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# Nonleptonic B-decays at NNLO

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The decay of B mesons can be predicted within the Heavy Quark Expansion as the decay of a free bottom quark plus corrections which are suppressed by powers of $1/m_b$. This contribution describes the calculation of the NNLO QCD corrections to nonleptonic decays of a free bottom quark including charm quark mass effects. In particular it outlines the challenges in connection to the computation of master integrals, the renormalization of the effective operators and the problems which arise from calculating traces with $\gamma_5$ in $d$ dimensions.

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## 1. Introduction

Lifetimes of $B$ mesons can be calculated in Heavy Quark Effective Expansion (HQE). In this effective theory, the decay width of the $B$ meson, $\Gamma(B)$, is decomposed into the decay of a free $b$ quark and additional contributions which are suppressed by powers of the heavy quark mass, $m_b$:

$$\Gamma(B) = \Gamma_3 + \Gamma_5 \frac{\langle O_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle O_6 \rangle}{m_b^3} + \cdots + 16\pi^2 \left( \Gamma_6 \frac{\langle \tilde{O}_6 \rangle}{m_b^3} + \Gamma_7 \frac{\langle \tilde{O}_7 \rangle}{m_b^4} + \ldots \right). \tag{1}$$

Since the bottom mass $m_b$ is relatively large compared to the energy scale of the decay, the main contribution to the decay width is $\Gamma_3$, the decay width of the free $b$ quark. In our work, we calculate QCD corrections to this quantity for weak decays of $B$ mesons with a massive charm quark in the final state. These decays can be divided into two different decay channels, the semileptonic and the nonleptonic one. For the semileptonic decay channel, $b \to c l \bar{\nu}$, QCD corections are known up to N$^3$LO [1–6].The nonleptonic decays include the two CKM favored decay channels $b \to c\bar{u}d$ and $b \to c\bar{c}s$ and CKM suppressed channels, for example $b \to u\bar{c}s$ and $b \to u\bar{u}d$. The calculation of these processes is more involved than the semileptonic case. The NLO corrections for $b \to c\bar{u}d$ and $b \to c\bar{c}s$ are known [7, 8]. At NNLO, first steps were made in Ref. [9], however only one effective operator has been considered and massless quarks in the final state have been assumed. The uncertainty contributions on B-meson lifetimes are dominated by the uncertainty induced by renormalization scale $\mu$. This uncertainty will be reduced once higher order corrections are known. In the following, the calculation of the NNLO corrections to all nonleptonic decay channels is outlined. A more detailed discussion can be found in [10]

## 2. Calculation Setup

The calculation is done by using the optical theorem. This leads to two loop diagrams at LO and therefore four loop diagrams at NNLO. However, only the imaginary part of these diagrams has to be calculated. The diagrams contributing to this process are generated with `qgraf` [11]. We find 1308 diagrams at NNLO for each of the decays $b \to c\bar{u}d$ and $b \to c\bar{c}s$, which are then mapped to scalar integral families using `tapir` [12] and `exp` [13, 14]. The diagrams for $b \to u\bar{c}s$ can be mapped to families of the $b \to c\bar{u}d$ diagrams and therefore also to the same set of master integrals. The $b \to u\bar{u}d$ decay channel is obtained by taking the massless limit of one of the other decays and adding an additional contribution originating from closed charm-loop insertions into a gluon propagator. We will call this contribution the $U_c$ contribution in the following. Using `Kira` [15, 16] we find 321 master integrals with non-vanishing imaginary parts for $b \to c\bar{u}d$, 527 for $b \to c\bar{c}s$ and 21 for the $U_c$ contribution. Their calculation is described in section 4.

## 3. Evanescent operators and $\gamma_5$

We describe the nonleptonic decays with the following effective Hamiltonian:

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \sum_{q_{1,3}=u,c} \sum_{q_2=d,s} V_{q_1 b} V^*_{q_2 q_3} \left( C_1(\mu_b) O_1^{q_1 q_2 q_3} + C_2(\mu_b) O_2^{q_1 q_2 q_3} \right) + \text{h.c.} \tag{2}$$

with the physical operators

$$O_1^{q_1q_2q_3} = (\bar{q}_1^\alpha\gamma^\mu P_L b^\beta)(\bar{q}_2^\beta\gamma_\mu P_L q_3^\alpha),$$
$$O_2^{q_1q_2q_3} = (\bar{q}_1^\alpha\gamma^\mu P_L b^\alpha)(\bar{q}_2^\beta\gamma_\mu P_L q_3^\beta), \tag{3}$$

and the matching coefficients $C_i(\mu_b)$. Starting from NLO, the evaluation of diagrams with insertions of these operators includes traces over $\gamma_5$ which have to be evaluated in $d \neq 4$ dimensions. Since the calculation of the anomalous dimension was done using anticommuting $\gamma_5$ [17, 18] we want to apply the same scheme for $\gamma_5$ to be consistent. To avoid the calculation of traces with one $\gamma_5$ we use Fierz identies [7]:

$$O_1^{q_1q_2q_3} = (\bar{q}_1^\alpha\gamma^\mu P_L b^\beta)(\bar{q}_2^\beta\gamma_\mu P_L q_3^\alpha) \xrightarrow{\text{Fierz}} (\bar{q}_2^\beta\gamma^\mu P_L b^\beta)(\bar{q}_1^\alpha\gamma_\mu P_L q_3^\alpha) = O_2^{q_2q_1q_3}. \tag{4}$$

After applying this transformation one of the operators in all diagrams, we are left with only one trace over Dirac matrices. In case of two $\gamma_5$ matrices appearing in this trace we can use anticommuting $\gamma_5$. In case of only one $\gamma_5$, we can discard this term, since the decay width we calculate is a parity-even quantity. Fierz identies are four dimensional but we can restore them order by order in perturbation theory by choosing the correct evanescent operators [17, 19]. In order to do this up to NNLO, we introduce terms proportional to $\epsilon^2$ multiplied with physical operators and undetermined coefficients $\{A_2, B_1, B_2\}$ to the definition of the evanescent operators:

$$E_1^{(1),q_1q_2q_3} = (\bar{q}_1^\alpha\gamma^{\mu_1\mu_2\mu_3} P_L b^\beta)(\bar{q}_2^\beta\gamma_{\mu_1\mu_2\mu_3} P_L q_3^\alpha) - (16 - 4\epsilon + A_2\epsilon^2)O_1^{q_1q_2q_3},$$
$$E_2^{(1),q_1q_2q_3} = (\bar{q}_1^\alpha\gamma^{\mu_1\mu_2\mu_3} P_L b^\alpha)(\bar{q}_2^\beta\gamma_{\mu_1\mu_2\mu_3} P_L q_3^\beta) - (16 - 4\epsilon + A_2\epsilon^2)O_2^{q_1q_2q_3},$$
$$E_1^{(2),q_1q_2q_3} = (\bar{q}_1^\alpha\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\beta)(\bar{q}_2^\beta\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\alpha) - (256 - 224\epsilon + B_1\epsilon^2)O_1^{q_1q_2q_3},$$
$$E_2^{(2),q_1q_2q_3} = (\bar{q}_1^\alpha\gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L b^\alpha)(\bar{q}_2^\beta\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} P_L q_3^\beta) - (256 - 224\epsilon + B_2\epsilon^2)O_2^{q_1q_2q_3}. \tag{5}$$

We now fix the coefficients $\{A_2, B_1, B_2\}$ by imposing a symmetric anomalous dimension matrix $\gamma$ [17, 19]

$$\mu\frac{\mathrm{d}C_i}{\mathrm{d}\mu} = \gamma_{ij}C_j, \qquad \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}, \quad \text{with } \gamma_{11} = \gamma_{22}, \quad \gamma_{12} = \gamma_{21} \tag{6}$$

This condition ensures the validity of the Fierz symmtry up to NNLO and we obtain

$$A_2 = -4, \qquad B_1 = -\frac{45936}{125}, \qquad B_2 = -\frac{115056}{115}. \tag{7}$$

## 4. Calculation of master integrals

We calculate the needed master integrals by using the method developed in Refs. [20, 21]. We construct expansions of the master integrals around different kinematic points using differential equations. To do this, we make an expansion ansatz for the integrals with undetermined coefficients. This ansatz is inserted in the differential equations which yields linear equations between the

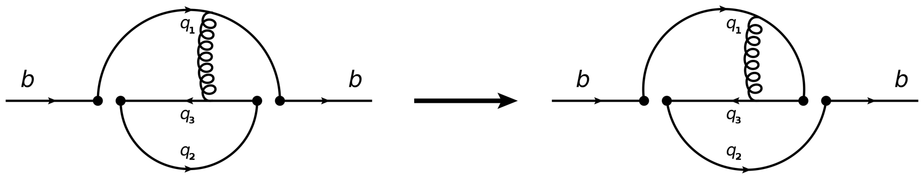


**Figure 1:** Applying Fierz identities to one of the effective operators leads to only one trace.

expansion coefficients. The linear equations can be solved for a small set of independent coefficients using `Kira` and `FireFly` [22].Tey can be determined by matching to precise numerical values of the integrals obtained with `AMFlow` [23].

The ansatz we use for the expansion of the integrals around $\rho = \rho_0$ depends on the expansion point and the singular points of the differential equation. For the various decay channels we have the following singular points:

- $b \to c\bar{u}d$: $\rho_{\text{sing}} \in \{0, 1/3, 1\}$
- $b \to c\bar{c}s$: $\rho_{\text{sing}} \in \{0, 1/4, 1/2\}$
- $U_c$: $\rho_{\text{sing}} \in \{0, 1/2\}$

For $\rho \neq \rho_{\text{sing}}$, we can use simple Taylor expansions:

$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+4} \sum_{n=0}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\rho_0 - \rho)^n , \tag{8}$$

where the coefficients $c[i,j,m,n]$ have both real and imaginary parts. For the expansion around the three-charm threshhold corresponding to the singular point at $\rho = m_c/m_b = 1/3$ in the $b \to c\bar{u}d$ decay channel, we use the ansatz

$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+4} \sum_{n=0}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\rho - \rho_0)^n \log^m(\rho - \rho_0) , \tag{9}$$

with $\rho_0 = 1/3$. When crossing the three-charm threshold at $\rho = 1/3$ from $\rho > 1/3$ to $\rho < 1/3$, the argument of the logarithm gets negative and produces an additional imaginary part, which corresponds to the three-charm contribution.

For the expansions around $\rho = 0$, we use the same ansatz as given in equation (9) with $\rho_0 = 0$.

For expansions around a threshold with an even number of massive particles in the final state, we need to include roots in our ansatz:

$$I_i(\rho, \rho_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+4} \sum_{n=0}^{n_{\max}} c[i,j,m,n]\, \epsilon^j \left(\sqrt{\rho - \rho_0}\right)^n \log^m(\rho - \rho_0) , \tag{10}$$

In our calculation we construct expansions for the master integrals around the following points:

- $b \to c\bar{u}d$: $\rho_0 \in \{0, 1/4, 1/3, 1/2, 7/10, 1\}$
- $b \to c\bar{c}s$: $\rho_0 \in \{0, 1/5, 1/3\}$
- $U_c$: $\rho_0 \in \{0, 1/3, 1/2, 7/10, 1\}$

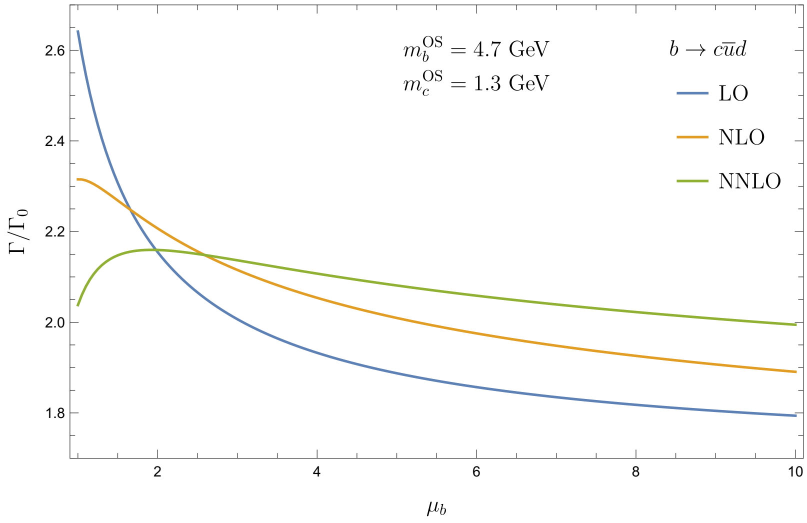


**Figure 2:** The decay width for the channel $b \to c\bar{u}d$ at LO, NLO and NNLO as function of the renormalization scale $\mu_b$. The mass ratio is set to $\rho = m_c/m_b = 1.3/4.7$. This figure is taken from [10].

## 5. Results

For simplicity we only show the numerically most important decay channel $b \to c\bar{u}d$ here. The result for the decay width can be written in the form

$$\Gamma\left(b \to c\bar{u}d\right) = \frac{G_f^2 m_b^5 \left|V_{bc}\right|^2}{192\pi^3}\left[C_1^2(\mu)G_{11} + C_1(\mu)C_2(\mu)G_{12} + C_2^2(\mu)G_{22}\right].$$

To get an estimate of the corrections, we set the on-shell masses of the quarks to $m_c = 1.3$GeV and $m_b = 4.7$GeV. Evaluating the decay width at $\mu = m_b$, we obtain

$$\Gamma\left(b \to c\bar{u}d\right) = \Gamma_0\left[1.89907 + 1.77538\left(\frac{\alpha_s}{\pi}\right) + 14.1081\left(\frac{\alpha_s}{\pi}\right)^2\right]\Bigg|_{\mu=m_b}, \tag{11}$$

where $\Gamma_0 = G_F^2 m_b^5 \left|V_{cb}\right|^2 \left|V_{ud}\right|^2/(192\pi^3)$. In Figure 2 we show the decay width as a function of the renormalization scale $\mu_b$. To estimate the uncertainty induced by the renormalization scale, we consider the region $m_b/2 < \mu_b < 2m_b$. One observes a relative uncertainty of $\approx 7\%$ at LO which reduces to $\approx 3.5\%$ at NNLO relative to the central value $\mu = m_b$.


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