PROCEEDINGS OF SCIENCE

Recent progress in the calculation of the N3LO splitting functions

Giulio Falcioni[a,b,*]

[a] *Physik Institute, Universität Zürich, 190 Winterthurerstrasse, Zürich, Switzerland*

[b] *Dipartimento di Fisica, Università di Torino, Via Pietro Giuria 1, Torino, Italy*

E-mail: giulio.falcioni@physik.uzh.ch, giulio.falcioni@unito.it

The four-loop DGLAP splitting functions, which govern the scale evolution of the Parton Distribution Functions (PDFs), are of the missing ingredients to push the frontier of QCD calculations to N3LO accuracy, as required to match the experimental precision at the LHC and at the forthcoming EIC. In this talk, I will report on the recent progress in the determination of the splitting functions from the direct calculation of a fixed number of moments of these quantities. This approach allows to control the evolution of the PDFs to percent-level precision over the relevant range of momentum fraction x.

31st International Workshop on Deep Inelastic Scattering (DIS2024)
8–12 April 2024
Grenoble, France

*Speaker

https://pos.sissa.it/

1. Introduction

The next decade of experiments at the Large Hadron Collider (LHC) will probe the Standard Model with unprecedented precision. The experimental uncertainties in the measurement of the Higgs production cross section are expected to approach the level of 1% [1]. In addition, new measurements of the Deep Inelastic Scattering process will be performed at the forthcoming Electron-Ion Collider facility. These are also expected to reach uncertainties of the order of 1% [3]. To compare this data with theoretical predictions at the same level of accuracy, it is necessary to compute the collider observables through N^3LO in QCD [2].

One necessary element to carry out consistent N^3LO phenomenology is the calculation of the partonic cross sections to such accuracy. In this respect, the N^3LO era began with the pioneering results of [4, 5] for DIS and, for LHC processes, with the work [6] on Higgs production in gluon fusion. The other ingredient for N^3LO phenomenology is the determination of the Parton Distribution Functions (PDFs) to such accuracy. Indeed, the error associated to employing only NNLO PDFs in N^3LO calculations has been estimated as [7]

$$\delta(\text{PDF-TH}) = \frac{1}{2}\left|\frac{\sigma^{\text{NNLO,NNLO-PDFs}} - \sigma^{\text{NNLO,NLO-PDFs}}}{\sigma^{\text{NNLO,NNLO-PDFs}}}\right| \sim \mathcal{O}(\%), \tag{1}$$

where $\sigma^{\text{NNLO,NNLO-PDFs}}$ is the cross section produced by convoluting the NNLO partonic cross sections with the NNLO PDFs, while $\sigma^{\text{NNLO,NLO-PDFs}}$ is the convolution of NNLO partonic cross sections and NLO PDFs. To remove this uncertainty, several PDF collaborations started working on fitting approximate N^3LO PDFs [8–13], using the theoretical information that is currently available.

A key ingredient is the scale evolution of the PDFs

$$\mu^2 \frac{d}{d\mu^2} f_{\text{i}}\left(x, \mu^2\right) \equiv \dot{f}_{\text{i}}\left(x, \mu^2\right) = \int_x^1 \frac{dy}{y} P_{\text{ij}}(\alpha_s, y)\, f_{\text{j}}\left(\frac{x}{y}, \mu^2\right), \quad \text{i}, \text{j} = \text{q}, \text{g}, \tag{2}$$

where the splitting functions P_{ij} admit the perturbative expansion

$$P_{\text{ij}}(\alpha_s, x) = a\, P_{\text{ij}}^{(0)} + a^2\, P_{\text{ij}}^{(1)} + a^3\, P_{\text{ij}}^{(2)} + a^4\, P_{\text{ij}}^{(3)} \quad, a = \frac{\alpha_s}{4\pi}, \tag{3}$$

such that the N^3LO evoultion requires the calculation of the splitting functions to four loops. Given the challenges of the calculation, only partial results are currently available. In particular there has been significant progress regarding the colour factors featuring (powers of) the number of fermions, n_f, [14–20] and in the planar limit of the flavour non-singlet splitting functions [21].

2. Fixed moments of the splitting functions

This talk focuses on the approximations of the splitting functions based on the calculation of the Mellin moments

$$\gamma_{\text{ij}}^{(k)}(N) = -\int_0^1 dx\, x^{N-1}\, P_{\text{ij}}^{(k)}(x, \alpha_s). \tag{4}$$

This approach has been used successfully in [21], which provides a high-precision approximation of the N^3LO evolution of the flavour non-singlet quark distributions. This parameterisation was

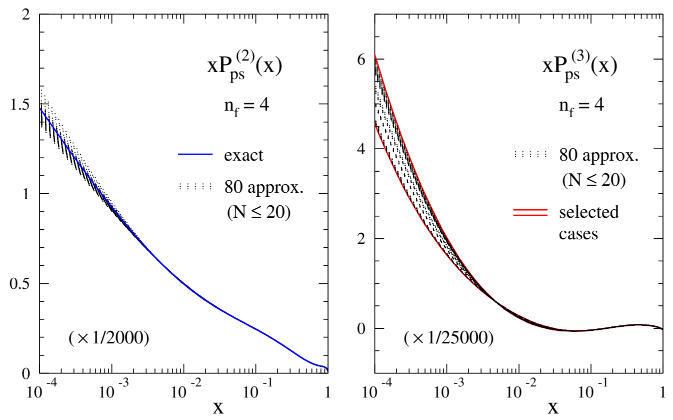

Figure 1: Left: approximation of $P_{\mathrm{ps}}^{(2)}$ compared to the exact result. Right: parameterisation of $P_{\mathrm{ps}}^{(3)}$.

constructed by fixing the first 8 even Mellin moments of $P_{\mathrm{ns}}(x)$ and the behaviour of the splitting function in the large-x and in the small-x limits.

The crucial theoretical tool employed in [21] is the Operator Product Expansion (OPE) approach [22], which allows to compute efficiently the moments $\gamma_{\mathrm{ij}}^{(k)}(N)$ as the anomalous dimensions of gauge invariant operators. Unfortunately, in the flavour singlet sector the OPE method is far from trivial. Its formulation at two-loop level was developed a long time ago [22–24], but its extension to three and four loop has been studied only recently [25–27]. Following the approach of ref. [25], the moments up to $N = 20$ were computed for the quark-to-quark splitting function [28] and for the gluon-to-quark splitting function [29]. As a check, these results reproduce the correct large-n_f limit [17]. The n_f^2 colour factor of $P_{\mathrm{qq}}^{(3)}$ has also been computed and is in agreement with [28]. In addition, the moments up to $N = 12$ of $P_{\mathrm{qq}}^{(3)}$ and up to $N = 10$ for the remaining splitting functions have been computed by expanding the DIS structure functions [30, 31] and agree with [28, 29].

In this talk, the calculation of the moments of $P_{\mathrm{gq}}^{(3)}$ for $N \le 20$ was announced for the first time. The results for these moments, which are provided in [32], agree with the large-n_f limit [17] and with the known result for the n_f^2 colour factor [20]. In addition, the moments up to $N = 10$ are checked with the DIS calculation [30, 31].

3. Approximate evolution

Following [21], the evolution of the PDF is approximated with analytic parameterisations that reproduce the known moments of the splitting functions [28, 29, 32] and match the behaviour for $x \to 1$ and $x \to 0$. The available information in the former limit is summarised as

$$P^{(n)}_{\mathrm{ij},\,x\to 1}(x) \;=\; \sum_{\ell=0}^{2n} \sum_{p=0}^{\infty} C^{\mathrm{ij}}_{n,\ell,p}\,(1-x)^p\,\ln^{2n-\ell}(1-x)\,. \tag{5}$$

Notably, the leading power contribution to $P_{\mathrm{ps}} = P_{\mathrm{qq}} - P_{\mathrm{ns}}^{(+)}$ vanishes, $C^{\mathrm{ps}}_{3,\ell,0} = 0$, and the coefficients $C^{\mathrm{ps}}_{3,\ell,1}$ for $\ell = 4, 3$ were predicted in [33]. The coefficients $C^{\mathrm{qg}}_{3,\ell,p}$ and $C^{\mathrm{gq}}_{3,\ell,p}$ have been predicted for

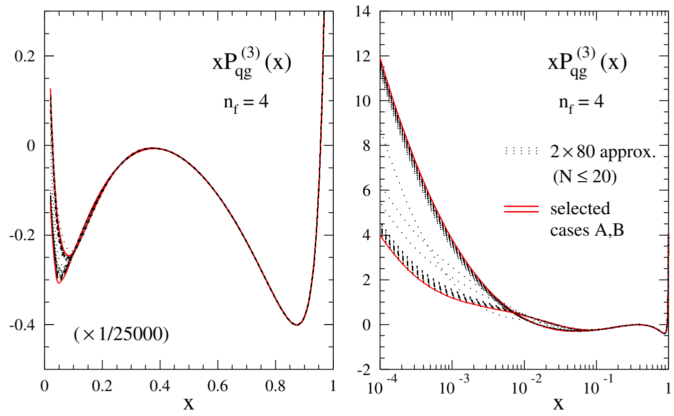

Figure 2: Approximation of $P^{(3)}_{\rm qg}$ in linear scale in x (left) and logarithmic scale (right).

$\ell = 0,\ 1,\ 2$ in ref. [33–35]. The small-x expansion reads

$$P^{(n)}_{\rm ij,\, x\to 0}(x) \;=\; \sum_{\ell=1}^{n} E^{\rm\, ij}_{n,\ell}\,\frac{\ln^{n-\ell} x}{x} \;+\; \sum_{\ell=0}^{2n} F^{\rm\, ij}_{n,\ell}\,\ln^{2n-\ell} x \;+\; O\,(x\,\ln^{a} x)\;. \tag{6}$$

The coefficients of the leading power logarithm $E^{\rm ij}_{n,1}$ are known [36], as well as the highest three sub-dominant x^0 logarithms, $F^{\rm ij}_{n,\ell}$, for $\ell = 0,\ 1,\ 2$ at $n = 3,\ 4$ [37]. By constructing families of curves that obey these constraints, we obtain a parameterisations of each splitting functions in x-space with its associated uncertainty, given by the spread of the curves. As a test of this procedure, Fig. 1 (left) shows the comparison between the exact pure singlet splitting function at NNLO [38] and the approximation constructed with this method. Fig. 1 (right) shows the approximation of $P^{(3)}_{\rm ps}$ and its uncertainty, which increases as $x \to 0$. The approximation of $P^{(3)}_{\rm qg}$ is given in Fig. 2. The errors are larger compared to $P^{(3)}_{\rm ps}$, because the unknown large-x logarithms in eq. (5) are not suppressed by a power of $1 - x$, contrary to the pure singlet case. We are now in the position to approximate the complete evolution of the quark density, using eq. (2), $\dot{f}_{\rm q} = P_{\rm qq} \otimes f_{\rm q} + P_{\rm qg} \otimes f_{\rm g}$. To this end, we use model quark and gluon PDFs [38]

$$x\, f_{\rm q} = 0.6\, x^{-0.3}(1-x)^{3.5}(1+5.0x^{0.8}), \tag{7a}$$

$$x\, f_{\rm g} = 1.6\, x^{-0.3}(1-x)^{4.5}(1-0.6x^{0.3}). \tag{7b}$$

Notice that the PDFs are suppressed at $x \to 1$. Therefore, in the convolution integral of eq. (2), the large uncertainties of the splitting functions at small x, $P_{\rm ij}(z \to 0)$, are significantly reduced by the factor $f_{\rm j}(\frac{x}{z} \to 1)$. The scale derivative of the singlet PDF, $\dot{f}_{\rm q}$, both at NNLO and at N^{3}LO, is reported in Fig. 3 (left). Each curve is normalised by the previous perturbative order, showing that the N^{3}LO calculation shifts the NNLO evolution by less than 1% for $x \geq 10^{-4}$ and by 2% at $x = 10^{-5}$. The error associated with the parameterisation is negligible for $x \gtrsim 10^{-4}$, it increases at small x and is $O(1\%)$ at $x = 10^{-5}$. The variation of $\dot{f}_{\rm q}$ with the renormalisation scale

$$\Delta_{\mu_{\rm r}}\, \dot{q}_{\rm s} \;\equiv\; \frac{\max\,[\,\dot{q}_{\rm s}(x, \mu_{\rm r}^2 = \lambda\,\mu_{\rm f}^2)] \;-\; \min\,[\,\dot{q}_{\rm s}(x, \mu_{\rm r}^2 = \lambda\,\mu_{\rm f}^2)]}{2|\,\text{average}\,[\,\dot{q}_{\rm s}(x, \mu_r^2 = \lambda\,\mu_{\rm f}^2)]\,|}, \tag{8}$$

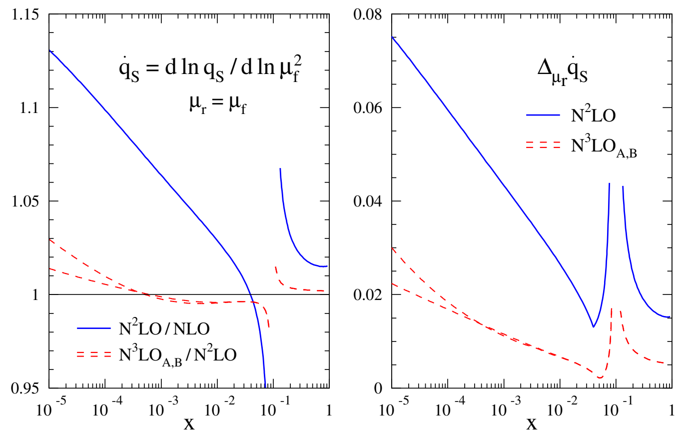

Figure 3: Left: Scale derivative of the quark singlet PDF at NNLO and at N^{3}LO normalised by the previous order. The normalisation introduces a pole at $x \sim 10^{-1}$, where $\dot{f}_{\rm q}$ changes sign, but it shows the impact of each perturbative order. Right: stability of the evolution under variations of the renormalisation scale.

with $\lambda = \frac{1}{4} \ldots 4$, gives an estimate on the theoretical error. As shown in Fig. 3 (right), the scale uncertainty improves at N^{3}LO, where it is below 2% for $x \gtrsim 10^4$, compared to $O(6\%)$ at NNLO.

Finally, two different approximations of $P^{(3)}_{\rm gq}$ are shown in Fig. 4. One (dashed blue) relies on the moments up to $N = 10$, the other (in red) includes the new moments computed in [32]. The former has small uncertainties for $x \gtrsim 10^{-1}$ that grow for $x \to 0$. When 5 additional moments are included, the uncertainty remains small for $x \gtrsim 10^{-2}$.

4. Conclusion

The level of precision of the forthcoming measurements at the LHC and at the EIC will approach $O(1\%)$. On theory side, matching this level of accuracy requires N^{3}LO calculations in QCD. The N^{3}LO splitting functions are key missing ingredients to compute collider observables to this level of accuracy. Recently, there was significant progress towards this goal.

This paper reports on the approximation of $P^{(3)}_{\rm qq}$, $P^{(3)}_{\rm qg}$ and $P^{(3)}_{\rm gq}$, based on the calculation of a set of Mellin moments of these splitting functions [28, 29, 32]. The approximation is obtained by constructing a family of functions that matches the computed Mellin moments, as well as the known limits of the splitting functions at large x and at small x. As a general feature, the uncertainty associated to this approximation increases for small values of x. The information of new fixed moments at $10 < N \le 20$ reduces the uncertainty band in the region $10^{-2} \lesssim x \lesssim 10^{-1}$, see Fig. 4.

The approximate splitting functions have been used determine the scale evolution of the flavour singlet quark density. Notably, the large uncertainty of the splitting functions at small-x is suppressed in the convolution with the PDFs, which vanish in the large-x limit. As a result, the evolution of the quark singlet PDF is determined with high precision over the region $x \gtrsim 10^{-4}$, which is relevant at the LHC and at the EIC. At $x = 10^{-5}$, the uncertainty on the approximate evolution is of $O(1\%)$, in line with the precision goal set by the experiments. The impact of the N^{3}LO calculation on the evolution of the quark PDF is at sub-percent level for $x \gtrsim 10^{-4}$ and is about 2% at $x = 10^{-5}$. Similarly, changing the renormalisation scale gives effects below $O(3\%)$ at $x \sim 10^{-5}$.

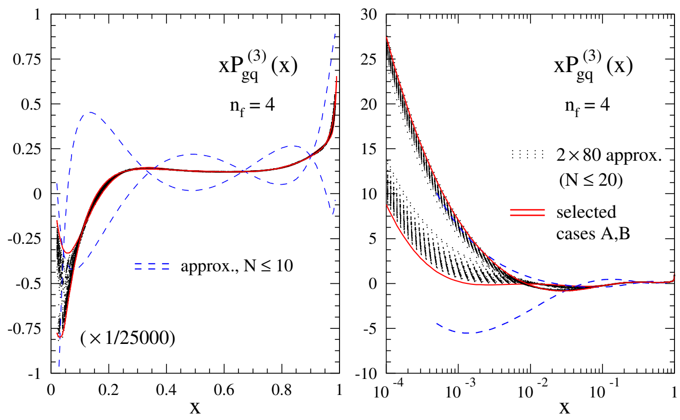

Figure 4: Approximation of $P_{gq}^{(3)}$ based on ten Mellin moments (in red), compared to the one obtained from five moments (dashed, blue), both in linear scale (left) and in logarithmic scale (right).

The splitting function P_{gg} is the missing ingredient to determine the approximate evolution of the gluon PDF. Currently, the moments up to $N = 10$ are available, but these are not sufficient to parameterise the splitting function with high precision over the whole range of values of x probed at the LHC. The calculation of further Mellin moments is now work in progress.

Acknowledgements

This work has been supported by the EU's Horizon Europe programme under the Marie Sklodowska-Curie grant 101104792, *QCDchallenge*.

References

[1] A. Dainese, M. Mangano, A. B. Meyer, A. Nisati, G. Salam and M. A. Vesterinen, CERN, 2019, ISBN 978-92-9083-549-3 doi:10.23731/CYRM-2019-007

[2] F. Caola, W. Chen, C. Duhr, X. Liu, B. Mistlberger, F. Petriello, G. Vita and S. Weinzierl, [arXiv:2203.06730 [hep-ph]].

[3] R. Abdul Khalek et al., Nucl. Phys. A1026 (2022) 122447 doi:10.1016/j.nuclphysa.2022.122447 [arXiv:2103.05419 [hep-ex]]

[4] J. A. M. Vermaseren, A. Vogt and S. Moch, Nucl. Phys. B **724** (2005), 3-182 doi:10.1016/j.nuclphysb.2005.06.020 [arXiv:hep-ph/0504242 [hep-ph]].

[5] S. Moch, J. A. M. Vermaseren and A. Vogt, Phys. Lett. B **606** (2005), 123-129 doi:10.1016/j.physletb.2004.11.063 [arXiv:hep-ph/0411112 [hep-ph]].

[6] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog and B. Mistlberger, Phys. Rev. Lett. **114** (2015), 212001 doi:10.1103/PhysRevLett.114.212001 [arXiv:1503.06056 [hep-ph]].

[7] J. Baglio, C. Duhr, B. Mistlberger and R. Szafron, JHEP **12** (2022), 066 doi:10.1007/JHEP12(2022)066 [arXiv:2209.06138 [hep-ph]].

[8] J. McGowan, T. Cridge, L. A. Harland-Lang and R. S. Thorne, Eur. Phys. J. C **83** (2023) no.3, 185 [erratum: Eur. Phys. J. C **83** (2023) no.4, 302] doi:10.1140/epjc/s10052-023-11236-0 [arXiv:2207.04739 [hep-ph]].

[9] A. Candido, F. Hekhorn and G. Magni, Eur. Phys. J. C **82** (2022) no.10, 976 doi:10.1140/epjc/s10052-022-10878-w [arXiv:2202.02338 [hep-ph]].

[10] F. Hekhorn and G. Magni, [arXiv:2306.15294 [hep-ph]].

[11] T. Cridge, L. A. Harland-Lang and R. S. Thorne, [arXiv:2312.07665 [hep-ph]].

[12] R. D. Ball *et al.* [NNPDF], Eur. Phys. J. C **84** (2024) no.7, 659 doi:10.1140/epjc/s10052-024-12891-7 [arXiv:2402.18635 [hep-ph]].

[13] A. Cooper-Sarkar, T. Cridge, F. Giuli, L. A. Harland-Lang, F. Hekhorn, J. Huston, G. Magni, S. Moch and R. S. Thorne, [arXiv:2406.16188 [hep-ph]].

[14] J.A. Gracey, Phys. Lett. B322 (1994) 141 doi:10.1016/0370-2693(94)90502-9, [hep-ph/9401214]

[15] J. A. Gracey, Nucl. Phys. B **480** (1996), 73-98 doi:10.1016/S0550-3213(96)00485-3 [arXiv:hep-ph/9609301 [hep-ph]].

[16] J. F. Bennett and J. A. Gracey, Nucl. Phys. B **517** (1998), 241-268 doi:10.1016/S0550-3213(98)00021-2 [arXiv:hep-ph/9710364 [hep-ph]].

[17] J. Davies, A. Vogt, B. Ruijl, T. Ueda and J.A.M. Vermaseren, Nucl. Phys. B915 (2017) 335 doi:10.1016/j.nuclphysb.2016.12.012, [arXiv:1610.07477 [hep-ph]]

[18] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T.Z. Yang, Phys. Lett. B849 (2024) 138427 doi:10.1016/j.physletb.2023.138427, [arXiv:2310.12240 [hep-ph]]

[19] T. Gehrmann, A. von Manteuffel, V. Sotnikov and T. Z. Yang, JHEP **01** (2024), 029 doi:10.1007/JHEP01(2024)029 [arXiv:2308.07958 [hep-ph]].

[20] G. Falcioni, F. Herzog, S. Moch, J. Vermaseren and A. Vogt, Phys. Lett. B **848** (2024), 138351 doi:10.1016/j.physletb.2023.138351 [arXiv:2310.01245 [hep-ph]].

[21] S. Moch, B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt, JHEP 10 (2017) 041 doi:10.1007/JHEP10(2017)041, [arXiv:1707.08315 [hep-ph]]

[22] D. J. Gross and F. Wilczek, Phys. Rev. D **9** (1974), 980-993 doi:10.1103/PhysRevD.9.980

[23] J. A. Dixon and J. C. Taylor, Nucl. Phys. B **78** (1974), 552-560 doi:10.1016/0550-3213(74)90598-7

[24] R. Hamberg and W. L. van Neerven, Nucl. Phys. B **379** (1992), 143-171 doi:10.1016/0550-3213(92)90593-Z

[25] G. Falcioni and F. Herzog, JHEP **05** (2022), 177 doi:10.1007/JHEP05(2022)177 [arXiv:2203.11181 [hep-ph]].

[26] G. Falcioni, PoS **LL2022** (2022), 049 doi:10.22323/1.416.0049 [arXiv:2207.09892 [hep-ph]].

[27] T. Gehrmann, A. von Manteuffel and T. Z. Yang, JHEP **04** (2023), 041 doi:10.1007/JHEP04(2023)041 [arXiv:2302.00022 [hep-ph]].

[28] G. Falcioni, F. Herzog, S. Moch and A. Vogt, Phys. Lett. B **842** (2023), 137944 doi:10.1016/j.physletb.2023.137944 [arXiv:2302.07593 [hep-ph]].

[29] G. Falcioni, F. Herzog, S. Moch and A. Vogt, Phys. Lett. B **846** (2023), 138215 doi:10.1016/j.physletb.2023.138215 [arXiv:2307.04158 [hep-ph]].

[30] S. Moch, B. Ruijl, T. Ueda, J. A. M. Vermaseren and A. Vogt, Phys. Lett. B **825** (2022), 136853 doi:10.1016/j.physletb.2021.136853 [arXiv:2111.15561 [hep-ph]].

[31] S. Moch, B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt, Phys. Lett. B **849** (2024), 138468 doi:10.1016/j.physletb.2024.138468 [arXiv:2310.05744 [hep-ph]].

[32] G. Falcioni, F. Herzog, S. Moch, A. Pelloni and A. Vogt, [arXiv:2404.09701 [hep-ph]].

[33] G. Soar, S. Moch, J. A. M. Vermaseren and A. Vogt, Nucl. Phys. B **832** (2010), 152-227 doi:10.1016/j.nuclphysb.2010.02.003 [arXiv:0912.0369 [hep-ph]].

[34] A. Vogt, Phys. Lett. B **691** (2010), 77-81 doi:10.1016/j.physletb.2010.06.010 [arXiv:1005.1606 [hep-ph]].

[35] A. A. Almasy, G. Soar and A. Vogt, JHEP **03** (2011), 030 doi:10.1007/JHEP03(2011)030 [arXiv:1012.3352 [hep-ph]].

[36] S. Catani and F. Hautmann, Nucl. Phys. B **427** (1994), 475-524 doi:10.1016/0550-3213(94)90636-X [arXiv:hep-ph/9405388 [hep-ph]].

[37] J. Davies, C. H. Kom, S. Moch and A. Vogt, JHEP **08** (2022), 135 doi:10.1007/JHEP08(2022)135 [arXiv:2202.10362 [hep-ph]].

[38] A. Vogt, S. Moch and J. A. M. Vermaseren, Nucl. Phys. B **691** (2004), 129-181 doi:10.1016/j.nuclphysb.2004.04.024 [arXiv:hep-ph/0404111 [hep-ph]].