Pole decomposition of BFKL eigenvalue

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We discuss the pole structure of the BFKL equation in the complex angular momentum plane. We argue that beyond the leading order, the proper framework for the BFKL dynamics is the Bethe–Salpeter equation. The Bethe–Salpeter equation was derived for describing evolution of the bound states and can provide a natural framework for the propagation of the bound state of two reggeized gluons. The Bethe–Salpeter approach to the BFKL dynamics sheds light on the internal structure of the BFKL eigenvalue beyond the leading order revealing the way the BFKL kernel with hermitian separability can result into the higher-order BFKL eigenvalue, where the hermitian separability is absent.

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The Balitsky-Fadin-Kuraev-Lipatov (BFKL) [1] equation can be written in the form of the linear Schrödinger equation

$$H\psi = E\psi \tag{1}$$

where H is the BFKL Hamiltonian, ψ is the eigenfunction and E is the BFKL eigenvalue related to the pomeron intercept. The eigenfunction ψ is a complex function of the transverse angular momentum or its canonical conjugate. The BFKL eigenvalue depends on two real valued degrees of freedom, the anomalous dimension ν and the conformal spin n that are introduced as the Mellin transform of the complex transverse momentum. It is convenient to introduce the complex variable

$$z = -\frac{1}{2} + i\nu + \frac{n}{2} \tag{2}$$

for continuous ν and discrete n. The term $-\frac{1}{2}$ reflects the wave function normalization condition for the color singlet BFKL equation.

The full analytic expression for the color singlet BFKL eigenvalue in QCD and $N = 4$ SYM is currently available only for the leading order (LO) and next-to-leading order (NLO) [2–5]. There is also some information available for the next-to-next-to-leading order (NNLO) in the $N = 4$ SYM, which follows from modern integrability techniques [1]. A useful property of the hermitian separability of the BFKL eigenvalue was first discussed by A. Kotikov and L. Lipatov [4, 5]. The property of the hermitian separability implies a way of writing a function of complex variable z and the complex conjugate $\bar{z}$ as a sum two contributions separately dependent on z and $\bar{z}$

$$f(z, \bar{z}) = F(z) + F(\bar{z}) \tag{3}$$

The property of the hermitian separability is less restrictive than the holomorphic separability as we discuss below. In eq. (3) the real function F is the same function for z and $\bar{z}$, which results in the real valued $f(z, \bar{z})$ and helps to reduce related two-dimensional calculations to a much simpler one-dimensional problem. For the BFKL eigenvalue the function $F(z)$ is the single valued function of complex variable so that $f(z, \bar{z})$ is always real for any value of z.

In the singlet color case the leading order BFKL eigenvalue is manifestly hermitian separable, whereas the next-to-leading eigenvalue [2] is not. It was demonstrated by A. Kotikov and L. Lipatov [4, 5] that color singlet NLO eigenvalue in $N = 4$ SYM can be written as a combination of a product of two hermitian separable functions and a hermitian separable function

$$f^{NLO}(z, \bar{z}) = f^{LO}(z, \bar{z})g(z, \bar{z}) + \rho(z, \bar{z}) \tag{4}$$

where $f^{LO}(z, \bar{z})$ is the corresponding LO eigenvalue. The function f^{LO}, f^{NLO}, g and ρ all have hermitian separable form of eq. (3). This non-linearity in eq. (4) has no obvious explanation in the Schrödinger equation approach to the BFKL dynamics, where the two degrees of freedom corresponding to the complex variables z and $\bar{z}$ are mixed at the level of the Hamiltonian.

Following the lines of A. Kotikov and L. Lipatov [4, 5] it is natural to consider the Bethe-Salpeter equation with the two degrees of freedom being separated at the level of the kernel. The BFKL

[1]See the review paper discussing aspects of integrability techniques applied to the BFKL evolution [7]

dynamics describes the evolution of the bound state of two reggeized gluons. This fact makes it natural to consider the Bethe-Salpeter equation, which was originally constructed to describe quantum bound states [8] and has a variety of applications in quantum field theory (positronium, mesons etc.). The Bethe-Salpeter equation can be schematically represented as follows

$$G = K \otimes S \otimes G \otimes S \tag{5}$$

where G is the propagator of the bound state under consideration, K is the kernel and S is the bare propagator (see Fig. 1).

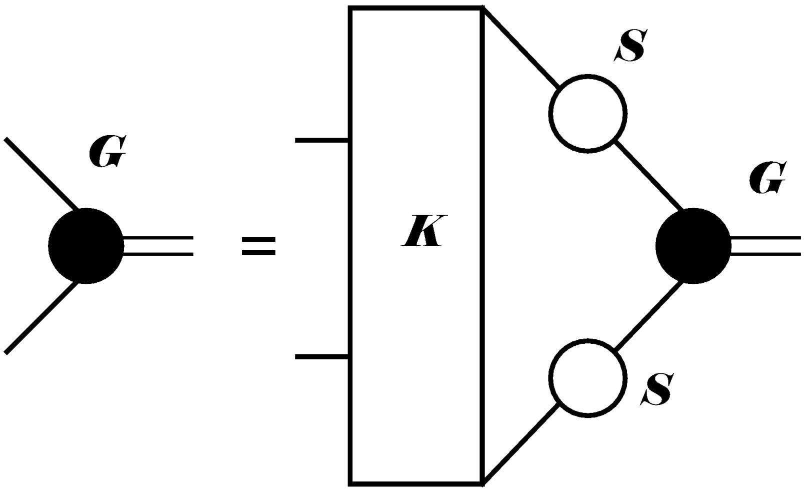

Figure 1: The figure depicts the structure of the Bethe-Salpeter equation in eq. 5.

In the Bethe-Salpeter approach one can represent the BFKL dynamics as pole decomposition of the scattering amplitude in the plane of complex angular momentum j with the leading singularity of $j \to 1$. It is customary to denote $j = 1 + \omega$ and make an expansion in powers of ω, which gives the leading Pomeron exchange. The leading-order (LO) in $\alpha_s \ln s$ corresponds to the simple pole $1/\omega$ of the BFKL amplitude, the next-to-leading (NLO) contributions includes a free term $(\omega)^0$, the next-to-next-to-leading (NNLO) stand for ω^1 and so forth. The recursive structure of the Bethe-Salpeter equation in eq. (5) implies that the sum of all those contributions equals ω itself. This can be written as follows [9]

$$1 = \frac{a}{\omega} \sum_{i=0}^{\infty} \omega^i \sum_{k=0}^{\infty} a^k f_{i,k}, \tag{6}$$

where $a = \frac{\alpha_s N_C}{2\pi}$ is the coupling constant. The functions $f_{i,k}$ are hermitian separable at any order and reproduce the structure of the next-to-leading eigenvalue making predictions for higher order corrections to the BFKL eigenvalue. Let us denote the leading order BFKL eigenvalue by

$$\omega_0 = a f^{LO}(z, \bar{z}), \tag{7}$$

the next-to-leading order BFKL eigenvalue by

$$\omega_1 = a^2 f^{NLO}(z, \bar{z}) \tag{8}$$

and so forth. The leading order BFKL eigenvalue in $N = 4$ SYM is rather simple

$$f^{LO}(z,\bar{z}) = 4\left(-\psi(z+1) - \psi(\bar{z}+1) + 2\psi(1)\right) \tag{9}$$

and the corresponding known NLO expression reads [4, 5]

$$f^{NLO}(z,\bar{z}) = \Phi(z+1) + \Phi(\bar{z}+1) - \frac{1}{2} f^{LO}(z,\bar{z}) \left(\beta'(z+1) + \beta'(\bar{z}+1) + \frac{\pi^2}{6}\right),$$

where

$$\beta'(z) = \sum_{r=0}^{\infty} \frac{(-1)^{r+1}}{(z+r)^2} \tag{10}$$

and

$$\Phi(z) = 3\zeta(3) + \psi''(z) + 2\Phi_2(z) + 2\beta'(z)\left(\psi(1) - \psi(z)\right). \tag{11}$$

as well as

$$\Phi_2(z) = \sum_{k=0}^{\infty} \frac{\beta'(k+1) + (-1)^k \psi'(k+1)}{k+z} - \sum_{k=0}^{\infty} \frac{(-1)^k \left(\psi(k+1) - \psi(1)\right)}{(k+z)^2} \tag{12}$$

We limit ourselves by the next-to-next-to-leading (NNLO) order in eq. (6) and get

$$1 = \frac{a(f_{0,0} + a f_{0,1} + a^2 f_{0,2})}{\omega} + a(f_{1,0} + a f_{1,1}) + a\omega f_{2,0} \tag{13}$$

Next, we plug

$$\omega = a(f^{LO} + a f^{NLO} + a^2 f^{NNLO} + ..) \tag{14}$$

and expand it in the powers of the coupling constant obtaining the first three orders in the perturbation theory as follows. The LO eigenvalue

$$\omega_0 = a f_{0,0} \tag{15}$$

the NLO eigenvalue of order a^2

$$\omega_1 = a\left(\omega_0 f_{1,0} + a f_{0,1}\right) = a^2\left(f_{0,0} f_{1,0} + f_{0,1}\right) \tag{16}$$

and finally the NNLO eigenvalue of order a^3.

$$\begin{aligned} \omega_2 &= a\left(\omega_0^2 f_{2,0} + a\omega_0 f_{1,1} + \omega_1 f_{1,0} + a^2 f_{0,2}\right) \\ &= a^3\left(f_{0,0}^2 f_{2,0} + f_{0,0} f_{1,1} + f_{0,0} f_{1,0}^2 + f_{1,0} f_{0,1} + f_{0,2}\right) \end{aligned} \tag{17}$$

At the NNLO level the function ω_2 is expressed in terms of three unknown functions $f_{1,1}$, $f_{0,2}$ and $f_{2,0}$. The functions $f_{0,0}$, $f_{0,1}$ and $f_{1,0}$ are known from the previous orders. The functions $f_{i,j}$ are single valued meromorphic functions each having a property of the hermitian separability . The complexity of $f_{i,j}$ can be related to the concept of transcendentality and a number of the nested summations. The transcendentality corresponds to the weight of the nested harmonic sums and the number of the nested summations stands for their depth.

The harmonic sums are defined through a nested summation with their argument being the upper limit in the outermost sum [10, 11]

$$S_{a_1,a_2,\ldots,a_k}(n) = \sum_{n\geq i_1\geq i_2\geq\ldots\geq i_k\geq 1} \frac{\mathtt{sign}(a_1)^{i_1}}{i_1^{|a_1|}}\ldots\frac{\mathtt{sign}(a_k)^{i_k}}{i_k^{|a_k|}}, \quad n\in\mathbb{N}^* \tag{18}$$

where a_i are integers, which build the alphabet of the possible negative and positive indices. In Eq. (18) k is the depth and $w = \sum_{i=1}^{k}|a_i|$ is the weight of the harmonic sum $S_{a_1,a_2,\ldots,a_k}(n)$. The nested harmonic sums are rich of identities between the sums of the same or multiple arguments (quasi-shuffle identity etc.). The argument of the nested harmonic sums is a natural number and they require analytic continuation to the complex plane in order to be used for calculations of the BFKL eigenvalue. The analytic continuation can be done using the integral representation as Mellin transform of Harmonic Polylogarithms. After analytic continuation, the nested harmonic sums are a single valued analytic functions with having pole singularities at negative integers. For example, the simplest harmonic sum $S_1(n)$ after analytic continuation for $n \to z$ corresponds to $\psi(z+1)-\psi(1)$, where $\psi(z)$ is the digamma function defined by $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$.

According to the Bethe-Salpeter form of the BFKL equation in eq. (6), the BFKL eigenvalue is constructed of functions $f_{i,j}$ that individually have property of hermitian separability, but their products do not. The decomposition of the products for any given n can be done using reflection identities of the nested harmonic sums [12–17] . The reflection identities are of the form $S_a(z)S_b(-1-z) = S_c(z) + S_d(-1-z) + \mathtt{constants}$ and they are useful for restoring the full dependence of the BFKL eigenvalue on n from a corresponding results for specific values of the conformal spin.

References

[1] L. N. Lipatov, Sov. J. Nucl. Phys. **23** (1976) 338;
V. S. Fadin, E. A. Kuraev and L. N. Lipatov, Phys. Lett. **B 60** (1975) 50;
E. A. Kuraev, L. N. Lipatov and V. S. Fadin, Sov. Phys. JETP **44** (1976) 443; **45** (1977) 199;
Ya. Ya. Balitskii and L. N. Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822.

[2] V. S. Fadin and L. N. Lipatov, "BFKL pomeron in the next-to-leading approximation," Phys. Lett. B **429**, 127 (1998) doi:10.1016/S0370-2693(98)00473-0 [hep-ph/9802290].

[3] A. V. Kotikov and L. N. Lipatov, "NLO corrections to the BFKL equation in QCD and in supersymmetric gauge theories," Nucl. Phys. B **582**, 19-43 (2000) doi:10.1016/S0550-3213(00)00329-1 [arXiv:hep-ph/0004008 [hep-ph]].

[4] A. V. Kotikov and L. N. Lipatov, "DGLAP and BFKL evolution equations in the N=4 supersymmetric gauge theory," hep-ph/0112346.

[5] A. V. Kotikov and L. N. Lipatov, "DGLAP and BFKL equations in the $N = 4$ supersymmetric gauge theory," Nucl. Phys. B **661**, 19 (2003) Erratum: [Nucl. Phys. B **685**, 405 (2004)] doi:10.1016/S0550-3213(03)00264-5, 10.1016/j.nuclphysb.2004.02.032 [hep-ph/0208220].

[6] N. Gromov, F. Levkovich-Maslyuk and G. Sizov, “Pomeron Eigenvalue at Three Loops in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory,” Phys. Rev. Lett. **115**, no. 25, 251601 (2015) doi:10.1103/PhysRevLett.115.251601 [arXiv:1507.04010 [hep-th]].

[7] M. Alfimov, N. Gromov and V. Kazakov, “$N = 4$ SYM Quantum Spectral Curve in BFKL regime,” [arXiv:2003.03536 [hep-th]].

[8] H. Bethe, E. Salpeter . "A Relativistic Equation for Bound-State Problems". Physical Review. 84 (6), 1232, 1951

[9] J. Bartels and A. Prygarin, “The Regge limit of N=4 SUSY Gauge Theories,” From the Past to the Future, Chapter 15, pp 407-433, doi:110.1142/97898112311240015

[10] A. Gonzalez-Arroyo, C. Lopez, and F.J. Yndurain, Nucl. Phys. **B153** (1979) 161;
A. Gonzalez-Arroyo and C. Lopez, Nucl. Phys. **B166** (1980) 429.

[11] J. A. M. Vermaseren, “Harmonic sums, Mellin transforms and integrals,” Int. J. Mod. Phys. A **14**, 2037 (1999) doi:10.1142/S0217751X99001032 [hep-ph/9806280].

[12] A. Prygarin, “Reflection identities of harmonic sums up to weight three,” [arXiv:1808.09307 [hep-th]].

[13] M. Joubat and A. Prygarin, “The analytic structure of the BFKL equation and reflection identities of harmonic sums at weight five,” Int. J. Mod. Phys. A **34**, no.11, 1950064 (2019) doi:10.1142/S0217751X19500647 [arXiv:1903.06773 [hep-th]].

[14] M. Joubat and A. Prygarin, “Reflection Identities of Harmonic Sums and pole decomposition of BFKL eigenvalue,” Int. J. Mod. Phys. A **36**, no.04, 2150025 (2021) doi:10.1142/S0217751X21500251 [arXiv:2011.08095 [hep-ph]].

[15] M. Joubat and A. Prygarin, “Real Valued Functions for the BFKL Eigenvalue,” Universe **7**, no.11, 444 (2021) doi:10.3390/universe7110444 [arXiv:2109.12904 [hep-th]].

[16] M. Joubat and A. Prygarin, “Universal transcendentality limit of BFKL eigenvalue,” Eur. Phys. J. C **82**, no.1, 45 (2022) doi:10.1140/epjc/s10052-021-09968-y [arXiv:2111.11664 [hep-th]].

[17] M. Joubat, C. C. M. Sandeu and A. Prygarin, “Pole decomposition of BFKL eigenvalue at zero conformal spin and the real part of digamma function,” Phys. Lett. B **848**, 138319 (2024) doi:10.1016/j.physletb.2023.138319 [arXiv:2307.12112 [hep-th]].