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Two-loop amplitude computation with HELAC

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We present recent progress in constructing a generic two-loop amplitude reduction algorithm within the computational framework HELAC. Following the HELAC tree- and one-loop paradigm, we have completed the generation and validation of the two-loop amplitude-integrand. Based on the well-known OPP reduction approach, we express the amplitude in terms of Feynman integrals which are further reduced to master integrals using IBP identities. We also discuss the numerical reconstruction of the amplitude in $d = 4 - 2\epsilon$ dimensions.

Loops and Legs in Quantum Field Theory - LL2024,
14-19 April, 2024
Wittenberg, Germany

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PoS(LL2024)051

1. Introduction

The upcoming High Luminosity upgrade of the LHC will provide us with experimental data of unprecedented precision. Making sense of these data and exploiting the machine's full potential will require theoretical predictions of equally high precision. In recent years, the theoretical community has made substantial effort to meet the challenge of performing notoriously difficult perturbative calculations in Quantum Field Theory. The current precision frontier for the QCD dominated processes studied at the LHC lies at Next-to-Next-to-Leading-Order (NNLO) for $2 \to 3$ scattering processes [1, 2].

A typical NNLO calculation involves the computation of two-loop amplitudes (VV), the one- (RV) and two-particle (RR) unresolved contributions,

$$\begin{aligned}\sigma_{NNLO} = &\int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left|M_m^{(1)}\right|^2\right) J_m(\Phi) && VV \\ &+ \int_{m+1} d\Phi_{m+1} \left(2Re\left(M_{m+1}^{(0)*} M_{m+1}^{(1)}\right)\right) J_{m+1}(\Phi) && RV \\ &+ \int_{m+2} d\Phi_{m+2} \left|M_{m+2}^{(0)}\right|^2 J_{m+2}(\Phi) && RR\end{aligned}$$

where $M_m^{(0)}$, $M_m^{(1)}$ and $M_m^{(2)}$ are the $m-$particle tree-order, one-loop and two-loop amplitudes and $J_m(\Phi)$ denote appropriate jet functions. The calculation of real-virtual RV and real-real RR contributions is well developed and automated, based mainly on subtraction schemes [3–8]. The computation of two-loop amplitudes constitutes the main bottleneck of the NNLO calculations.

The key aspects of a two-loop amplitude computation can be summarized as follows:

1. *construction* of two-loop integrands. This can be achieved either by computing individual Feynman diagrams or by using recursion relations (see *e.g.* Refs. [9–13];

2. *reduction* of two-loop amplitudes in terms of Master Integrals [14–32]. This can be achieved at the integrand or at the integral level;

3. *calculation* of Master Integrals [33–50] based on analytical or (semi-)numerical methods.

Each of these steps comes with its own challenges, and the developments during the last years have yielded a number of results for $2 \to 3$ cross sections at NNLO accuracy [51–56].

In this contribution, we present the steps towards the construction of HELAC-2LOOP, a framework for automated two-loop calculations. We focus in particular on items 1 and 2 in the list above. In Section 2 we illustrate the basic details of the algorithm for the computation of two-loop integrand functions. In Section 3 we sketch the approach for the reduction of two-loop amplitudes that we plan to implement in our framework. Finally in Section 4 we discuss the $d = 4 - 2\epsilon$ reconstruction of loop amplitudes.

2. Construction of two-loop integrands

The construction of two-loop integrands is described in Ref. [57]. For completeness we briefly present the main steps of the algorithm. The first step towards the construction of two-loop

integrands consists in the generation of *loop topologies*. A key observation is that all two-loop topologies describing arbitrary processes in the Standard Model[1] (SM) fall into one of the three master categories, that we name "*Theta*", "*Infinity*" and "*Dumbbell*" for brevity. The topologies generated at this stage carry no information about the flavor and color of the propagators in the loop. This information is provided in the next step, which we refer under the name of *color-flavor dressing* (schematically illustrated in Figure 1 in the context of a simple example). In the last step,

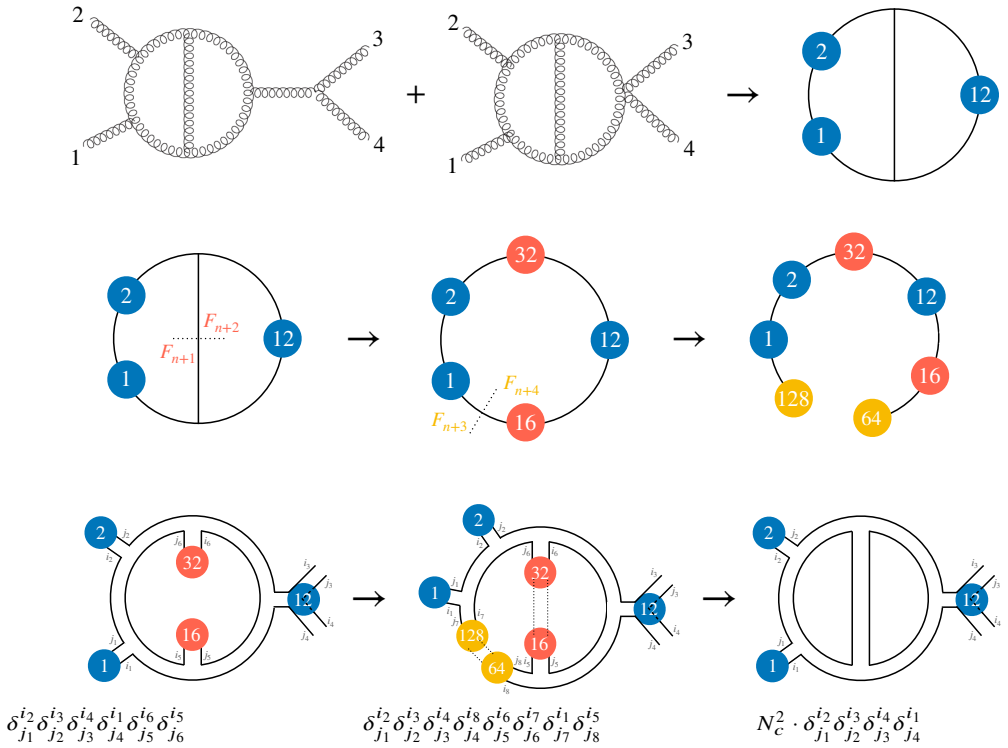

Figure 1: Schematic illustration of color-flavor dressing for a simple two-loop topology.

a set of recursion relations is constructed for each two-loop topology. These relations encode the instructions to compute recursively the numerators that will be used by HELAC during numerical evaluations. They can be generated once and stored in a file named *skeleton*. An example of a two-loop numerator stored in the *skeleton* is provided in Fig. 4 of reference [57].

In Table 1 the data regarding the skeleton for several two-loop processes is presented. We also include one-loop results for comparison.

3. Reduction of two-loop amplitudes at integrand level

Let us consider a two-loop contribution, generated according to the algorithm described in Section 2 and characterised by N_P loop propagators for a generic process with n external particles. Let $\{D_1, \ldots, D_{N_P}\}$ be the denominators of the loop propagators (we will refer to them as *propagators* in the following for brevity), $\{p_1, \ldots, p_n\}$ the four-dimensional momenta of the external particles

[1] More generally, this statement is true for any model whose Feynman rules include up to four-particle vertices

gray ***Process***	#	***Loop-Flavors***	***Color***	***Size***	***Time***	***Nums***
$gg \to gg$	2	$\{g, c, \bar{c}\}$	Lead.	8.9 MB	15.017s	4560
$gg \to gg$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	110.6 MB	6m 54.574s	89392
$gg \to q\bar{q}$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	16.1 MB	3m 14.509s	13856
$gg \to ggg$	2	$\{g, c, \bar{c}\}$	Lead.	300.0 MB	21m 42.609s	81480
$gg \to q\bar{q}g$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	686.1 MB	400m 31.591s	318964
$gg \to gg$	1	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	537.8 kB	2.386s	768
$gg \to ggg$	1	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	15.1 MB	8m 53.349s	11496
$gg \to gggg$	1	$\{g, c, \bar{c}\}$	Lead.	394.0 MB	104m 14.95s	19680

Table 1: Table containing information about the skeleton of some QCD processes at one- and two-loops. Therein, the column # refers to the number of loops, ***Loop-Flavors*** denotes the flavor of the particles included in the loops, and ***Color*** indicates the color order, with Lead. and Full referring to leading- and full-color approximation, respectively. The columns ***Size*** and ***Time***, indicate the size of the skeleton and the real-time consumed for its construction, respectively. The last column (***Nums***) reports the number of separate contributions (numerators) to the amplitude.

and $\{k_1, k_2\}$ the loop momenta. The latter, expressed in $d = 4 - 2\epsilon$ space-time dimensions, are decomposed as $k_i = \bar{k}_i + k_i^*$ where $\bar{k}_i$ is the four-dimensional and k_i^* the extra-dimensional part. The scalar product reads $k_i \cdot k_j = \bar{k}_i \cdot \bar{k}_j + \mu_{ij}$ $(i, j = 1, 2)$, where we have defined $\mu_{ij} = k_i^* \cdot k_j^*$. The integrand is understood to be a rational function of the form

$$\mathcal{R} = \frac{\mathcal{N}}{\mathcal{D}} \equiv \frac{\sum_a c_a(\vec{s}, \varepsilon) \left(z_1^{(a)}\right)^{\beta_1} \cdots \left(z_{n_a}^{(a)}\right)^{\beta_{n_a}}}{D_1 \cdots D_{N_P}}, \tag{1}$$

where the β's are integers, $\vec{s}$ denotes generically scalar products of the form $p_i \cdot p_j$ and $z^{(a)} \in S = \{k_i \cdot k_j,\ k_i \cdot p_j,\ k_i \cdot \eta_j\}$. The η_j are *transverse vectors* defined such that $\eta_i \cdot p_j = 0$. Following Eq.(1), the scalars $z^{(a)}$ that can be decomposed as linear combinations of propagators D_i cancel with the denominator. Then, the integrand can be recast in the form

$$\mathcal{R} = \sum_{m=0}^{N_P} \sum_{\sigma} \frac{\sum_j \tilde{c}_j^{(\sigma)}(\vec{s}, \varepsilon) \prod_{k=1}^{n_{\text{T+ISP}}^{(m)}} \left(\bar{z}_k^{(\sigma)}\right)^{\alpha_k^{(j)}}}{D_{\sigma_1} \cdots D_{\sigma_m}} \tag{2}$$

where the α's are integers and σ denotes any possible subset of $\{1, \ldots, N_P\}$ consisting of m elements. The residual scalar products appearing in the numerator (that we label as $\bar{z}_k^{(j)}$ for clarity) are either *transverse scalar products* (T) of the kind $k_i \cdot \eta_j$ or *irreducible scalar products* (ISP) which cannot be decomposed in terms of linear combinations of D_i.

Based on Eq.(2), one can express the numerator function $\mathcal{N}$ using the following equation:

$$\mathcal{N} = \sum_{m=0}^{N_P} \sum_{\sigma} \sum_{j} \tilde{c}_j^{(\sigma)}(\vec{s}, \varepsilon) \prod_{k=1}^{n_{\text{T+ISP}}^{(m)}} \left(\bar{z}_k^{(\sigma)}\right)^{\alpha_k^{(j)}} \prod_{i \notin \sigma} D_i \,. \tag{3}$$

or in a more schematic but compact notation,

$$\mathcal{N} = P_{\text{max-cut}} + \sum_i P_{\text{next-to-max-cut}}\, D_i + \sum_{ij} P_{\text{next-to-next-to-max-cut}}\, D_i D_j + \cdots . \tag{4}$$

All the P's are understood to be *polynomials* in the irreducible and transverse scalar products mentioned above.

Eqs.(3)-(4) allow us to easily sketch the necessary steps in order to complete a reduction for a given numerator:

- Identify the maximal set of loop propagators that can be set to zero (*maximal cut*) and find a set of solutions for k_1, k_2 which put all propagators simultaneously on shell;
- Write the equations for the coefficients $\mathbf{M} \cdot \vec{c} = \vec{\mathcal{N}}$ where $\mathbf{M}$ is a matrix of all monomials appearing in Eq.(3), evaluated on different values of cut-solutions, $\vec{c}$ is the vector of coefficients to be determined and $\vec{\mathcal{N}}$ is a vector of equal length with values of the numerator evaluated on these cut-solutions;
- Solve the system of equations, $\vec{c} = \mathbf{M}^{-1} \cdot \vec{\mathcal{N}}$, for the set of coefficients $\vec{c}$ and determine the polynomial $P_{\text{max-cut}}$;
- Subtract $P_{\text{max-cut}}$ from $\mathcal{N}$ and repeat the procedure iteratively to find $P_{\text{next-to-max-cut}}$ and so on;
- Once all the polynomials have been determined, verify that the right-hand side of Eq.(4) matches numerically with the input numerator for arbitrary values of loop momenta. This is the so-called $N = N$ test. At this point the reduction is complete.

Let us note that this is an entirely algebraic procedure that holds for any loop order. At the end of the reduction, the two-loop amplitude is expressed as

$$\mathcal{A} = \sum_i C_i F_i \tag{5}$$

where C_i are coefficients which only depend on the external momenta and wave functions and F_i are Feynman integrals of the form

$$F_i \equiv F_{a_1 \ldots a_N} = \int d^d k \frac{\overbrace{(D_{m+1})^{a_{m+1}} \cdots (D_N)^{a_N}}^{ISP}}{\underbrace{(D_1)^{a_1} \cdots (D_m)^{a_m}}_{RSP}} \tag{6}$$

The latter will be decomposed in terms of Master Integrals using IBP tables and then evaluated to obtain fully numerical results for all terms in the $\epsilon-$expansion of the amplitude.

It is illuminating to consider the steps sketched above for a specific $gg \to gg$ numerator example, corresponding to the double-box topology.

In $d = 4 - 2\epsilon$ dimensions, there are 11 degrees of freedom: 8 from the 4-dimensional components of the loop momenta k_1, k_2, and 3 from the ϵ-dimensional contributions of k_1^2, k_2^2 and $k_1 \cdot k_2$, respectively μ_{11}, μ_{22} and μ_{12}. In the case of the maximal cut, imposing the on-shell condition sets 7 cut equations, hence we are left with 4 free parameters, and the right-hand side of Eq.(4) consists of 70 monomials, *i.e.* there are 70 coefficients to be fitted. Using these 4 free parameters we can construct a full-rank matrix $\mathbf{M}$ and solve the system. We have completed a Mathematica

simulation of this fit, for all (sub-)topologies of the two-loop $gg \to gg$ amplitude and get agreement with the known results from `Caravel` [16].

In $d = 4$ dimensions, we begin with 8 degrees of freedom which we can use to construct solutions to the cut equations, therefore after imposing the on-shell condition only 1 parameter is left to build solutions with. Cut-solution sets with 1 free parameter cannot generate a matrix of rank 70. In $d = 4$ dimensions however, we can use Gram determinant relations to reduce the number of monomials and the number of the corresponding coefficients that we need to fit. Indeed after taking into account the Gram determinant relations we find that the matrix is again full-rank and this allows the determination of all 28 indepedent coefficients for the example of the $2 \to 2$ double-box [58].

To avoid the complication of including the Gram determinant relations, we can adopt a hybrid scheme, as in the one-loop OPP reduction [59], where the on-shell solutions are parametrized by the 11 degrees of freedom, as a result of solving the on-shell equations for the propagators by including appropriate mass terms proportional to $\mu_{11}, \mu_{22}, \mu_{12}$, as in $d = 4 - 2\epsilon$, and perform the fit, reconstructing the residues in Eq. (3), in $d = 4$ dimensions. A detailed description of the algorithm will be given in a forthcoming publication.

4. Numerical reconstruction of the amplitude in $d = 4 - 2\epsilon$

A solution of the reduction problem in $d = 4$ dimensions, as described in Section 3, is still missing contributions related to the explicit dependence of the numerators upon ϵ and μ_{ij}. Enabling `HELAC` to perform numerical computations of numerators in $d = 4 - 2\epsilon$ dimensions is therefore a highly desirable path. In this Section we briefly sketch the procedure that we are developing to this purpose, considering the case of a $n-$gluon amplitude at one loop as an example. In particular we focus on the computation of the numerator of the graph shown in Figure 2.

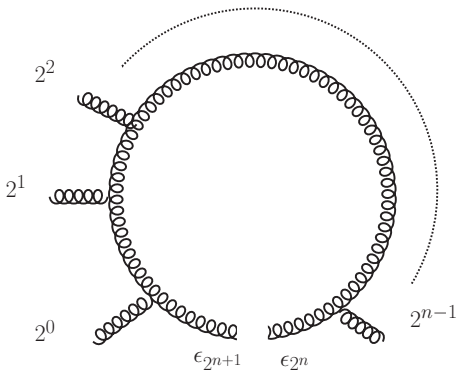

Figure 2: `HELAC` representation of a one-loop $n-$gluon amplitude.

Following standard `HELAC` notation [60–63], each external particle is labelled with an integer 2^i ($i \in [0, n-1]$), and the corresponding polarization vector is ε_{2^i}. Let $\varepsilon^{(\lambda)}_{2^n}$ and $\varepsilon^{(\lambda)}_{2^{n+1}}$ denote the polarization vectors of the two extra gluons originating from the cut of the one loop propagator, which satisfy the relation $\sum_\lambda \varepsilon^{(\lambda)\,\alpha}_{2^n} \varepsilon^{(\lambda)\,\beta}_{2^{n+1}} = g^{\alpha\beta}$. Following reference [64], we decompose the loop momentum q, gamma matrices and the metric as follows,

$$\bar{q}^\alpha = q^\alpha + \tilde{q}^\alpha, \quad \bar{\gamma}^\alpha = \gamma^\alpha + \tilde{\gamma}^\alpha, \quad \bar{g}^{\alpha\beta} = g^{\alpha\beta} + \tilde{g}^{\alpha\beta} \tag{7}$$

where the barred quantities live in $d = 4 - 2\epsilon$ dimensions and the tilded quantities refer to the $(d-4)$-dimensional contribution. Furthermore, we define

$$\mu = \tilde{q} \cdot \tilde{q} = \tilde{\not{q}}\tilde{\not{q}} \tag{8}$$

$$d - 4 = \tilde{g}^{\alpha\beta}\tilde{g}_{\alpha\beta} = \tilde{\gamma}^{\alpha}\tilde{\gamma}_{\alpha} \tag{9}$$

Our goal is to compute the numerator in $d = 4 - 2\epsilon$ dimensions [65]:

$$\mathcal{N}(q, \mu, \epsilon) \tag{10}$$

Knowing that the parts proportional to μ and ϵ arise from the following contributions

$$\begin{gathered} q^2 X \to \mu X \\ \sum_{\lambda} \varepsilon^{(\lambda)}_{2^n} \cdot \varepsilon^{(\lambda)}_{2^{n+1}} X \to (d-4)\, X \\ \sum_{\lambda} \left(\varepsilon^{(\lambda)}_{2^{n+1}} \cdot q\right) \left(\varepsilon^{(\lambda)}_{2^{n+1}} \cdot q\right) X \to \mu X \end{gathered} \tag{11}$$

the question is how to get X's from the recursive equations.
With the notation $X|_Y$ denoting the coefficient of an arbitrary quantity X with respect to an arbitrary quantity Y, we define

$$J_N[q] = J_N|_q \quad J_N[\varepsilon_{2^n}] = J_N|_{\varepsilon_{2^n}} \quad J^{\alpha}_N[\varepsilon_{2^n} \cdot q] = J_N|_{\varepsilon_{2^n} \cdot q} \quad Y_N[q] = J_N[\varepsilon_{2^n} \cdot q]\big|_q \tag{12}$$

and

$$c_1 = p_{N_1}\big|_q \quad c_2 = p_{N_2}\big|_q \tag{13}$$

where N labels the current, J^{α}_N, at a given stage of the recursion. The above defined objects (assuming that the label N_1 is associated to the current containing the loop propagator) satisfy the following recursive equations:

$$J^{\alpha}_N = V^{\alpha}\left(J_{N_1}, p_{N_1}; J_{N_2}, p_{N_2}\right) + \mu\,(c_1 + 2c_2)\, J^{\alpha}_{N_2} J_{N_1}[q] \tag{14}$$

$$J_N[q] = (c_1 - c_2)\, J_{N_1} \cdot J_{N_2} - \left(2p_{N_1} + p_{N_2}\right) \cdot J_{N_2}\, J_{N_1}[q] \tag{15}$$

$$J_N[\varepsilon_{2^n}] = -\left(2p_{N_1} + p_{N_2}\right) \cdot J_{N_2} J_{N_1}[\varepsilon_{2^n}] \tag{16}$$

$$\begin{aligned} J^{\alpha}_N[\varepsilon_{2^n} \cdot q] &= V^{\alpha}\left(J_{N_1}[\varepsilon_{2^n} \cdot q], p_{N_1}; J_{N_2}, p_{N_2}\right) + J_{N_1}[\varepsilon_{2^n}]\,(c_1 + 2c_2)\, J^{\alpha}_{N_2} \\ &+ \mu\, Y_{N_1}[q]\,(c_1 + 2c_2)\, J^{\alpha}_{N_2} \end{aligned} \tag{17}$$

$$Y_N[q] = J_{N_1}[\varepsilon_{2^n} \cdot q] \cdot J_{N_2} - \left(2p_{N_1} + p_{N_2}\right) \cdot J_{N_2} Y_{N_1}[q] \tag{18}$$

where $N = N_1 + N_2$, and

$$\begin{aligned} V^{\alpha}\left(J_{N_1}, p_{N_1}; J_{N_2}, p_{N_2}\right) \equiv\ & -\left(J_{N_2} \cdot (2p_{N_1} + p_{N_2})\right) J^{\alpha}_{N_1} \\ & + \left(J_{N_1} \cdot (p_{N_1} + 2p_{N_2})\right) J^{\alpha}_{N_2} \\ & + \left(J_{N_1} \cdot J_{N_2}\right) \left(p_{N_1} - p_{N_2}\right)^{\alpha} \end{aligned} \tag{19}$$

represents the three-gluon vertex current.
The final form of the numerator takes the form

$$N(q,\mu,\epsilon) = J_{2^{n+2}-2}\cdot\varepsilon_1 + (d-4)\,J_{2^{n+2}-2}[\varepsilon_{2^n}]\,(p_{2^{n+1}-2}-p_{2^{n+1}})\cdot\varepsilon_1 + \mu\left(Y_{2^{n+1}-2}[q]\,(p_{2^{n+1}-2}-p_{2^{n+1}})\cdot\varepsilon_1 - \left(J_{2^{n+1}-2}[\varepsilon_{2^n}\cdot q]\right)\cdot\varepsilon_1\right) \tag{20}$$

Similar equations hold for all possible currents, including four-gluon vertices, quarks and ghosts. Details on the numerical reconstruction of the amplitude in $d = 4 - 2\epsilon$ dimensions will appear in a forthcoming publication.

5. Summary

We have presented an approach for computing two-loop integrands for arbitrary processes in a fully automated way. Furthermore we have sketched the algorithm for the reduction of two-loop amplitudes to Master Integrals. We also discussed the numerical reconstruction of the amplitude in $d = 4 - 2\epsilon$ dimensions. These results are part of the ongoing efforts to develop HELAC-2LOOP, a framework for automated two-loop amplitude calculations.

6. Acknowledgment

The work of G. Bevilacqua, C. G. Papadopoulos, and A. Spourdalakis is supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the "2nd Call for H.F.R.I. Research Projects to support Faculty Members & Researchers" (Project Number: 02674 HOCTools-II). The work of D. Canko is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program grant agreement 101040760, *High-precision multi-leg Higgs and top physics with finite fields* (ERC Starting Grant FFHiggsTop).

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