similar equation.

Consequently we have, [21],

$$\begin{aligned}\beta^{\phi^3}_{\widetilde{\text{MOM}}}(a) &= \frac{3}{4}a^2 - \frac{125}{144}a^3 + [-1296\zeta_3 + 26741]\frac{a^4}{10368} \\ &+ [-1370736\zeta_3 + 2177280\zeta_5 - 2304049]\frac{a^5}{186624} \\ &+ [389670912\zeta_3^2 + 3307195440\zeta_3 + 89151840\zeta_5 \\ &\quad -5640570432\zeta_7 + 2190456157]\frac{a^6}{26873856} + O(a^7)\end{aligned} \tag{10}$$

where ζ_n is the Riemann zeta function. One interesting property is manifest and that is the absence of the even zetas, ζ_4 and ζ_6, which are present in the $\overline{\text{MS}}$ scheme β-function. Indeed this property is not restricted to (1) as it has been noted previously in QCD in [16, 26–29] and more recently checked for the core renormalization group functions, [17]. More generally the criteria for the absence of π in β-functions was formulated in the no-π theorem, [28, 29], having been motivated by observations in [30], and also discussed more recently in [31] in the multicoupling context.

One application of the six dimensional Forcer masters is to use them to explore a generalization of the $\widetilde{\mathrm{MOM}}$ scheme definition. That scheme removed the finite part of the 2- and 3-point functions at the subtraction point, where one leg of the latter had a nullified momentum, in addition to the poles in ϵ. This can be modified to the case where all the higher order powers in the ϵ expansion are removed too with the scheme being designated the maximal subtraction scheme and labelled by $\overline{\mathrm{MaxS}}$, [21]. The result of renormalizing ϕ^3 theory in this scheme is to produce renormalization group functions in six dimensions which are formally equivalent to those of the $\widetilde{\mathrm{MOM}}$ scheme. However this is not the case in the regularized theory where $\epsilon \neq 0$ when $\beta^{\phi^3}_{\overline{\mathrm{MaxS}}}(a) \neq \beta^{\phi^3}_{\widetilde{\mathrm{MOM}}}(a)$ since the coefficient of the $O(\epsilon^n)$ term of a renormalization group function is related to the coefficient of the $O(\epsilon^{n-1})$ term of the corresponding renormalization constant for $n \geq 1$. This ϵ dependence in the β-function and other renormalization group functions is necessary to ensure that the critical exponent $\hat{\omega} = \beta'(a^\star)$ is the same in all schemes where $a^\star$ is the Wilson-Fisher fixed point in d-dimensions. For (1) we note that $\hat{\omega}$ only depends on rationals and ζ_n for $3 \leq n \leq 5$ to $O(\epsilon^5)$.

The $\widetilde{\mathrm{MOM}}$ scheme is not restricted to (1) and was actually introduced for QCD in [16] being developed originally at two loops. As QCD has more than one cubic vertex with several involving different fields means there are several ways of nullifying external legs leading to quite a few $\widetilde{\mathrm{MOM}}$ schemes. Subsequently the β-functions of several of these schemes were provided to four loops [26, 27]. More recently the renormalization group functions for eight possible $\widetilde{\mathrm{MOM}}$ schemes were determined to five loops in [17]. The same renormalization group method as discussed for ϕ^3 theory above was used. In other words QCD was renormalized to four loops in the respective $\widetilde{\mathrm{MOM}}$ schemes using the bare 2- and 3-point functions provided in [32] which were computed using the Forcer package. Since the five loop QCD $\overline{\mathrm{MS}}$ scheme renormalization group functions are available for an arbitrary colour group in [32] all the ingredients are known to deduce the same data for the $\widetilde{\mathrm{MOM}}$ schemes of [16]. For instance, the Yang-Mills (YM) β-function for the $\widetilde{\mathrm{MOM}}_{ggg0g}$ scheme is, [17],

$$\begin{aligned}
\beta^{\mathrm{YM}}_{\widetilde{\mathrm{MOM}}_{ggg0g}}(a,0) &= -\frac{11}{3}C_A a^2 - \frac{34}{3}C_A^2 a^3 + \left[-\frac{6499}{48}C_A^3 + \frac{253}{12}\zeta_3 C_A^3 \right] a^4 \\
&+ \left[-\frac{10981313}{5184}C_A^4 - \frac{3707}{8}\zeta_3 \frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{8}{9}\frac{d_A^{abcd}d_A^{abcd}}{N_A} \right. \\
&\left. + \frac{6215}{24}\zeta_5 \frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{97405}{576}\zeta_5 C_A^4 + \frac{1116929}{1728}\zeta_3 C_A^4 \right] a^5 \\
&+ \left[-\frac{8598255605}{165888}C_A^5 - \frac{1161130663}{73728}\zeta_7 C_A^5 \right. \\
&- \frac{35208635}{3072}\zeta_7 C_A \frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{28905223}{2304}\zeta_3 C_A \frac{d_A^{abcd}d_A^{abcd}}{N_A} \\
&- \frac{15922907}{9216}\zeta_3^2 C_A^5 + \frac{131849}{3456}C_A \frac{d_A^{abcd}d_A^{abcd}}{N_A} \\
&+ \frac{4595789}{384}\zeta_3^2 C_A \frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{7284505}{1152}\zeta_5 C_A \frac{d_A^{abcd}d_A^{abcd}}{N_A}
\end{aligned}$$

$$+\frac{30643529}{2048}\zeta_3 C_A^5 + \frac{1667817635}{55296}\zeta_5 C_A^5\Big] a^6 + O(a^7) \quad (11)$$

where the second argument of the β-function is the gauge parameter α, $\widetilde{\text{MOM}}_{ggg0g}$ denotes a $\widetilde{\text{MOM}}$ scheme based on the triple gluon vertex, C_A, C_F, T_F are the usual colour factors and $d_A^{abcd}d_A^{abcd}$ is the rank four Casimir in the adjoint representation of dimension N_A. Clearly (11) is devoid of even zetas as are all the other five loop QCD $\widetilde{\text{MOM}}$ renormalization group functions, [17].

4. Generalities

Having provided an instance of a class of schemes with particular properties it is worthwhile considering scheme changes from a more general perspective. First we define the $\overline{\text{MS}}$ coupling renormalization constant and that of another scheme $\mathcal{S}$ by

$$Z_g = 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{n} z_{g\,nm}\frac{a^n}{\epsilon^m} \quad , \quad Z_g^{\mathcal{S}} = 1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n} z^{\mathcal{S}}_{g\,nm}\frac{a_{\mathcal{S}}^n}{\epsilon^m} \quad (12)$$

where there are finite contributions at each loop order in the scheme $\mathcal{S}$. The respective couplings are perturbatively related by

$$a_{\mathcal{S}} = \sum_{n=0}^{\infty} c_n a^{n+1} . \quad (13)$$

It is straightforward to show the connection the coefficients have with $Z_g^{\mathcal{S}}$ with the few terms given by

$$\begin{aligned} c_0 &= 1 \ , \ c_1 = -2z^{\mathcal{S}}_{g\,10} \ , \ c_2 = 7(z^{\mathcal{S}}_{g\,10})^2 - 2z^{\mathcal{S}}_{g\,20} \\ c_3 &= -30(z^{\mathcal{S}}_{g\,10})^3 + 18z^{\mathcal{S}}_{g\,10}z^{\mathcal{S}}_{g\,20} - 2z^{\mathcal{S}}_{g\,30} \end{aligned} \quad (14)$$

illustrating that the c_i depend solely on the finite parts of $Z_g^{\mathcal{S}}$. For instance the coupling constant mapping from the $\overline{\text{MS}}$ scheme to the $\widetilde{\text{MOM}}_{ggg0gg}$ scheme in the Landau gauge is, [17],

$$\begin{aligned} a_{\widetilde{\text{MOM}}_{ggg0gg}} &= a + 16a^2 + \left[\frac{93427}{192} - \frac{169}{4}\zeta_3\right] a^3 \\ &+ \left[\frac{129114635}{6912} - \frac{1822913}{576}\zeta_3 - \frac{124835}{192}\zeta_5\right] a^4 \\ &+ \left[\frac{4050665663}{4608} - \frac{393488663}{2304}\zeta_3 + \frac{980775}{512}\zeta_3^2 \right. \\ &\left. + \frac{1055749471}{36864}\zeta_7 - \frac{1387483355}{9216}\zeta_5 + 1335\zeta_4\right] a^5 + O(a^6) \end{aligned} \quad (15)$$

for the $SU(3)$ colour group and three active quarks. While this contains ζ_4 at four loops it is known that this term is key to the absence of ζ_4 in the $\widetilde{\text{MOM}}_{ggg0gg}$ renormalization group functions. The location of the various ζ_n terms in (15) is mirrored in the coupling constant maps for the other QCD $\widetilde{\text{MOM}}$ schemes.

One reason for the $\widetilde{\text{MOM}}_{ggg0gg}$ example rests in the potential connection with the C-scheme of [33]. That is a scheme which has its roots in the relation of the Λ parameters of two different schemes being related exactly by a one loop calculation [11]. The relation depends on the one loop finite part of the coupling renormalization or $z^{\mathcal{S}}_{g\,10}$ for an arbitrary scheme. In [33] the four loop coupling constant map is provided for the C-scheme to $\overline{\text{MS}}$ for $SU(3)$ and three quark flavours. It too shares the property of the QCD $\widetilde{\text{MOM}}$ scheme renormalization group functions in that no even zetas appear to five loops in various physical observables. So there is a possibility that one of the $\widetilde{\text{MOM}}$ schemes could correspond to the C-scheme. The specifics of the C-scheme renormalization prescription have not been recorded. Instead only the coupling constant map has been provided, [33], for three quarks. In particular, [33],

$$a_C(a) \; = \; a \; - \; \frac{9}{4} C\, a^2 \; - \; \left[\frac{3397}{2592} + 4C - \frac{81}{16} C^2 \right] a^3 \; + \; O(a^4) \tag{16}$$

where C is a free parameter within the C-scheme framework that can be tuned to reduce uncertainties on observables. Its origin can be traced back to the Λ ratio between two schemes. In that respect it is akin to $z^{\mathcal{S}}_{g\,10}$ and therefore we can use (16) to see if a connection can be made to one of the $\widetilde{\text{MOM}}$ schemes. Examining (15) we note that ζ_3 appears at $O(a^3)$ but is not present in (16) at the same order. A ζ_3 could be manufactured with a suitable choice of C but that would mean ζ_3 would be present at $O(a^2)$. There are no such contributions in any of the QCD $\widetilde{\text{MOM}}$ scheme mappings at that order even when we consider those with a non-zero gauge parameter. So we believe the C-scheme does not correspond to any of the $\widetilde{\text{MOM}}$ schemes. However what all the $\widetilde{\text{MOM}}$ schemes and C-scheme coupling mappings have in common is the ζ_4 term at $O(a^5)$ in (15) with the same coefficient. It can be shown, [17, 28, 29, 31], that this term is directly responsible for the absence of ζ_4 in the β-functions of these schemes.

We can also consider another general approach but in a different direction which is to extend the iMOM scheme. Instead of a scheme depending on one variable ω this can be replaced by the two independent parameters

$$x \; = \; \frac{p_1^2}{p_3^2} \quad , \quad y \; = \; \frac{p_2^2}{p_3^2} \tag{17}$$

for 3-point vertices. Consequently the renormalization functions will depend on x and y. For example the two loop Feynman gauge $SU(3)$ Yang-Mills β-function derived from the quark-gluon vertex is

$$\begin{aligned} \beta^{qqg}_{xy}(a,1)\Big|_{\text{YM}} \;=\; & -\, 11a^2 \\ & + \Big[[9x^3 - 9x^2y - 54x^2 - 9xy^2 + 64xy + 81x + 9y^3 \\ & \quad -54y^2 + 81y - 36]\Phi_1(x,y)\Delta - \frac{1064}{5}\Delta^2 \\ & \quad -[27x^2 - 68xy - 54x + 41y^2 - 68xy + 27]\ln(xy)\Delta \Big] \frac{5a^3}{12\Delta^2} \end{aligned} \tag{18}$$

where

$$\begin{aligned} \Phi_1(x,y) \;=\; & \frac{1}{\lambda} \Bigg[2\text{Li}_2(-\rho x) + 2\text{Li}_2(-\rho y) + \ln\left(\frac{y}{x}\right) \ln\left(\frac{(1+\rho y)}{(1+\rho x)}\right) \\ & + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3} \Bigg] \end{aligned} \tag{19}$$

with

$$\Delta(x,y) = x^2 - 2xy + y^2 - 2x - 2y + 1$$
$$\lambda(x,y) = \sqrt{\Delta} \quad , \quad \rho(x,y) = \frac{2}{[1 - x - y + \lambda(x,y)]} \ . \qquad (20)$$

The x and y dependence in the two loop term does not contradict any general properties since the β-function is gauge dependent. It is only in the $\overline{\text{MS}}$ scheme of a single coupling theory that the β-function is scheme independent to two loops. In schemes such as that which produced (18) one can determine the perturbative expansion of observables as a function of x and y. These parameters can then be varied to explore the uncertainty properties of the observable. In this respect x and y play a similar role to that of the free parameter C of the C-scheme but have a different origin. Moreover they could be restricted to a particular domain by constraints on the kinematics.

5. Conclusions

One of the main observations is the renormalization group equations of both QCD and scalar ϕ^3 theory are free of even zetas up to five loops in the $\widetilde{\text{MOM}}$ schemes. Moreover it has been shown that in the critical dimension of the latter theory the $\overline{\text{MaxS}}$ and $\widetilde{\text{MOM}}$ scheme renormalization group functions are equivalent. Indeed given the nature of both scheme definitions this property is probably applicable to theories other than ϕ^3 theory. In terms of usefulness of the results the availability of data on more schemes will mean that the estimate on the uncertainty deriving from the truncation of the perturbative expansion of an observable could in principle be improved. Finally while it is encouraging that the no-π theorem of [28, 29] appears to hold to five loops in the $\widetilde{\text{MOM}}$ schemes of [16] it remains to be seen whether this continues to high loop order. There is evidence that the theorem may breakdown at very high loop order from the analysis of [34][1].

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