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Neutrino oscillations in the interaction picture: a brief introduction

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We provide a brief overview of the main results of the interaction picture approach to neutrino oscillations. In this framework, mixing is treated as an interaction between different neutrino flavors. The oscillation formula is derived by calculating the survival probability of a specific flavor neutrino. Notably, this method yields the same modified oscillation formula as the flavor Fock space approach, exhibiting dependence on both the difference and the sum of neutrino frequencies.

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1. Neutrino oscillations: a perturbative approach

Our starting point is the effective weak-interaction Lagrangian[1] including a neutrino-mixing term

$$\mathcal{L} \ = \ \mathcal{L}_0 + \mathcal{L}_{mix} + \mathcal{L}_{wint} \, , \tag{1}$$

with

$$\mathcal{L}_0 \ = \ \sum_{\sigma=e,\mu} \overline{\nu}_\sigma \left(i\partial\!\!\!/ - m_\sigma\right) \nu_\sigma + \sum_{\sigma=e,\mu} \bar{l}_\sigma \left(i\partial\!\!\!/ - \tilde{m}_\sigma\right) l_\sigma \, , \tag{2}$$

$$\mathcal{L}_{mix} \ = \ -m_{e\mu} \left(\overline{\nu}_e \nu_\mu + \overline{\nu}_\mu \nu_e\right) \, , \tag{3}$$

$$\mathcal{L}_{wint} \ = \ -\frac{g}{2\sqrt{2}} \sum_{\sigma=e,\mu} \left[W_\mu^+ \, \overline{\nu}_\sigma \, \gamma^\mu \, (1-\gamma^5) \, l_\sigma + h.c.\right] \tag{4}$$

Here we limit to the case of two-flavors.

The neutrino kinetic term $\mathcal{L}_0 + \mathcal{L}_{mix}$ can be diagonalized by means of the *mixing transformation* [1, 2]

$$\begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix} \ = \ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix} \, , \tag{5}$$

with $\tan 2\theta = 2m_{e\mu}/(m_\mu - m_e)$. This transformation connects the flavor/gauge basis to the mass/energy basis, where the weak interaction described by $\mathcal{L}_{wint}$ is non-diagonal. In such a case the effect of mixing is entirely included in the weak-interaction vertex. However, it is crucial to establish a proper definition of flavor states [3–7], as neutrinos are produced with a definite flavor by (charged current) weak interactions.

Here, following Ref. [8], we proceed in a different way, working in the flavor basis. We split the Lagrangian as

$$\mathcal{L} \ = \ \mathcal{L}_0 + \mathcal{L}_{int} \, , \tag{6}$$

with $\mathcal{L}_{int} = \mathcal{L}_{mix} + \mathcal{L}_{wint}$. In other words, we now consider the mixing as an interaction among different flavored neutrinos and not as the off-diagonal part of the mass matrix. In this way we can apply the usual techniques of the perturbation theory to deal with interacting field theories. From now on we will work at the zeroth-order in g, i.e. we only consider $\mathcal{L}_{int} = \mathcal{L}_{mix}$.

Using the standard Dyson expansions of the time evolution operator up to the second order in $m_{e\mu}$, one has

$$U(t_i, t_f) \ = \ \mathbb{I} - i\, m_{e\mu} \int_{t_i}^{t_f} \mathrm{d}^4 x \ : \overline{\nu}_e(x)\nu_\mu(x) + \overline{\nu}_\mu(x)\nu_e(x) : \tag{7}$$

$$- \ \frac{m_{e\mu}^2}{2} \int_{t_i}^{t_f} \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \, \mathcal{T}\Big[\left(: \overline{\nu}_e(x_1)\nu_\mu(x_1) + \overline{\nu}_\mu(x_1)\nu_e(x_1) :\right) \left(: \overline{\nu}_e(x_2)\nu_\mu(x_2) + \overline{\nu}_\mu(x_2)\nu_e(x_2) :\right) \Big] + \ldots$$

[1] Here we do not consider the neutral-current interactions because they do not play an active role in neutrino oscillations.

The second order piece can be further expanded using Wick's theorem:

$$U^{(2)}(t_i,t_f) \;=\; -\frac{m_{e\mu}^2}{2}\int_{t_i}^{t_f} \mathrm{d}^4x_1 \int_{t_i}^{t_f} \mathrm{d}^4x_2 \,\Big[:\overline{\nu}_e(x_1)\nu_\mu(x_1)\overline{\nu}_e(x_2)\nu_\mu(x_2): \\ + :\overline{\nu}_e(x_1)\nu_\mu(x_1)\overline{\nu}_\mu(x_2)\nu_e(x_2): + :\overline{\nu}_\mu(x_1)\nu_e(x_1)\overline{\nu}_e(x_2)\nu_\mu(x_2): + :\overline{\nu}_\mu(x_1)\nu_e(x_1)\overline{\nu}_\mu(x_2)\nu_e(x_2): \\ +\, 2\,i\left(S^e_{\alpha\beta}(x_2-x_1)\; :\overline{\nu}^\beta_\mu(x_2)\nu^\alpha_\mu(x_1): + S^\mu_{\alpha\beta}(x_2-x_1)\; :\overline{\nu}^\beta_e(x_2)\nu^\alpha_e(x_1):\right)\Big]\,, \tag{8}$$

where $S^\sigma_{\alpha\beta}(x)$ is the Dirac propagator for the σ-flavor neutrino.

In the interaction picture, ν_σ ($\sigma = e,\mu$) can be expanded as free fields, whose evolution is governed by $\mathcal{L}_0$:

$$\nu_\sigma(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k},r}\left[u^r_{\mathbf{k},\sigma}(t)\,\alpha^r_{\mathbf{k},\sigma} + v^r_{-\mathbf{k},\sigma}(t)\,\beta^{r\dagger}_{-\mathbf{k},\sigma}\right]e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{9}$$

with $u^r_{\mathbf{k},\sigma}(t) \;=\; e^{-i\omega_{\mathbf{k},\sigma}t}\,u^r_{\mathbf{k},\sigma}$, $v^r_{\mathbf{k},\sigma}(t) \;=\; e^{i\omega_{\mathbf{k},\sigma}t}\,v^r_{\mathbf{k},\sigma}$, $\omega_{\mathbf{k},\sigma} \;=\; \sqrt{|\mathbf{k}|^2+m_\sigma^2}$. Annihilation operators satisfy

$$\alpha^r_{\mathbf{k},\sigma}|0\rangle = 0 = \beta^r_{\mathbf{k},\sigma}|0\rangle\,. \tag{10}$$

Our aim is to compute the the survival probability for a flavor neutrino (say $\sigma = e$), i.e. the probability of the process $|\nu^r_{\mathbf{p},e}\rangle \;\to\; |\nu^r_{\mathbf{p},e}\rangle$. The amplitude can be written as

$$\mathcal{A}_{e\to e}(\mathbf{p};t_i,t_f) = 1 + \mathcal{A}^{(2)}_{e\to e}(\mathbf{p};t_i,t_f)\,, \tag{11}$$

where $\mathcal{A}^{(2)}_{e\to e}(\mathbf{p};t_i,t_f)$ is the second-order piece, which is proportional to $m^2_{e\mu}$. Taking the square, we disregard the pieces proportional to $m^4_{e\mu}$. Therefore, we get

$$\mathcal{P}_{e\to e}(\mathbf{p};\Delta t) \;=\; 1 + 2\,\Re e\left(\mathcal{A}^{(2)}_{e\to e}(\mathbf{p};t_i,t_f)\right)\,. \tag{12}$$

Explicitly one finds [8]

$$\mathcal{P}_{e\to e}(\mathbf{p};\Delta t) \;=\; 1 - 4m^2_{e\mu}\left[\frac{W^2_\mathbf{p}}{\left(\omega_{\mathbf{p},e}-\omega_{\mathbf{p},\mu}\right)^2}\sin^2\left(\frac{\left(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e}\right)\Delta t}{2}\right) \right. \\ \left. + \frac{Y^2_\mathbf{p}}{\left(\omega_{\mathbf{p},e}+\omega_{\mathbf{p},\mu}\right)^2}\sin^2\left(\frac{\left(\omega_{\mathbf{p},\mu}+\omega_{\mathbf{p},e}\right)\Delta t}{2}\right)\right]. \tag{13}$$

where we introduced the coefficients

$$W_\mathbf{p} \;=\; \overline{u}^s_{\mathbf{p},\mu}u^s_{\mathbf{p},e}\,, \quad Y^2_\mathbf{p} \;=\; \sum_\mathbf{s}\left(Y^{rs}_\mathbf{p}\right)^* Y^{rs}_\mathbf{p}\,, \quad Y^{ss'}_\mathbf{p} \;=\; \overline{u}^s_{\mathbf{p},\mu}v^{s'}_{-\mathbf{p},e}\,. \tag{14}$$

If we now define

$$U_\mathbf{p} \;=\; W_\mathbf{p}\frac{m_\mu - m_e}{\omega_{\mathbf{p},e}-\omega_{\mathbf{p},\mu}} \;=\; \sqrt{\frac{\left(\omega_{\mathbf{p},e}+m_e\right)\left(\omega_{\mathbf{p},\mu}+m_\mu\right)}{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(1 + \frac{|\mathbf{p}|^2}{\left(\omega_{\mathbf{p},e}+m_e\right)\left(\omega_{\mathbf{p},\mu}+m_\mu\right)}\right), \tag{15}$$

$$V_\mathbf{p} \;=\; Y_\mathbf{p}\frac{m_\mu - m_e}{\omega_{\mathbf{p},e}+\omega_{\mathbf{p},\mu}} \;=\; \sqrt{\frac{\left(\omega_{\mathbf{p},e}+m_e\right)\left(\omega_{\mathbf{p},\mu}+m_\mu\right)}{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(\frac{|\mathbf{p}|}{\omega_{\mathbf{p},e}+m_e} - \frac{|\mathbf{p}|}{\omega_{\mathbf{p},\mu}+m_\mu}\right), \tag{16}$$

we can write the survival probability (13) as

$$\mathcal{P}_{e\to e}(\mathbf{p};\Delta t) \;=\; 1-\sin^2 2\theta\left[U_{\mathbf{p}}^2\sin^2\left(\frac{(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e})\,\Delta t}{2}\right)+V_{\mathbf{p}}^2\sin^2\left(\frac{(\omega_{\mathbf{p},\mu}+\omega_{\mathbf{p},e})\,\Delta t}{2}\right)\right]\,. \qquad (17)$$

where we used that $m_{e\mu}/(m_\mu - m_e) = \tan 2\theta/2 \approx \theta \approx \sin\theta$. In the approximation we adopted, this expression non-trivially coincides with the oscillation probability originally derived in Ref. [9] (see also the review [10]). With respect to the usual Pontecorvo formula [1, 2, 11], which only depends on the difference of neutrino energies[2] the above expression also contains a term describing fast oscillations, depending on $\omega_e + \omega_\mu$. Moreover, Eq.(17) contains the two *Bogoliubov coefficients* $U_{\mathbf{p}}$ and $V_{\mathbf{p}}$ so that $U_{\mathbf{p}}^2 + V_{\mathbf{p}}^2 = 1$. From the expressions (15),(16) one can see that $U_{\mathbf{p}} \to 1$ while $V_{\mathbf{p}} \to 0$ in the relativistic limit $m_\sigma/|\mathbf{p}| \to 0$, thus recovering the standard formula.

It is important to highlight the central role of finite time in the present approach. Specifically, the primary focus of the analysis is the time evolution operator (8), rather than the S-matrix. This distinction is closely tied to the time-energy uncertainty relation [12, 13], which can also be interpreted as a *flavor-energy* uncertainty relation in the present case [14] (see also the recent review [15]). In the limit $t_i \to -\infty$ and t_f, energy would be strictly conserved, leading to the suppression of flavor oscillations [16]. This scenario is analogous to the case of unstable particles [17–19], where computations at finite times are standard.

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