PROCEEDINGS OF SCIENCE

Towards four-loop splitting functions in QCD

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In this talk, we provided an update on our progress in computing four-loop splitting functions. The computations are based on off-shell operator matrix elements, where under renormalization physical twist-two operators mix with previously unidentified gauge-variant operators. We presented a method to systematically extract the counterterm Feynman rules resulting from those gauge-variant operators. Then we applied the Feynman rules to derive the full three-loop splitting functions as well as certain four-loop splitting functions with specific color structures.

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1. Introduction

QCD predictions for high-energy observables at hadron colliders depend on the factorization theorem, which encapsulates the structure of hadrons through universal parton distribution functions (PDFs). The evolution of these PDFs with energy scale is determined by the DGLAP equations [1–3], where splitting functions play a key role. Achieving high precision in both PDFs and splitting functions is essential for accurate phenomenological predictions for the Large Hadron Collider (LHC) and the upcoming Electron-Ion Collider (EIC). Current cutting-edge calculations of splitting functions extend to the four-loop level, with several partial results recently published [4–16]. These findings have already contributed to deriving approximate N^{3}LO PDFs [17, 18].

A significant portion of the current four-loop results for splitting functions comes from calculations of four-loop off-shell operator matrix elements (OMEs). OMEs refer to matrix elements with a single insertion of twist-two quark or gluon operators, for the case of two partons in the external states it is

$$A_{ij} = \langle j(p)|O_i|j(p)\rangle \quad \text{with } p^2 < 0 \tag{1}$$

with O_i being a twist-two operator. The off-shell method approach has proven to be particularly efficient for obtaining splitting functions. However, renormalization presents a complication: physical twist-two operators mix with unidentified gauge-variant (GV) operators. Identifying these unknown GV operators or determining the associated counterterm Feynman rules remains one of the significant challenges in calculating four-loop splitting functions. Despite extensive efforts in this field [19–24], a complete solution has yet to be found. Recently, generalized BRST and anti-BRST constraints on the structure of GV operators for any value of moments n were derived in [25]. A method for solving these constraints and computing the corresponding Feynman rules (at leading order) for up to five external legs, with all-n dependence, was subsequently presented in [26]. In [27], an alternative approach was proposed by us to derive all-n GV counterterm Feynman rules. Within our approach, all leading GV counterterm Feynman rules with all-n dependence for five external legs was recently available in [28]. In this talk, we will list all required OMEs that are necessary to extract the four-loop splitting functions and discuss their computations.

In Section 2, we briefly review the method described in [27]. We then list all required OMEs and discuss their computations in Section 3. In Section 4, we present some of the results achieved so far in our approach. Finally, we conclude in Section 5.

2. Review of the method to renormalize the twist-two operators

Twist-two operators are divided into two categories: non-singlet and singlet operators, based on the flavor symmetry group SU(N_f). The non-singlet sector contains a single quark operator for each spin-n:

$$O_{\rm ns}(n) = \frac{i^{n-1}}{2}\left[\bar{\psi}_{i_1}\Delta\cdot\gamma(\Delta\cdot D)_{i_1 i_2}(\Delta\cdot D)_{i_2 i_3}\cdots(\Delta\cdot D)_{i_{n-1}i_n}\frac{\lambda_k}{2}\psi_{i_n}\right],\ k=3,\cdots N_f^2-1\,. \tag{2}$$

Additionally, there are two physical singlet operators: one for quarks and one for gluons:

$$O_q(n) = \frac{i^{n-1}}{2}\left[\bar{\psi}_{i_1}\Delta\cdot\gamma(\Delta\cdot D^{\mu})_{i_1 i_2}(\Delta\cdot D)_{i_2 i_3}\cdots(\Delta\cdot D)_{i_{n-1}i_n}\psi_{i_n}\right],$$

$$O_g(n) = -\frac{i^{n-2}}{2}\left[\Delta_{\mu_1}\cdot G^{\mu_1}_{a_1,\mu}(\Delta\cdot D)_{a_1a_2}\cdots(\Delta\cdot D)_{a_{n-2}a_{n-1}}\Delta_{\mu_n}G^{\mu_n\mu}_{a_{n-1}a_n}\right]. \tag{3}$$

In these expressions, $\lambda_k/2$ represents the diagonal generators of the flavor group $SU(N_f)$, while Δ is a light-like vector satisfying $\Delta^2 = 0$. The notations ψ and G correspond to the quark field and gluon field strength tensor, respectively. The covariant derivative $D^\mu = \partial_\mu - ig_s\mathbf{T}^a A^a_\mu$ acts in either fundamental or adjoint representations of the gauge group $\mathrm{SU}(N_c)$ with A^a_μ being the gauge filed.

The non-singlet sector allows a multiplicative renormalization:

$$O^{\mathrm{R}}_{\mathrm{ns}}(\mu,n) = Z_{\mathrm{ns}}(\mu,n)O^{\mathrm{B}}_{\mathrm{ns}}(n), \tag{4}$$

where O^{B} and O^{R} refer to the bare and renormalized operators, respectively.

As stated in the previous Section, the renormalization in the singlet sector is much more involved. A naive renormalization with the following form

$$\begin{pmatrix} O_q \\ O_g \end{pmatrix}^{\mathrm{R,naive}} = \begin{pmatrix} Z_{qq} & Z_{qg} \\ Z_{gq} & Z_{gg} \end{pmatrix}\begin{pmatrix} O_q \\ O_g \end{pmatrix}^{\mathrm{B}} \tag{5}$$

should be generalized to the renormalization involving GV operators (counterterms):

$$\begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^{\mathrm{R}} = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qA} \\ Z_{gq} & Z_{gg} & Z_{gA} \\ 0 & 0 & Z_{AA} \end{pmatrix}\begin{pmatrix} O_q \\ O_g \\ O_{ABC} \end{pmatrix}^{\mathrm{B}} + \begin{pmatrix} [ZO]^{\mathrm{GV}}_q \\ [ZO]^{\mathrm{GV}}_g \\ [ZO]^{\mathrm{GV}}_A \end{pmatrix}^{\mathrm{B}}. \tag{6}$$

Some explanations of symbols and notations are in order. The shorthand notation O_{ABC} denotes the GV operators $O_{ABC} = O_A + O_B + O_C$ with O_A, O_B, O_C being the pure-gluon, quark-gluon, and ghost-gluon GV operators respectively. At higher-loop orders, three twist-two GV counterterms $[ZO]^{\mathrm{GV}}_q$, $[ZO]^{\mathrm{GV}}_g$, $[ZO]^{\mathrm{GV}}_A$ are also involved in the renormalization procedure. Here, Z and O are written as a whole since it is difficult or not possible (also not required for practical calculations) to disentangle the renormalization constants Z from the associated operators O when retaining the all-n dependence. Notice that the subscript i in $[ZO]^{\mathrm{GV}}_i$ is simply a label concerning the renormalization of the O_i operator. The counterterm Feynman rules for each $[ZO]^{\mathrm{GV}}_i$ could involve pure-gluon, quark-gluon, and ghost-gluon vertices. The GV counterterms allow a formal expansion in a_s as the following form

$$[ZO]^{\mathrm{GV}}_i = \sum_l^\infty a_s^l\,[ZO]^{\mathrm{GV},\,(l)}_i\,, \text{with } i = q, g, A\,, \tag{7}$$

where $a_s = \alpha_s/(4\pi)$. The eq. (6) is designed such that the GV counterterms are separated from GV operators O_{ABC} by the different loop orders they start to contribute, i.e.,

$$\begin{aligned} Z_{qA} &= O(a_s^2), \quad [ZO]^{\mathrm{GV}}_q = O(a_s^3), \\ Z_{gA} &= O(a_s), \quad [ZO]^{\mathrm{GV}}_g = O(a_s^2), \\ Z_{AA} &= O(a_s^0), \quad [ZO]^{\mathrm{GV}}_A = O(a_s)\,. \end{aligned} \tag{8}$$

A crucial observation in [27] is that the (counterterm) Feynman rules of the associated GV operators or counterterms can be derived by computing the all-off-shell OMEs with insertions of eq. (6). Without loss of generality, we consider the following physical gluon operator renormalization:

$$O_g^{\mathrm{R}} = Z_{gq} O_q^{\mathrm{B}} + Z_{gg} O_g^{\mathrm{B}} + Z_{gA} O_{ABC}^{\mathrm{B}} + [ZO]_g^{\mathrm{GV}} \; . \tag{9}$$

As we are considering the renormalization of a leading-twist operator, it suffices to deal with the following one-particle-irreducible (1PI) OMEs with off-shell external states consisting of two partons of type j plus m gluons,

$$\begin{aligned} \langle j|O_g|j + m\, g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,\mathrm{R}} &= Z_j(\sqrt{Z_g})^m \Big[\langle j|(Z_{gq} O_q + Z_{gg} O_g)|j + m\, g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,\mathrm{B}} \Big] \\ &+ Z_j(\sqrt{Z_g})^m \Big[Z_{gA} \langle j|O_{ABC}|j + m\, g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,\mathrm{B}} + \langle j|[ZO]_g^{\mathrm{GV}}|j + m\, g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,\mathrm{B}} \Big] . \end{aligned} \tag{10}$$

Here j could be a quark (q), gluon (g), or ghost (c), and $\sqrt{Z_j}$ is the associated field renormalization constant. To make the extraction of (counterterm) Feynman rules transparent, we expand the all-off-shell OMEs according to the number of legs $m + 2$ and loops l,

$$\langle j|O|j + m\, g\rangle^{\mu_1\cdots\mu_m} = \sum_{l=0}^{\infty} \left[\langle j|O|j + m\, g\rangle^{\mu_1\cdots\mu_m,\,(l),\,(m)} \right] a_s^l\, g_s^m \; . \tag{11}$$

The left-hand side of the above equation is ultraviolet-renormalized and infrared finite (because all external partons are off-shell). Therefore, the summation on the right-hand side of the equation should be finite, i.e., free from poles in the dimensional regulator ϵ. And due to the hierarchy between the physical operators and GV operators (counterterms) shown in equation (8), it provides a method for deriving the GV (counterterm) Feynman rules order by order by computing the corresponding off-shell OMEs. For example, the two-ghost plus m-gluon Feynman rules for the O_C operator can be written in the following compact form,

$$\langle c|O_C|c + m\, g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,(0),\,(m)} = -\frac{1}{Z_{gA}^{(1)}} \left[\langle c|O_g|c + m\, g\rangle_{\mathrm{1PI}}^{\mu_1\cdots\mu_m,\,(1),\,(m),\,\mathrm{B}} \right]_{\mathrm{div}} , \tag{12}$$

where the subscript 'div' stands for the pole part in dimensional regulator ϵ, and only the one-loop OMEs are required to extract the Feynman rules as shown on the left-hand side. In the above equation, $Z_{gA}^{(1)}$ is the leading GV renormalization constant

$$Z_{gA}^{(1)} = \frac{1}{\epsilon} \frac{C_A}{n(n-1)} \tag{13}$$

which are free from the number of legs and can be determined by setting $m = 0$. In practice, we found that the numerator on the right-hand side of eq. (12) is proportional to $Z_{gA}^{(1)}$ which cancels against the denominator. Therefore, the renormalization constant Z_{gA} can be separated from the operator O_C. It is also true in the cases of O_A and O_B, and this is the reason that we call O_{ABC} as operators instead of counterterms.

The aforementioned approach can also be directly applied to derive the counterterm Feynman rules at two-loop order and beyond. The only difference is that at two-loop order or beyond, we

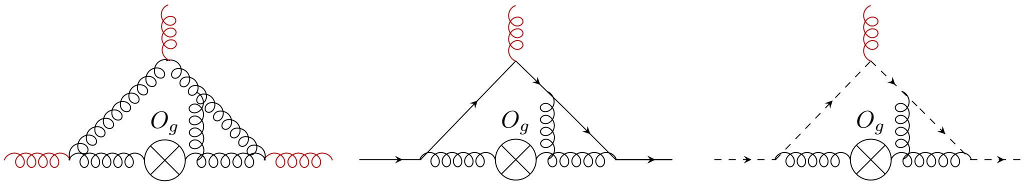

Figure 1: Representative 2-loop Feynman diagrams to derive the counterterm Feynman rules with 3 legs resulting from $[ZO]_g^{\mathrm{GV},(2)}$.

Loops \ Legs	2	3	4	5	6
0	A. D.	$[ZO]_g^{\mathrm{GV},(3)}$	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_q, O_g
1	$[ZO]_g^{\mathrm{GV},(3)}$	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g	
2	$[ZO]_g^{\mathrm{GV},(2)}$	O_{ABC}	O_g		
3	O_{ABC}	O_g			
4	O_q, O_g				

Table 1: All required OMEs to derive four-loop splitting functions. The A. D. represents the anomalous dimensions or splitting functions.

cannot disentangle the renormalization constants from the assisted operators. For further details, please refer to [27]. Figure 1 illustrates several sample diagrams used in the extraction of two-loop counterterm Feynman rules with three legs. We emphasized again that the above method is general and applies to any number of loops and legs.

3. Required OMEs and their computations

In the previous section, we reviewed a method for renormalizing physical twist-two operators. This method allows us to derive the necessary GV (counterterm) Feynman rules through the computations of all-off-shell multi-loop multi-leg OMEs. In tab. 1, we list all required OMEs that allow the extraction of four-loop splitting functions. The workflow for deriving four-loop splitting functions, as outlined in the table (read it from right to left and from bottom to top), consists of several sequential steps:

- Write down all-n Feynman rules for physical operators O_q and O_g up to six legs.
- Derive the all-n Feynman rules for O_{ABC} by computing the one-loop all-off-shell matrix elements with O_g insertion up to five legs.
- Derive two-loop and three-loop all-n counterterm Feynman rules for $[ZO]_g^{\mathrm{GV},(2)}$ and $[ZO]_g^{\mathrm{GV},(3)}$ by computing all-off-shell OMEs with O_g insertions up to four legs (two loops) and three legs (three loops), respectively.

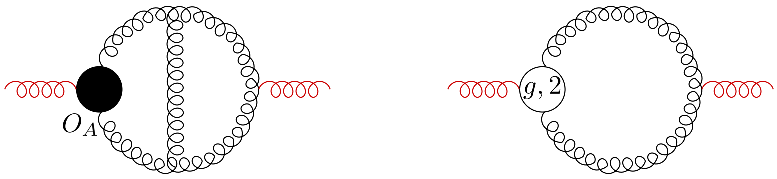

Figure 2: Representative diagrams with the insertion of GV operator O_A or two-loop GV counterterm $[ZO]_g^{\mathrm{GV},(2)}$, they enter the calculations of splitting functions starting from three-loop order.

- Once all GV (counterterm) Feynman rules are obtained, one can derive the four-loop anomalous dimensions with all-n dependence by inserting $[ZO]_g^{\mathrm{GV},(3)}$, $[ZO]_g^{\mathrm{GV},(2)}$, O_{ABC} and (O_q, O_g) into all-off-shell matrix elements up to one-loop, two-loop, three-loop, four-loop order, respectively. Some sample diagrams with the insertion of GV operator O_A or two-loop counterterms can be found in fig. 2.

To retain the all-n dependence when computing the off-shell OMEs, we adopted a generation function method proposed in [29, 30]. By introducing a tracing parameter t, the method sums the non-standard terms like $(\Delta \cdot p)^{n-1}$, which appear in the Feynman rules of physical operators as well as GV operators O_{ABC}, into linear propagators. For example

$$(\Delta \cdot p)^{n-1} \to \sum_{n=1}^{\infty} (\Delta \cdot p)^{n-1} t^n = \frac{t}{1 - t\Delta \cdot p} \,. \tag{14}$$

Then we work in t-space throughout, which allows us to perform standard integration-by-parts (IBP) reductions [31, 32]. The desired n-space results can be extracted by expanding t around $t = 0$ in the final step. The computational steps of these OMEs follow a standard chain, which involves applying different techniques like IBP, (canonical) differential equations [33, 34], and several tools like `QGRAF` [35], `FORM` [36], `Reduze 2` [37], `FeynCalc` [38], `Apart` [39], `MultivariateApart` [40], `Singular_pfd` [41], `LiteRed` [42], `FIRE6` [43], `Kira` [44], `CANONICA` [45], `Libra` [46], `FiniteFlow` [47], `HarmonicSums` [48–53], `HPL` [54] as well as `Finred` based on finite field sampling and rational reconstruction [47, 55, 56].

4. Results

With the above methods, in [27] we extracted the Feynman rules for operators O_{ABC} to g_s^2 (four legs), as well as the counterterm Feynman rules for $[ZO]_g^{\mathrm{GV}}$ to $a_s^2 g_s$ (three legs), where a_s^2 is from Z and g_s is from O. Quite recently, the Feynman rules with five legs for O_{ABC} were available in [16, 28].

Like the Feynman rules for physical operators O_q and O_g, the Feynman rules for the operator O_{ABC} also consist of multiple summations without denominators in the kernel, for example,

$$\sum_{j=0}^{n-3} (\Delta \cdot p_1)^{n-3-j} (\Delta \cdot p_2)^j \,.$$

For this reason, it allows us to make use of the trick as shown in eq. (14) to straightforwardly compute the corresponding OMEs with O_{ABC} insertion.

However, the two-loop counterterm Feynman rules with three legs follow a more intricate structure, involving transcendental functions: harmonic sums [57, 58] and generalized harmonic sums [59], as a simple example

$$S_1\left(z_1+1;n\right)=\sum_{x_1=1}^{n}\frac{(1+z_1)^{x_1}}{x_1}\,,\ \text{with}\ z_1=\frac{\Delta\cdot p_2}{\Delta\cdot p_1}\ \text{and}\ \Delta\cdot p_1=1\,. \tag{15}$$

Inserting these terms into the two-point OMEs and summing them according to eq. (14) results in polylogarithmic functions, which lie beyond the capabilities of current IBP methods. Specifically, the generalized harmonic sum in eq. (15) is summed into the following logarithmic function

$$\sum_{n=1}^{\infty}t^n S_1\left(z_1+1;n\right)=\frac{\log\left(1-t(1+z_1)\right)}{-1+t}\,. \tag{16}$$

For higher transcendentality-weight generalized harmonic sums, one gets higher transcendentality-weight polylogarithmic functions. An interesting fact is that the generalized harmonic sums collapse into polynomial functions for fixed moments, for example,

$$\begin{aligned}S_1(z_1+1;2)&=\frac{z_1^2}{2}+2z_1+\frac{3}{2}\,,\\ S_1(z_1+1;4)&=\frac{z_1^4}{4}+\frac{4z_1^3}{3}+3z_1^2+4z_1+\frac{25}{12}\,.\end{aligned} \tag{17}$$

The above fact implies that one can rewrite the generalized harmonic sums as the following form

$$S_1(z_1+1;n)=\sum_{m=1}^{n}a_{mn}\,z_1^m \tag{18}$$

with a_{mn} being some rational numbers. The above equation also works directly for higher transcendentality-weight generalized harmonic sums. To overcome the difficulty of performing IBP reductions for OMEs with the insertion of generalized harmonic sums depending on z_1, we introduce an additional tracing parameter t_1, such that

$$h(t,t_1)=\sum_{n=1}^{\infty}t^n\sum_{m=1}^{n}t_1^m z_1^m=\frac{t\,t_1\,z_1}{(1-t)\,(1-t\,t_1\,z_1)}\,. \tag{19}$$

Now it's straightforward to perform IBP reductions with the above linear propagators insertion, and the differential equation method can be used to expand t to very high powers for corresponding OMEs, for example,

$$\langle g|h(t,\,t_1)|g\rangle^{(1)}=\sum_{n=1}^{\infty}t^n\sum_{m=1}^{n}c_{mn}\,t_1^m\,. \tag{20}$$

Similarly with a_{mn} in eq. (18), c_{mn} above is also rational numbers. Combing eqs. (18), (19), (20), we have

$$\langle g|S_1(z_1+1;n)|g\rangle^{(1)}=\sum_{m=1}^{n}a_{mn}\,c_{mn}\,. \tag{21}$$

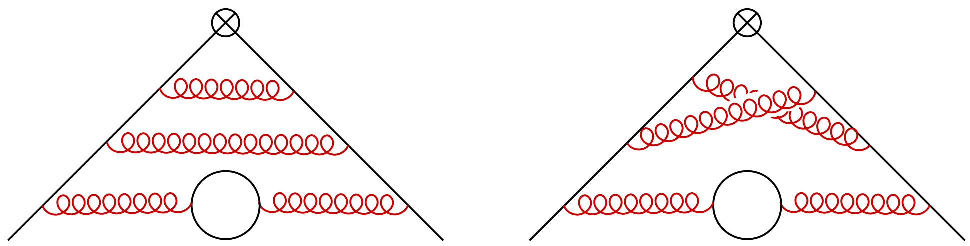

Figure 3: Representative diagrams for the $N_f C_F^3$ contribution to the four-loop, non-singlet OME with two external quarks. The crossed circle denotes the non-singlet operator O_{ns}.

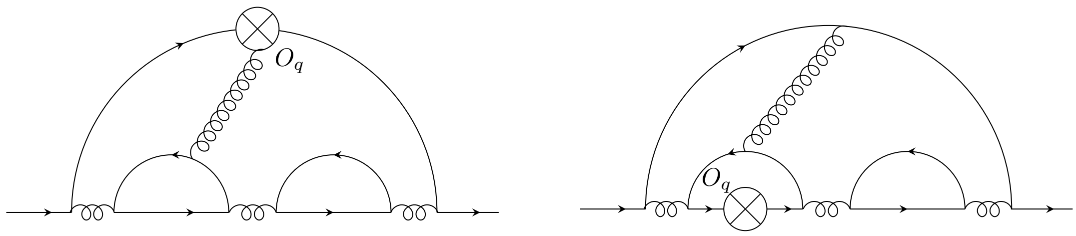

Figure 4: Representative diagrams for N_f^2 contributions to $\langle q|O_q^{\mathrm{B}}|q\rangle$ at four-loop order. The first diagram contributes to the non-singlet anomalous dimension, while the second one contributes to the pure-singlet anomalous dimension.

Therefore, the computations of off-shell OMEs with generalized harmonic sums insertion or two-loop counterterm operator insertion are converted into a linear algebra problem. Above, we discussed the generalized harmonic sums with a singlet variable dependence (z_1). The method can also be easily generalized to the case with multiple variables dependence. For example, if the generalized harmonic sums depend on two variables z_1, z_2, then two additional tracing parameters t_1, t_2 are sufficient to sum them into linear propagators. In practice, we applied the above method to compute the one-loop OMEs with two-loop counterterm Feynman rules insertion to $n = 500$ and reconstruct the final result with all-n dependence in terms of harmonic sums. A sample Feynman diagram can be found in fig. 2 (the second diagram).

With all the above tricks, we can compute all required off-shell OMEs in principle. In the non-singlet sector, the GV operators (counterterms) are absent, and the first three-loop calculations from off-shell OMEs were given in [60]. We repeated the calculation in our notation in [27] and found full agreement with them. In [14], we derived the $N_f C_F^3$ contribution to non-singlet splitting functions at four-loop order for the first time. Some sample Feynman diagrams are shown in Fig. 3. We successfully checked our results with fixed moments to $n = 16$ [7]. In the singlet sector, we computed the unpolarized three-loop off-shell OMEs with all-n dependence for the first time, the extracted splitting functions agree with the results [61, 62] derived from forward deep inelastic scattering. In [8], we determined the four-loop pure-singlet splitting functions with N_f^2 color structure for the first time. Some sample Feynman diagrams can be found in fig 4. We cross-validated this result against the fixed-n results up to $n = 20$ in [10].

5. Conclusions

In the renormalization of off-shell OMEs with a twist-two operator insertion, the physical operators O_q and O_g mix with the previously unidentified GV operators. We proposed a systematic framework to extract the all-n counterterm Feynman rules associated with these GV counterterm operators. Within this framework, we identified all OMEs necessary for deriving the four-loop splitting functions and discussed their computations. Using the off-shell OME method, we have employed several techniques to obtain, for the first time, the three-loop unpolarized singlet, the four-loop N_f^2 contributions to the pure-singlet, and the four-loop $N_f\, C_F^3$ contributions to the non-singlet splitting functions.

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