PROCEEDINGS OF SCIENCE

Anomalous dimensions of leading twist operators to four loops

Giulio Falcioni[a,b,*]

[a] *Physik Institute, Universität Zürich,*
190 Winterthurerstrasse, Zürich, Switzerland
[b] *Dipartimento di Fisica, Università di Torino,*
Via Pietro Giuria 1, Torino, Italy

E-mail: giulio.falcioni@physik.uzh.ch, giulio.falcioni@unito.it

The anomalous dimensions of the twist-2 operators control the scale evolution of the Parton Distribution Functions. The precision targets set by the forthcoming experiments at the LHC and at the EIC require the calculation of the four-loop anomalous dimensions. I will describe a theoretical framework, based on the renormalisation of off-shell correlators, which allows to determine these quantities efficiently. Finally, I will discuss recent progress obtained within this approach.

Loops and Legs in Quantum Field Theory (LL2024)
14-19, April, 2024
Wittenberg, Germany

*Speaker

© Copyright owned by the author(s) under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License (CC BY-NC-ND 4.0).

PoS(LL2024)058

1. Introduction

The precision physics programme at the Large Hadron Collider (LHC) [1] and at the forthcoming Electron Ion Collider (EIC) [2] is pushing the frontier of the calculation of QCD radiative corrections towards the Next-to-Next-to-Next-to Leading Order ($\mathrm{N^3LO}$) accuracy [3]. In recent years, there has been significant progress towards the calculation of the scale evolution of the Parton Distribution Functions (PDFs) to $\mathrm{N^3LO}$, which is one of the building blocks of the collider observables together with the partonic cross sections. Plenty of information is now available regarding the $\mathrm{N^3LO}$ splitting function that governs the evolution of the flavour non-singlet PDFs: its leading [4] and next-to-leading [5] contributions in the limit of large number of quark flavours, n_f, are known analytically, as well as the QED-like term $n_f C_F^3$ [6]; the leading colour (planar) limit was computed in ref. [7]. In addition, the first 11 even Mellin moments ($N = 22$) of the non-singlet splitting function are available [8] and can be used to construct high-precision approximations of this splitting functions in a wide region of the LHC and EIC phase space [7].

The evolution of the flavour singlet PDFs, namely the gluon density $\mathrm{g}(x, \mu^2)$ and the combination of quark densities

$$\mathrm{q_s}(x, \mu^2) = \sum_{i=u,d,s} f_i(x, \mu^2) + f_{\bar{i}}(x, \mu^2), \tag{1}$$

with f_i labelling the PDF of the quark i, is governed by a 2-by-2 matrix of splitting functions

$$\mu^2 \frac{d}{d\mu^2} \begin{pmatrix} \mathrm{q_s}(x, \mu^2) \\ \mathrm{g}(x, \mu^2) \end{pmatrix} = \int_x^1 \frac{dy}{y} \begin{pmatrix} P_{\mathrm{qq}}(y) & P_{\mathrm{qg}}(y) \\ P_{\mathrm{gq}}(y) & P_{\mathrm{gg}}(y) \end{pmatrix} \begin{pmatrix} \mathrm{q_s}(\frac{x}{y}, \mu^2) \\ \mathrm{g}(\frac{x}{y}, \mu^2) \end{pmatrix}, \tag{2}$$

with the perturbative expansion

$$P_{\mathrm{ij}}(x) = \sum_{k>0} a^k\, P_{\mathrm{ij}}^{(k-1)}, \qquad a = \frac{\alpha_s}{4\pi}. \tag{3}$$

The $\mathrm{N^3LO}$ flavour singlet splitting functions $P_{\mathrm{ij}}^{(3)}$ are known analytically in the large-n_f limit [5, 9, 10]. Recently, the analytic form of the terms n_f^2 of $P_{\mathrm{qq}}^{(3)}$ [11] and $P_{\mathrm{gq}}^{(3)}$ [12] has been computed too. The first 5 even Mellin moments have been calculated using the structure functions in deep-inelastic scattering [13, 14]. A different approach relies on the relation between the Mellin moments of the PDFs and the Operator Matrix Elements (OMEs) [15] of the gauge invariant operators of leading twist, defined as the difference between the canonical mass dimension of the operator and its spin N. At twist 2 we have

$$O_{\mathrm{q}}^{(N)} = \frac{1}{2}\, \bar{\psi}\, \not{\Delta} (i\Delta.D)^{N-1} \psi, \tag{4}$$

$$O_{\mathrm{g}}^{(N)} = \frac{1}{2}\, (\Delta^\rho F^a_{\mu\rho})\, (i\Delta.D)^{N-2}\, (\Delta_\sigma F^{a\,\mu\sigma}), \tag{5}$$

where Δ_μ is an arbitrary lightlike vector, $F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f^{abc} A_b^\mu A_c^\nu$ is the field strength tensor and $D^\mu_{ab} = \delta_{ab} - igA_c^\mu (\mathbf{T}^c)_{ab}$ is the covariant derivative with $\mathbf{T}^c$ generator of the gauge group: the quark operator O_{q} has $(\mathbf{T}^c)_{ab} = (T_F^c)_{ab}$ and the gluon operator O_{g} has $(\mathbf{T}^c)_{ab} = i f^{acb}$. The anomalous dimensions of O_{q} and O_{g} agree, up to a sign, with the moments of the splitting functions

$$\gamma_{\mathrm{ij}}^{(k)}(N) = -\int_0^1 dx\, x^{N-1}\, P_{\mathrm{ij}}^{(k)}(x). \tag{6}$$

PoS(LL2024)058

In minimal subtraction, the anomalous dimensions $\gamma_{\rm ij}(N)$ are extracted from the single pole of the renormalisation constants

$$O_{\rm i}^{(N)} = Z_{\rm ij}(N) O_{\rm j}^{(N),\rm bare}. \tag{7}$$

Indeed, using the ϵ-expansion $Z_{\rm ij}(N) = \delta_{\rm ij} + \sum_{k=1}^{\infty} \frac{\delta Z_{\rm ij}^{(k)}(N)}{\epsilon^k}$, one has

$$\gamma_{\rm ij}(N) = a \frac{d}{da} \delta Z_{\rm ij}^{(1)}(N). \tag{8}$$

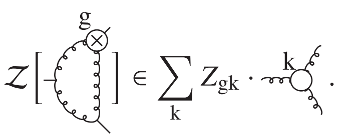
In turn, $Z_{\rm ij}$ is obtained by renormalising the off-shell OMEs $\mathrm{A}_{\rm ij} = \langle \mathrm{j}(p)|O_{\rm i}|\mathrm{j}(p)\rangle$, with $\mathrm{i}, \mathrm{j} = \mathrm{q}, \mathrm{g}$, by imposing

$$Z_{\rm j} \left[Z_{\rm ij} \mathrm{A}_{\rm jj}^{\rm bare}(g_0, \xi_0) + \sum_{\rm k \neq j} Z_{\rm ik} \mathrm{A}_{\rm kj}^{\rm bare}(g_0, \xi_0) \right] = \text{finite}, \tag{9}$$

where $Z_{\rm j}$ indicates the field strength renormalisation, if $\mathrm{j} = \mathrm{g}$, or the quark wave function renormalisation, if $\mathrm{j} = \mathrm{q}$, and g_0, ξ_0 are the bare coupling and gauge parameter. The sum over k in eq. (9) not only includes the gauge invariant operators $O_{\rm g}$ and $O_{\rm q}$, but it must extend to a set of unphysical operators [16], often dubbed *aliens*. These vanish when inserted in physical S-matrix elements, but they contribute to the off-shell OMEs. The alien operators that enter eq. (9) at two loops were determined long ago [17, 18]. However, the (off-shell) renormalisation of $O_{\rm g}$ and $O_{\rm q}$ beyond two-loop level has been developed only recently [19–21].

Sections 2 and 3 summarise the construction of the alien operators and the renormalisation of the physical operators in Yang Mills and in QCD, respectively. This step provides the necessary Feynman rules to compute all the OMEs $\mathrm{A}_{\rm kj}$ that contribute to eq. (9). The calculation of such OMEs boils down to evaluating 2-point integrals, which can by automated up to four-loop order by using the package `Forcer` [22], written in `FORM` [23–25].

The second ingredient of eq. (9) is the *mixing* among operators, due to the off-diagonal elements $Z_{\rm ik}$. Such mixing can be interpreted diagrammatically in terms of the operation $\mathcal{Z}$, which extract the UV counterterm of a Feynman diagram using the Bogoliubov R-operation. The counterterm of a general diagram featuring the insertion of the gauge invariant operators typically corresponds to a sum over several different operators with appropriate renormalisation constants, as shown below



(10)

In order to compute the mixing renormalisation constants $Z_{\rm ik}$, as in the example above, we must renormalise all the divergent Green functions that feature an insertion of O_q and $O_{\rm g}$, by imposing conditions analogous to eq. (9), see [19, 20]. The calculation of these off-shell Green functions at multiloop level is far from trivial and is not automated beyond the 2-point case, mentioned above. This introduces a potential bottleneck in the determination of the anomalous dimensions.

2. The alien operators in Yang Mills theory

The alien operators that mix with the gluon operator up to four loops in Yang Mills theory were determined in ref. [20]. The latter builds upon general theorems [26, 27] that show that the aliens

PoS(LL2024)058

receive contributions from *Equation of Motion* (EOM) and *BRST exact* operators. The former can be generated from the gauge invariant part of the Yang Mills Lagrangian via field redefinitions

$$\mathcal{L} = -\frac{1}{4}F^{a;\mu\nu}F^{a}_{\mu\nu} + c_{\rm g}O_{\rm g} \underset{A\to A+\delta A}{\longrightarrow} \mathcal{L} + O_{\rm EOM}, \tag{11}$$

where δA is the general field transformation of maximal twist

$$\delta A^a_\mu = \Delta_\mu \mathcal{G}^a(\Delta.A^a, \Delta.\partial\Delta.A^a, \dots), \tag{12}$$

yielding

$$\begin{aligned} O^{(N)}_{\rm EOM} \equiv (D.F)^a\,\mathcal{G}^a = (D.F)^a\Big[&\eta\,\partial^{N-2}A^a + g f^{aa_1a_2}\sum_{i_1+i_2=N-3}\kappa_{i_1i_2}(\partial^{i_1}A^{a_1})(\partial^{i_2}A^{a_2}) \\ &+ g^2\sum_{\substack{i_1+i_2+i_3\\ N-4}}\left(\kappa^{(1)}_{i_1i_2i_3}f^{aa_1z}f^{a_2a_3z} + \kappa^{(2)}_{i_1i_2i_3}d_4^{aa_1a_2a_3} + \kappa^{(3)}_{i_1i_2i_3}d_{4\widehat{ff}}^{aa_1a_2a_3}\right)(\partial^{i_1}A^{a_1})..(\partial^{i_3}A^{a_3}) \\ &+ g^3\sum_{\substack{i_1+..+i_4\\ N-5}}\left(\kappa^{(1)}_{i_1\dots i_4}(f\,f\,f)^{aa_1a_2a_3a_4} + \kappa^{(2)}_{i_1\dots i_4}d_{4f}^{aa_1a_2a_3a_4}\right)(\partial^{i_1}A^{a_1})..(\partial^{i_4}A^{a_4}) + O(g^4)\Big], \end{aligned} \tag{13}$$

where we use the notation $\Delta.A^a \equiv A^a$ (and similarly for the other Lorentz tensors) and where

$$\begin{aligned} &(f\,f\,f)^{aa_1a_2a_3a_4} = f^{aa_1b}f^{ba_2c}f^{ca_3a_4}, \qquad\qquad d_{4f}^{aa_1a_2a_3a_4} = d_4^{aa_1a_2b}f^{ba_3a_4}, \\ &d_{4\widehat{ff}}^{aa_1a_2a_3} = d_4^{aa_1bc}f^{ba_2d}f^{da_3c} - \frac{C_A}{3}d_4^{aa_1a_2a_3}. \end{aligned} \tag{14}$$

In the equations above, d_4^{abcd} is the symmetrised trace of the generators of the gauge group in the adjoint representation. The coefficients η, $\kappa_{ij}\dots$ in eq. (13) play the role of coupling constants, one for each operator. The contributions of $O(g^4)$ to eq. (13) are given in ref. [20].

Notably, the operator $O_{\rm EOM}$ *breaks* the gauge invariance of the Lagrangian $\mathcal{L}$. However, the extended Lagrangian in eq. (11) retains the symmetry under the *generalised* gauge transformation [17, 20], $A^a_\mu \longrightarrow A^a_\mu + \delta_\omega A^a_\mu + \tilde{\delta}_\omega A^a_\mu$, with

$$\delta_\omega A^a_\mu = D^{ab}_\mu\omega^b, \tag{15}$$

$$\tilde{\delta}_\omega A^a_\mu = -\Delta_\mu\left(\delta_\omega\mathcal{G}^a - g\,f^{abc}\mathcal{G}^b\omega^c\right). \tag{16}$$

The first equation defines the standard gauge transformation. The second equation is its generalisation, proportional to the lightlike source Δ_μ. By applying the BRST method [28, 29], eqs. (15) and (16) define *generalised BRST* transformations

$$s(A^a_\mu) = D^{ab}_\mu c^b, \tag{17}$$

$$\tilde{s}(A^a_\mu) = -\Delta_\mu\left(s(\mathcal{G}^a) - g\,f^{abc}\mathcal{G}^b c^c\right), \tag{18}$$

where it can be shown that the operator $s' = s + \tilde{s}$ is nihilpotent [20]. Therefore, eqs. (17) and (18) are employed to complete the Lagrangian in eq. (11) with the appropriate gauge-fixing and ghost terms. Eq. (17), which is independent on Δ, gives the standard BRST construction of the gauge

PoS(LL2024)058

fixing and ghost terms of the Yang-Mills Lagrangian [28, 29]. Eq. (18) generates the ghost operator of leading twist

$$O_G^{(N)} \equiv \tilde{s}\left(\bar{c}^a \partial^\mu A_\mu^a\right) = \bar{c}^a \partial\left(s\left(\mathcal{G}^a\right) - g\, f^{abc}\, \mathcal{G}^b\, c^c\right), \tag{19}$$

where $\mathcal{G}$ is the same polynomial in the field and its derivatives that appears in O_{EOM}, eq. (13).

The complete set of alien operators is given by the sum of O_{EOM} and O_G in eqs. (13) and (19), respectively. In particular, the terms multiplying each coupling constant, e.g. η, κ_{ij} for every allowed value of i, j, .., should be regarded as parts of the same alien, consisting of a gluonic and a ghost contribution. However, it turns out [20] that not all the coefficients in eq. (13) are independent, because there are relations imposed by the symmetry of the colour factors in eq. (13) and by the so-called antiBRST symmetry [30–33]. In [20], such relations were derived and solved for every fixed value $N \leq 16$, now extended to $N \leq 20$, in order to determine the relevant bases of independent alien operatators, which are closed under renormalisation [26]. Notably, the basis at $N = 2(N = 4)$ has only 1(2) aliens to all orders in perturbation theory. This was checked explicitly by computing the anomalous dimensions $\gamma_{\text{gg}}^{(3)}(2)$ and $\gamma_{\text{gg}}^{(3)}(4)$ [20]. As we increase N, depending on the loop order, the number of independent aliens increases, because of two effects. On the one hand, there are more terms that contribute to the sums in eq. (13). On the other hand, the operators involving many gluons, e.g. those associated to the couplings $\kappa_{ijk}^{(1)}$, start contributing only at higher loop order, due to their suppression in the coupling constant g. As check on the construction of the aliens, we verified that the three-loop alien counterterms computed for all-N in ref. [21] have a unique decomposition in every basis up to $N \leq 20$. The proliferation of aliens is not an issue for the calculation of the OMEs A_{kj} in eq. (9), since this is already automated. More severe computational issues potentially arise in the calcultion of the mixing constants Z_{ik} at multiloop level, when k is an alien involving many gluons. Indeed, as discussed at the end of sec. 1, this requires the renormalisation of multi-leg Green functions. However, this complication is largely absent in QCD, when we consider the fermionic contributions. For these, the calculation of the anomalous dimensions can be performed to four-loop order, as discussed in the next section.

3. From Yang Mills to QCD

The construction of the alien operators for QCD follows similar steps to those described in sec.2. The EOM operators of eq. (13) are modified by including the quark term in the equation of motion. It is convenient to organise the EOM operators in classes of operators featuring an increasing number of external gluons

$$O_{\text{EOM}}^{I} = \eta(N)\left(D.F^a + g\bar{\psi}\,\Delta\!\!\!/\, t^a \psi\right)\left(\partial^{N-2} A_a\right), \tag{20}$$

$$O_{\text{EOM}}^{II} = g\, f^{a a_1 a_2}\left(D.F^a + g\bar{\psi}\,\Delta\!\!\!/\, t^a \psi\right) \sum_{\substack{n_1, n_2 \geq 0 \\ n_1 + n_2 = N-3}} \kappa_{n_1\, n_2}^{(1)}\left(\partial^{n_1} A_{a_1}\right)\left(\partial^{n_2} A_{a_2}\right), \tag{21}$$

where the vertex of lowest valence in O_{EOM}^{I} has 2 gluons, the one in O_{EOM}^{II} has 3 gluons and similarly for class III, involving the terms multiplying $\kappa_{i_1 i_2 i_3}^{(1)}$, $\kappa_{i_1 i_2 i_3}^{(2)}$ in eq. (13), etc. Besides the prefactor featuring the equation of motion of the gauge invariant part of the Lagrangian, the operators in eqs. (20) and (21) are identical to the Yang Mills case, eq. (13). Indeed, the polynomial $\mathcal{G}$ that

PoS(LL2024)058

defines $O_{\rm EOM}$ cannot include monomials in the quark/antiquark fields, because these would give subleading twist. Hence, the polynomial $\mathcal{G}$ is not modified going from Yang Mills to QCD. Since the ghost alien operators are defined in terms of the polynomial $\mathcal{G}$, see eq. (19), the ghost operators of QCD are also identical to the Yang Mills ones and they are organised as follows

$$O_{\rm c}^{I} = -\eta(N)\,(\partial\,\bar{c}^{a})\left(\partial^{N-1}c_{a}\right), \tag{22}$$

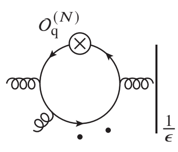
$$O_{\rm c}^{II} = -g\,f^{aa_1a_2}\,(\partial\,\bar{c}^{a})\sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2=N-3}}\eta^{(1)}_{n_1,n_2}\,(\partial^{n_1}A_{a_1})\left(\partial^{n_2+1}c_{a_2}\right), \tag{23}$$

where the constants $\eta^{(1)}_{n_1,n_2}$ are written in terms of $\eta(N)$ and $\kappa^{(1)}_{n_1\,n_2}$ [17, 18, 20].

The origin of the alien contributions to eq. (9) is ultimately related to the breaking of gauge invariance in the off-shell OMEs [17], hence it's not surprising that the inclusion of fermions does not alter significantly the renormalisation of the leading twist operators described in sec. 2. In fact, the renormalisation of the fermionic contributions is much simpler compared to the gluonic diagrams. The remaining two parts of this section will focus on two such examples: the calculation of the anomalous dimensions of the quark operator $\gamma_{\rm qi}$ and the determination of the terms proportional to n_f^2 in $\gamma_{\rm gq}$, respectively.

3.1 Renormalisation of the quark operator

The mixing between the gauge invariant quark operator $O_{\rm q}$ and the alien operators is much simpler than the mixing pattern of its gluonic counterpart. Notably, there isn't any alien operator featuring a quark-antiquark 2-point vertex, because eqs. (20) and (21) involve at least one gluon field. Thererefore, at leading order, i.e. at one loop, there is no mixing between $O_{\rm q}$ and the unphysical operators [16], contrary to the gluon case. The same argument applies to all the pure singlet diagrams with any number of external gluons



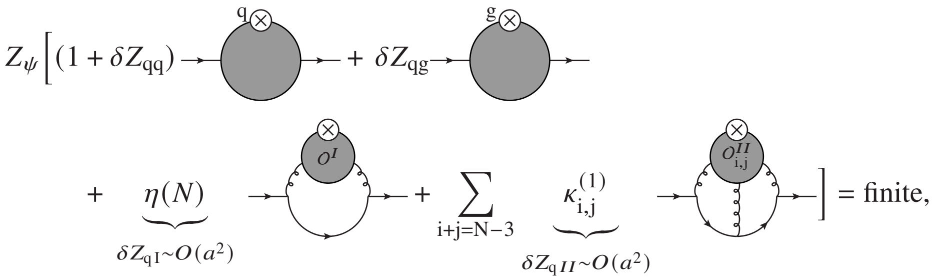
$\propto O_{\rm g}^{(N)}$,

where the RHS indicates the corresponding multi-gluon vertex in the gauge invariant gluonic operator $O_{\rm g}^{(N)}$, eq. (5). Such feature follows directly from the BRST structure of the alien operators. Indeed, any mixing between $O_{\rm q}$ and the gluonic EOM alien, $O_{\rm EOM}^{(N)}$, would also imply mixing with the ghost alien operators $O_{G}^{(N)}$, via eq. (19). However, at leading order there are no pure singlet diagrams with external ghosts, hence no such mixing can occur.

This fact has important consequences for the computation of the anomalous dimensions of the quark singlet operator. For instance, the anomalous dimensions $\gamma_{\rm qq}^{(N)}$ are determined from the

PoS(LL2024)058

renormalisation of A_{qq} according to eq. (9), which reads



(24)

where Z_ψ is the quark wave function renormalisation. The diagrams in the first line of the equation above represent the OMEs of the physical operators, A_{qq}, A_{gq}; those in the second line are the OMEs with external quarks and an insertion of the aliens of class I and II, eqs. (20)-(23). As a shorthand notation, the latter OMEs are labelled $A_{\eta q}$, $A_{\kappa^{(1)}_{ij} q}$ etc, where η and $\kappa^{(1)}_{ij}$ are the allowed couplings entering eqs. (20)-(23) for each value of N. Similarly, the mixing renormalisation constants of each alien, $\delta Z_{q\eta}$, $\delta Z_{q\kappa^{(1)}_{ij}}$ etc., are indicated with $\eta(N)$ and $\kappa^{(1)}_{ij}$.

By imposing the finiteness of eq. (24), the renormalisation constant δZ_{qq} is determined to L-loop level in terms of the poles of the OME A_{qq} at L loops and of the products of quantities at lower loop order. These include the alien contributions in the second line of eq. (24). As discussed above, the mixing $\eta(N)$, $\kappa^{(1)}_{ij}(N)$ must be *at least* of $O(a^2)$. Therefore, the OMEs $A_{\eta q}$, $A_{\kappa^{(1)}_{ij} q}$ are computed up to $O(a^2)$, to reach four-loop accuracy in δZ_{qq}. These OMEs are computed straightforwardly, e.g. with `Forcer`. In turn, the maximal loop order required in $\eta(N)$ and $\kappa^{(1)}_{ij}$ is fixed by the lowest order possible in $A_{\eta q}$ and $A_{\kappa^{(1)}_{ij} q}$: since $A_{\eta q}$ starts at one loop, $\eta(N)$ is computed to $O(a^3)$ at each value of N. This can be achieved by renormalising the OMEs A_{qc}, where c is an external ghost-antighost pair. The constants $\kappa^{(1)}_{ij}$ are needed only at the leading order $O(a^2)$. To this accuracy one has [18, 21, 34]

$$\kappa^{(1)}_{i,j} = \frac{\eta(N)}{8}\left[(-1)^N - 3\binom{N-2}{i} + 3\binom{N-2}{i+1}\right]. \tag{25}$$

Therefore, all the terms in eq. (24) can be computed to the required accuracy to determine $\gamma^{(3)}_{qq}(N)$ for each fixed value of N. Indeed, in this channel only few aliens can contribute and their mixing constants are determined systematically. The results of ref. [34] cover $N \le 20$ and the calculation can be pushed to higher N. The limiting factor is the memory required for the reduction of A_{qq} to master integrals with `Forcer`. With a similar approach, the anomalous dimensions $\gamma^{(3)}_{qg}(N)$ were computed through $N = 20$ [35], providing key information on the top line of the matrix in eq. (2).

3.2 Double fermionic contributions to $\gamma^{(3)}_{gq}$

Now we turn to the renormalisation of the gluon operator O_g. Ref. [12] focuses on the contributions proportional to n_f^2 in $\gamma^{(3)}_{gq}$. These are compuuted using eq. (9), wiht i = g and j = q and by restricting to the terms that feature two closed quark loops, either in the OMEs or in the mixing constants Z_{jq}.

PoS(LL2024)058

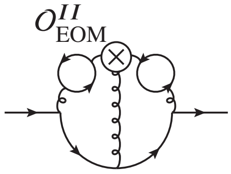


Figure 1: Double fermionic contributions to $\mathrm{A}_{\kappa^{(1)}_{\mathrm{ij}}c}$.

The n_f^2 terms of the physical OME A_{gq} at four loops are computationally cheaper to compute, compared to the fully gluonic diagrams, because the most complicated `Forcer` topologies do not enter this colour factor. The alien contributions to eq. (9) can be classified by using a counting in loops and powers of n_f, which slightly generalises the loop-counting performed below eq. (24). Beginning with the simplest class of aliens, O^{I}_{EOM} in eq. (20), one finds that the mixing $\eta(N)$ between O_{g} and O^{I}_{EOM} starts at $O(an_f^0)$. Therefore, the OMEs $\mathrm{A}_{\eta\mathrm{q}}$ must be computed up to three-loop order, where only the contributions $O(a^3 n_f^2)$, $O(a^2 n_f)$ and $O(an_f^0)$ will be relevant. In turn, also the mixing $\eta(N)$ will enter up to three loops, following the same pattern in the required coefficients of n_f. These values are obtained by renormalising the OME $\mathrm{A}_{\mathrm{g}c}$, with an external ghost-antighost pair and an insertion of O_{g}. Hence, the necessary 2-point OMEs can be computed automatically with `Forcer` for fixed N.

By repeating the argument above for the aliens of class II, eq. (20), we find that they do not contribute to $\gamma^{(3)}_{\mathrm{gq}}$ at $O(n_f^2)$ in the n_f-expansion. Indeed, the mixing $\kappa^{(1)}_{\mathrm{ij}}$ starts also at $O(an_f^0)$, as in the case of $\eta(N)$. However, requiring the two closed fermion lines in the OME $\mathrm{A}_{\kappa^{(1)}_{\mathrm{ij}}\mathrm{q}}$ singles out diagrams with least four loops, like the one in Fig. 1. Thus, O^{II}_{EOM} starts contributing only at $O(a^5 n_f^2)$, rather than $O(a^4 n_f^2)$.

The setup described in this section was applied in ref. [12] to compute the n_f^2 terms in the anomalous dimensions $\gamma^{(3)}_{\mathrm{gq}}$ for $N \leq 60$. As it has been presented at this conference, such data is sufficient to reconstruct the analytic dependence of $\gamma^{(3)}_{\mathrm{gq}}(N)|_{n_f^2}$. In turn, the inverse Mellin transform, see eq. (6), provides the n_f^2 contribution in $P^{(3)}_{\mathrm{gq}}(x)$ with its exact x-dependence.

4. Conclusion and Outlook

The anomalous dimensions of the gauge invariant operators of leading twist play a key role for the precision physics programme at the LHC and at the EIC, due to their connection with the Mellin moments of the splitting functions that govern the scale evolution of the PDFs. The four-loop anomalous dimensions $\gamma^{(3)}_{\mathrm{ij}}(N)$ are required to develop collider phenomenology to N^{3}LO in QCD. An efficient method to compute the anomalous dimensions relies on the renormalisation of the off-shell OMEs of the operators in eqs. (4) and (5). This approach presents a conceptual difficulty in the mixing between gauge invariant and unphysical *alien* operators under renormalisation. In this contribution to the proceedings, the construction of the relevant aliens up to four loops has been summarised in the case of Yang Mills theory and extended to QCD.

In both theories, the proliferation of the alien operators poses challenges for the calculation of the anomalous dimensions. However, in QCD such complications are less severe when we look at the fermionic contributions. Due the simplified mixing pattern of the operator O_{q}, the

PoS(LL2024)058

anomalous dimensions $\gamma_{\rm qq}^{(3)}(N)$ and $\gamma_{\rm qg}^{(3)}(N)$ were computed up to $N = 20$, as described in sec. 3.1. Such calculations can be extended for $N \geq 20$, without any new theoretical ingredient, given the availability of computational resources. In the case of the gluonic anomalous dimensions, the colour factors featuring powers of n_f do not present the full complexity of alien mixing. The n_f^2-terms in $\gamma_{\rm gq}^{(3)}$ require only one alien class, which allowed to compute these contributions to the anomalous dimensions up to $N = 60$.

Beyond the n_f-enhanced contributions, more classes of aliens become relevant. Ref. [36] tackled the calculation of $\gamma_{\rm gq}^{(3)}(N)$ for $N \leq 20$. In this case, the operators of class II, eq. (21), which did not contribute at $O(n_f^2)$, do play a role and their mixing, $\kappa_{\rm ij}^{(1)}$, must be computed to 2 loops. As discussed in the example in eq. (10), these correspond to the counterterms of three-point Green functions at 2 loops. Such calculation can be performed, for example, with the R* operation [37–39], giving results in agreement with ref. [21]. In addition, we must take into account the following new classes of operators

$$O_{\rm A_1}^{III} = g^2 f^{a a_1 x} f^{a_2 a_3 x} \left(D.F^a + g\bar\psi \not{\Delta} T^a \psi\right) \sum_{\substack{n_1,n_2,n_3\geq 0\\ n_1+n_2+n_3=N-4}} \kappa^{(1)}_{n_1 n_2 n_3} \left(\partial^{n_1} A_{a_1}\right)\left(\partial^{n_2} A_{a_2}\right)\left(\partial^{n_3} A_{a_3}\right), \tag{26}$$

$$O_{\rm A_2}^{III} = g^2 d^{a a_1 a_2 a_3} \left(D.F^a + g\bar\psi \not{\Delta} T^a \psi\right) \sum_{\substack{n_1,n_2,n_3\geq 0\\ n_1+n_2+n_3=N-4}} \kappa^{(2)}_{n_1 n_2 n_3} \left(\partial^{n_1} A_{a_1}\right)\left(\partial^{n_2} A_{a_2}\right)\left(\partial^{n_3} A_{a_3}\right), \tag{27}$$

$$O_{\rm c}^{III} = -g^2 f^{a a_1 x} f^{a_2 a_3 x} \left(\partial \bar c^a\right) \sum_{\substack{n_1,n_2,n_3\geq 0\\ n_1+n_2+n_3=N-4}} \eta^{(1)}_{n_1 n_2 n_3} \left(\partial^{n_1} A_{a_1}\right)\left(\partial^{n_2} A_{a_2}\right)\left(\partial^{n_3+1} c_{a_3}\right), \tag{28}$$

where the relation between the coefficients $\kappa^{(1)}_{n_1 n_2 n_3}$ and $\eta^{(1)}_{n_1 n_2 n_3}$ are given in [20]. These quantities, together with $\kappa^{(2)}_{n_1 n_2 n_3}$ can be interpreted as the mixing between $O_{\rm g}$ and the corresponding alien and were computed at one loop, finding consistent results with the counterterms given in [21].

The calculation of $\gamma_{\rm gg}^{(3)}(N)$ presents the full complexity of operator mixing. However, the n_f-enhanced contributions show the same type of simplifications that were discussed in sec. 3.2 in the context of $\gamma_{\rm gq}^{(3)}(N)|_{n_f^2}$. The study of the fully gluonic colour factors in $\gamma_{\rm gg}^{(3)}$ is in progress.

Acknowledgements

This work has been supported by the EU's Horizon Europe programme under the Marie Sklodowska-Curie grant 101104792, *QCDchallenge*.

References

[1] A. Dainese, M. Mangano, A. B. Meyer, A. Nisati, G. Salam and M. A. Vesterinen, CERN, 2019, ISBN 978-92-9083-549-3 doi:10.23731/CYRM-2019-007

[2] R. Abdul Khalek et al., Nucl. Phys. A1026 (2022) 122447 doi:10.1016/j.nuclphysa.2022.122447 [arXiv:2103.05419 [hep-ex]]

PoS(LL2024)058

[3] F. Caola, W. Chen, C. Duhr, X. Liu, B. Mistlberger, F. Petriello, G. Vita and S. Weinzierl, [arXiv:2203.06730 [hep-ph]].

[4] J.A. Gracey, Phys. Lett. B322 (1994) 141 doi:10.1016/0370-2693(94)90502-9, [hep-ph/9401214]

[5] J. Davies, A. Vogt, B. Ruijl, T. Ueda and J.A.M. Vermaseren, Nucl. Phys. B915 (2017) 335 doi:10.1016/j.nuclphysb.2016.12.012, [arXiv:1610.07477 [hep-ph]]

[6] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T.Z. Yang, Phys. Lett. B849 (2024) 138427 doi:10.1016/j.physletb.2023.138427, [arXiv:2310.12240 [hep-ph]]

[7] S. Moch, B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt, JHEP 10 (2017) 041 doi:10.1007/JHEP10(2017)041, [arXiv:1707.08315 [hep-ph]]

[8] S. Moch, J. Vermaseren and A. Vogt, *to appear*

[9] J. A. Gracey, Nucl. Phys. B **480** (1996), 73-98 doi:10.1016/S0550-3213(96)00485-3 [arXiv:hep-ph/9609301 [hep-ph]].

[10] J. F. Bennett and J. A. Gracey, Nucl. Phys. B **517** (1998), 241-268 doi:10.1016/S0550-3213(98)00021-2 [arXiv:hep-ph/9710364 [hep-ph]].

[11] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T. Z. Yang, JHEP **01** (2024), 029 doi:10.1007/JHEP01(2024)029 [arXiv:2308.07958 [hep-ph]].

[12] G. Falcioni, F. Herzog, S. Moch, J. Vermaseren and A. Vogt, Phys. Lett. B **848** (2024), 138351 doi:10.1016/j.physletb.2023.138351 [arXiv:2310.01245 [hep-ph]].

[13] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, Phys. Lett. B825 (2022) 136853 doi:10.1016/j.physletb.2021.136853, [arXiv:2111.15561 [hep-ph]]

[14] S. Moch, B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt, Phys. Lett. B849 (2024) 138468 doi:10.1016/j.physletb.2024.138468, [arXiv:2310.05744 [hep-ph]]

[15] J. C. Collins and D. E. Soper, Nucl. Phys. B **194** (1982), 445-492 doi:10.1016/0550-3213(82)90021-9

[16] D. J. Gross and F. Wilczek, Phys. Rev. D **9** (1974), 980-993 doi:10.1103/PhysRevD.9.980

[17] J.A. Dixon and J.C. Taylor, Nucl. Phys. B78 (1974) 552 doi:10.1016/0550-3213(74)90598-7

[18] R. Hamberg and W.L. van Neerven, Nucl. Phys. B379 (1992) 143 doi:10.1016/0550-3213(92)90593-Z

[19] G. Falcioni, PoS **LL2022** (2022), 049 doi:10.22323/1.416.0049 [arXiv:2207.09892 [hep-ph]].

[20] G. Falcioni and F. Herzog, JHEP 05 (2022) 177 doi:10.1007/JHEP05(2022)177, [arXiv:2203.11181 [hep-ph]]

PoS(LL2024)058

[21] T. Gehrmann, A. von Manteuffel and T.Z. Yang, JHEP 04 (2023) 041 doi:10.1007/JHEP04(2023)041, [arXiv:2302.00022 [hep-ph]]

[22] B. Ruijl, T. Ueda and J. A. M. Vermaseren, Comput. Phys. Commun. **253** (2020), 107198 [arXiv:1704.06650 [hep-ph]].

[23] J. A. M. Vermaseren, [arXiv:math-ph/0010025 [math-ph]].

[24] J. Kuipers, T. Ueda, J. A. M. Vermaseren and J. Vollinga, Comput. Phys. Commun. **184** (2013), 1453-1467 [arXiv:1203.6543 [cs.SC]].

[25] B. Ruijl, T. Ueda and J. Vermaseren, [arXiv:1707.06453 [hep-ph]].

[26] S.D. Joglekar and B.W. Lee, Annals Phys. 97 (1976) 160 doi:10.1016/0003-4916(76)90225-6

[27] H. Kluberg-Stern and J.B. Zuber, Phys. Rev. D12 (1975) 3159 doi:10.1103/PhysRevD.12.3159

[28] C. Becchi, A. Rouet and R. Stora, Annals Phys. **98** (1976), 287-321 doi:10.1016/0003-4916(76)90156-1

[29] I. V. Tyutin, [arXiv:0812.0580 [hep-th]].

[30] G. Curci and R. Ferrari, Nuovo Cim. A **32** (1976), 151-168 doi:10.1007/BF02729999

[31] I. Ojima, Prog. Theor. Phys. **64** (1980), 625 doi:10.1143/PTP.64.625

[32] L. Baulieu and J. Thierry-Mieg, Nucl. Phys. B **197** (1982), 477-508 doi:10.1016/0550-3213(82)90454-0

[33] D. Binosi and A. Quadri, Phys. Rev. D **88** (2013) no.8, 085036 doi:10.1103/PhysRevD.88.085036 [arXiv:1309.1021 [hep-th]].

[34] G. Falcioni, F. Herzog, S. Moch and A. Vogt, Phys. Lett. B **842** (2023), 137944 doi:10.1016/j.physletb.2023.137944 [arXiv:2302.07593 [hep-ph]].

[35] G. Falcioni, F. Herzog, S. Moch and A. Vogt, Phys. Lett. B **846** (2023), 138215 doi:10.1016/j.physletb.2023.138215 [arXiv:2307.04158 [hep-ph]].

[36] G. Falcioni, F. Herzog, S. Moch, A. Pelloni and A. Vogt, [arXiv:2404.09701 [hep-ph]].

[37] F. Herzog and B. Ruijl, JHEP 05 (2017) 037 doi:10.1007/JHEP05(2017)037 [arxiv:1703.03776 [hep-ph]].

[38] K.G. Chetyrkin and F.V. Tkachov, Phys. Lett. B114 (1982) 340 doi:10.1016/0370-2693(82)90358-6

[39] K.G. Chetyrkin and V.A. Smirnov, Phys. Lett. B144 (1984) 419 doi:10.1016/0370-2693(84)91291-7

PoS(LL2024)058