PROCEEDINGS OF SCIENCE

Truncation orders, external constraints, and the determination of $|V_{cb}|$

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We present a model selection framework for the extraction of the CKM matrix element $|V_{cb}|$ from exclusive $B \to D^* l\nu$ decays. By framing the truncation of the Boyd-Grinstein-Lebed (BGL) parameterization as a model selection task, we apply the Akaike Information Criterion (AIC) to choose the optimal truncation order. We demonstrate the performance of our approach through a comprehensive toy study, comparing it to the Nested Hypothesis Test (NHT) method used in previous analyses. Our results show that the AIC-based approach produces unbiased estimates of $|V_{cb}|$, albeit with some issues of undercoverage. We further investigate the impact of unitarity constraints and explore model averaging using the Global AIC (gAIC) approach, which produced unbiased results with correct coverage properties. Our findings suggest that model selection techniques based on information criteria and model averaging offer a promising path towards more reliable $|V_{cb}|$ determinations.

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1. Introduction

The precise determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{cb}|$ remains a crucial challenge in flavor physics. A long-standing tension exists between the values obtained from inclusive and exclusive measurements of semileptonic B meson decays. This discrepancy, often referred to as the $|V_{cb}|$ puzzle, has significant implications for our understanding of quark flavor physics and potential new physics beyond the Standard Model.

In recent years, much attention has focused on the exclusive decay channel $B \to D^* l\nu$, which offers a promising avenue for precise $|V_{cb}|$ determination due to its experimental accessibility and theoretical cleanliness (Bordone and Juttner 2024). However, the extraction of $|V_{cb}|$ from this channel is sensitive to the parameterization of hadronic form factors, which describe the strong interaction effects in the decay.

The results presented in this paper are preliminary findings from a more comprehensive study (F. Bernlochner et al. forthcoming).

1.1 BGL Parameterization and the Truncation Dilemma

The decay rate in the $B \to D^* l\nu$ channel can be expressed in terms of three form factors for massless leptons. These form factors can be parameterized using the Boyd-Grinstein-Lebed (BGL) expansion (Boyd, Grinstein, and Lebed 1995), which exploits the analytic properties of the form factors and incorporates constraints from unitarity. The BGL parameterization expresses each form factor as an infinite series:

$$f(z) = \frac{1}{P(z)\phi(z)} \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

where z is a kinematic variable, $P(z)$ is a Blaschke factor, $\phi(z)$ is an outer function, and a_n are the expansion coefficients (Simons, Gustafson, and Meurice 2024). Similar expansions with coefficients b_n and c_n are used for the other two form factors.

In practice, these infinite series must be truncated at some finite order, represented by three integers (N_a, N_b, N_c). This truncation presents a fundamental dilemma: truncating too early may introduce bias in our estimates of $|V_{cb}|$, while truncating too late leads to an overly complex model, unnecessarily increasing the variance of the fit results. This situation exemplifies the classic bias-variance trade-off in statistical modeling. The choice of truncation order can significantly impact the extracted value of $|V_{cb}|$, as demonstrated in previous studies (F. U. Bernlochner, Ligeti, and Robinson 2019; Gambino, Jung, and Schacht 2019).

2. Model Selection Framework

In the present paper we propose framing this truncation problem as a model selection task, drawing on the well-established field of statistical model selection. In this framework, each possible BGL truncation order (N_a, N_b, N_c) represents a distinct model. The goal is to choose both the best model (truncation order) and the best parameters within that model.

By framing our problem in these terms, we can apply established statistical techniques and provide a principled, rigorous procedure for choosing the truncation order. This approach aims to minimize arbitrary choices on the part of the researcher, enhancing the reproducibility and reliability of $|V_{cb}|$ determinations.

2.1 Components of Model Selection

To clarify the choices involved in model selection, we propose the following taxonomy:

1. **Model evaluation metrics**: These assess model performance quantitatively. Common examples include the sum of squared errors, chi-square statistic, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and cross-validation error.

2. **Model selection decision rules**: These determine how to choose between competing models based on their evaluation metrics. For instance, one might select the model with the lowest AIC or choose a more complex model only if it improves the metric by a certain threshold.

3. **Model space search algorithms**: These define how to navigate through the space of possible models. Examples include forward stepwise selection, backward elimination, and exhaustive search.

Previous work in this field can be analyzed using this taxonomy. Bernlochner et al. (F. U. Bernlochner, Ligeti, and Robinson 2019) used sum of squared residuals (SSR) as their metric, chose a more complex model only if the improvement in SSR exceeded 1, and employed forward stepwise selection as their search algorithm. Gambino et al. (Gambino, Jung, and Schacht 2019) adopted a similar approach, but imposed unitarity constraints and moved towards more complex models until $|V_{cb}|$ estimates stabilized.

In contrast, we propose using the Akaike Information Criterion (AIC) (Akaike 1974) as our metric, choosing the model with the lowest AIC, and employing an exhaustive search within feasible models. We argue that these choices are more less arbitrary, simpler to implement, and easier to interpret statistically than previous approaches.

2.2 Akaike Information Criterion (AIC)

The Akaike Information Criterion (AIC) is a well-established model selection tool in statistics that provides a principled way to balance model complexity against goodness-of-fit.

The AIC is defined as:

$$\text{AIC} = -2\,log(L) + 2k \tag{2}$$

where L is the maximum likelihood of the model and k is the number of parameters.

Despite its apparent simplicity, the AIC has strong theoretical motivations (Blankenshipa, Perkinsb, and Johnsonc 2002; Burnham and Anderson 1998). It has deep connections to information theory, specifically to the Kullback-Leibler divergence between the true data-generating process and the model. It provides a principled way to balance model complexity against goodness-of-fit, addressing the bias-variance trade-off. Furthermore, it is widely applicable across various fields and model types, making it a versatile tool for model selection.

In the context of BGL parameterization, the AIC offers a natural way to determine the optimal truncation order, potentially leading to more robust and reliable $|V_{cb}|$ extractions.

3. Toy Study Results

To evaluate our AIC-based approach against the Nested Hypothesis Test (NHT) method used in (F. U. Bernlochner, Ligeti, and Robinson 2019), we conducted a comprehensive toy study. We simulated $B \to D^* l\nu$ decay data based on realistic parameters, assumed an underlying true BGL order to generate the data, and used the Belle covariance matrix (Heavy Flavor Averaging Group 2024) to incorporate a realistic error structure. We then fitted models using both the NHT and AIC procedures. Our primary goal was to demonstrate that our method produces unbiased estimates of $|V_{cb}|$ with correct coverage properties.

We present our results primarily through "pull distributions", defined as:

$$\text{pull} = \frac{\text{estimated value} - \text{true value}}{\text{estimated uncertainty}}. \tag{3}$$

Figure 1 shows the pull distribution for the NHT approach without unitarity constraints. The NHT approach produced unbiased estimates of $|V_{cb}|$, but showed some under-coverage issues, indicating that the uncertainty might be underestimated. As expected, the NHT tended to select simpler models (e.g., order (2,1,1)). Without unitarity constraints, some fits saturated the unitarity bounds.

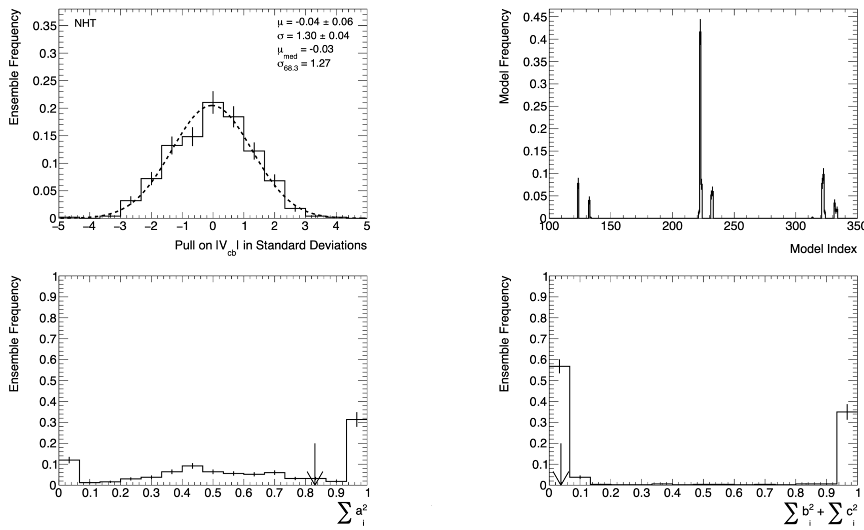

Figure 1: Pull plot of NHT, without unitarity constraints

Figure 2 presents the pull distribution for the AIC approach without unitarity constraints. The AIC approach showed competitive performance compared to NHT; it produced similarly unbiased estimates of $|V_{cb}|$, but suffered from under-coverage issues similar to NHT.

These results suggest that our AIC-based approach is a viable alternative to the NHT method, offering comparable performance while providing a simpler and statistically more rigorous framework for model selection.

3.1 Unitarity Constraints

In addition to comparing the NHT and AIC approaches, we investigated the impact of imposing unitarity constraints on our model selection procedures.

For both NHT and AIC approaches, imposing unitarity constraints ameliorated the under-coverage issues observed in the unconstrained fits. Figure 3 shows the pull distribution for the NHT approach with unitarity constraints, while Figure 4 presents the corresponding results for the AIC approach.

The relationship between unitarity constraints and coverage properties is a key area for future investigation (F. Bernlochner et al. forthcoming). Our results currently suggest that imposing unitarity constraints can improve the coverage properties of $|V_{cb}|$ estimates, but the underlying reasons for this improvement remain unclear.

3.2 Global AIC: Model Averaging

Moving beyond single model selection, we also explored a model averaging approach using the concept of Global AIC (gAIC) (Burnham and Anderson 1998). The key idea of gAIC is to weigh multiple models based on their

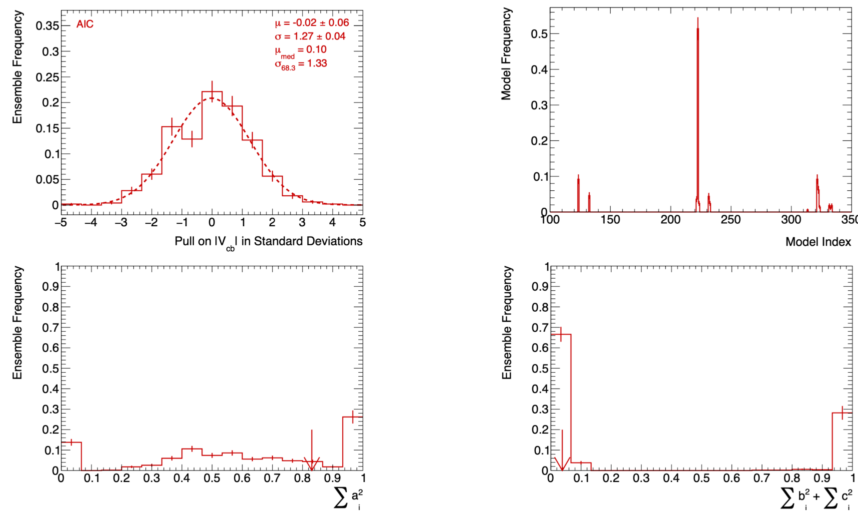

Figure 2: Pull plot of AIC, no unitarity constraints

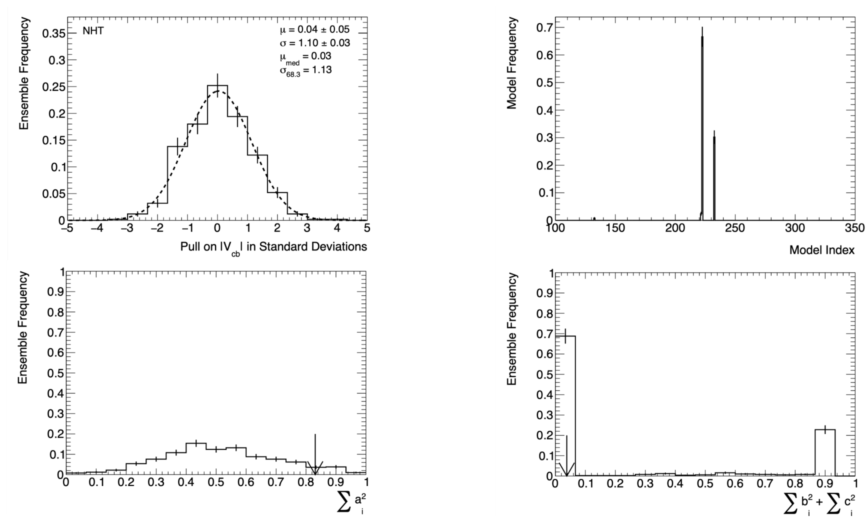

Figure 3: Pull plot of NHT, with unitarity constraints

relative support from the data, rather than selecting a single "best" model. This approach can be formalized as follows:

1. Calculate the AIC for each model.
2. Compute model weights based on the relative likelihood:

$$w_i = exp(-\frac{1}{2}\Delta_i)/\sum_j exp(-\frac{1}{2}\Delta_j) \tag{4}$$

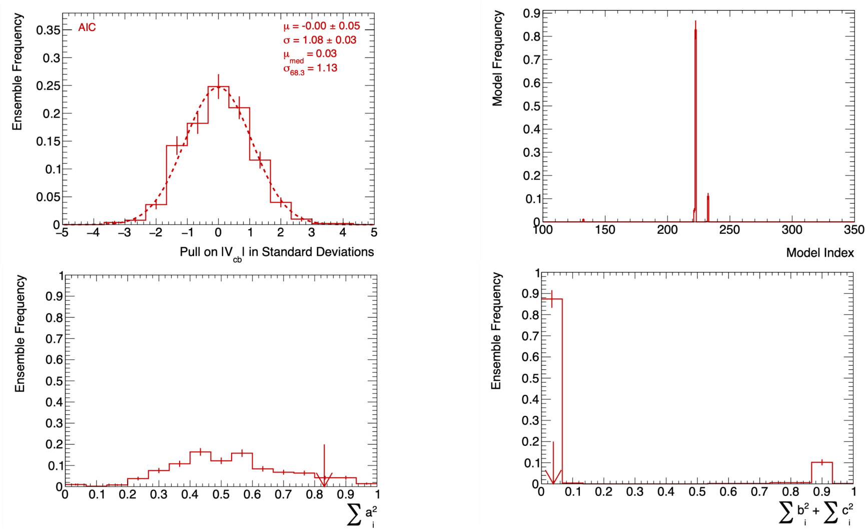

Figure 4: Pull plot of AIC, with unitarity constraints

where

$$\Delta_i = \text{AIC}_i - \text{AIC}_{\text{min}} \tag{5}$$

3. Calculate a weighted average of $|V_{cb}|$ estimates across models:

$$V_{cb}| = \sum_i w_i |V_{cb}|_i \tag{6}$$

The estimator for the variance of the gAIC is given by:

$$\widehat{\text{var}}(\hat{\bar{\theta}}) = \left[\sum_{i=1}^{R} w_i \sqrt{\widehat{\text{var}}(\hat{\theta}_i | g_i) + (\hat{\theta}_i - \hat{\bar{\theta}})^2} \right]^2 \tag{7}$$

where $\hat{\bar{\theta}} = \sum_{i=1}^{R} w_i \hat{\theta}_i$.

The gAIC approach offers several potential advantages. By considering multiple models, it can better account for model uncertainty, potentially leading to more robust estimates of $|V_{cb}|$. It also provides a natural way to incorporate information from all plausible models, rather than relying on a single selected model which may not be definitively better than its close competitors.

Figure 5 shows the pull distribution for the gAIC approach, both with and without unitarity constraints. The results from the gAIC approach are particularly encouraging, since it produced unbiased estimates of $|V_{cb}|$ with correct coverage properties, both with and without unitarity constraints.

The superior performance of gAIC, particularly in terms of coverage properties, suggests that model averaging may be a valuable tool for future $|V_{cb}|$ determinations.

4. Discussion and Future Work

Our results demonstrate the potential of rigorous model selection techniques, particularly those based on information criteria and model averaging, for improving $|V_{cb}|$ extraction from exclusive $B \to D^* l\nu$ decays. The

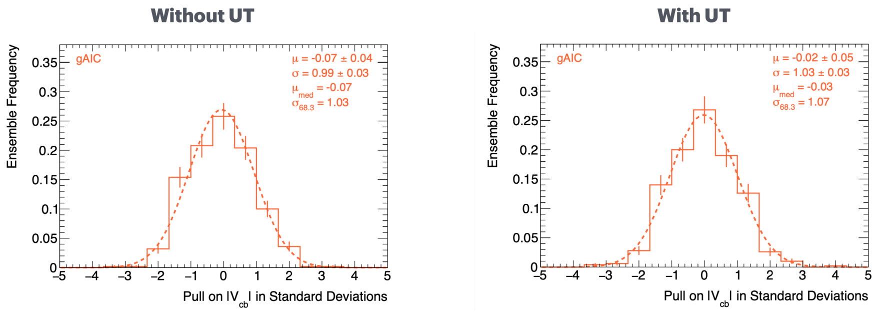

Figure 5: Pull plot of gAIC, with and without unitarity constraints

AIC-based approach, especially when combined with model averaging in the form of gAIC, shows promise in producing unbiased estimates of $|V_{cb}|$ with correct coverage properties.
However, several areas require further investigation:

1. The source of under-coverage in non-averaged approaches, particularly when unitarity constraints are not imposed, needs to be better understood.
2. While our results support the use of AIC, it would be valuable to evaluate other model selection metrics to confirm AIC's superiority in this context. Potential alternatives include the Bayesian Information Criterion (BIC) (Schwarz 1978) or cross-validation approaches.
3. Incorporating external constraints, such as lattice QCD results, into our model selection framework presents both an opportunity and a challenge. Recent lattice QCD calculations (Bazavov et al. 2022; Harrison and Davies 2024; Aoki et al. 2024) provide valuable information about the form factors, but integrating this information into our model selection procedure requires careful consideration.

5. Conclusion

In this work, we have presented a comprehensive study of model selection techniques for $|V_{cb}|$ extraction from exclusive $B \to D^* l\nu$ decays. We have demonstrated that an approach based on the Akaike Information Criterion, particularly when combined with model averaging, can produce unbiased estimates of $|V_{cb}|$ with correct coverage properties.
Our results highlight the importance of rigorous model selection in precision flavor physics. By providing a principled way to choose between different truncations of the BGL expansion, our approach reduces the impact of arbitrary choices on $|V_{cb}|$ determinations. The use of model averaging further enhances the robustness of our results by accounting for model uncertainty.

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