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Massive twistor worldline on electromagnetic fields

Jung-Wook Kim*

Max Planck Institute for Gravitational Physics (Albert Einstein Institute),
Am Mühlenberg 1, D-14476 Potsdam, Germany

E-mail: jung-wook.kim@aei.mpg.de

The (ambi-)twistor model for spinning particles interacting via electromagnetic field is studied as a toy model for studying classical dynamics of gravitating bodies including effects of both spins to all orders. The all-orders-in-spin effects are encoded as a dynamical implementation of the Newman-Janis shift. It is found that the expansion in both spins can be resummed to simple expressions in special kinematic configurations, at least up to next-to-leading order. It is also observed that cutting rules associated with causality prescription for worldline propagators can be viewed as Poisson brackets of subdiagrams.

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*Speaker

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1. Introduction

Since the first detection of gravitational waves [4], gravitational wave science is gradually becoming a precision science as observatories keep improving their signal-to-noise ratios. However, the inaccurate modelling of spin effects in current waveform models is expected to impede progress, as they are not accurate enough for data analysis in next-generation gravitational wave observatories [5]. This motivates the study of spin effect resummation in binary dynamics; more precise understanding of spin effects will lead to more precise waveform models for future observatories.

The twistor worldline description of spinning particles offers an avenue to explore resummation of spin effects, where spin effects to all orders are implemented as a dynamical Newman-Janis shift [1, 2]. The findings of ref. [1] are reported, where spin effect resummation in binary dynamics was studied using electromagnetic $2 \to 2$ scattering of $\sqrt{\text{Kerr}}$ particles as a toy model for the gravitational dynamics. Intriguingly, the spinning dynamics described by the eikonal was found to resum into simple closed form expressions at next-to-leading order for special configurations.

2. Twistor worline model for spinning particles

The twistor worldline model uses twistor variables ($\lambda_\alpha{}^I$, $\mu^{\dot\alpha I}$, and their complex conjugates) to describe a spinning particle as a constrained Hamiltonian system [1, 2]. The free action is

$$S_{\text{free}} = \int \left[\lambda_\alpha{}^I \dot{\bar\mu}_I{}^\alpha + \bar\lambda_{I\dot\alpha}\dot\mu^{\dot\alpha I} - \kappa^0\phi_0 - \kappa^1\phi_1\right] d\sigma \,, \tag{1}$$

where $\kappa^{0,1}$ are Lagrange multipliers imposing the constraint conditions $\phi_{0,1} = 0$.

$$\begin{aligned} \phi_0 &= \frac{1}{2}(m^2 - \Delta\bar\Delta) = \frac{1}{2}(p^2 + m^2)\,, \quad \phi_1 = \frac{1}{2i}(\bar\lambda_{I\dot\alpha}\mu^{\dot\alpha I} - \bar\mu_I{}^\alpha\lambda_\alpha{}^I) = p\cdot y\,, \\ \Delta &= \det(\lambda) = -\frac{1}{2}\epsilon^{\alpha\beta}\epsilon_{IJ}\lambda_\alpha{}^I\lambda_\beta{}^J\,, \quad \bar\Delta = \det(\bar\lambda) = \frac{1}{2}\epsilon^{IJ}\epsilon^{\dot\alpha\dot\beta}\bar\lambda_{I\dot\alpha}\bar\lambda_{J\dot\beta}\,, \end{aligned} \tag{2}$$

Here, p^μ is the momentum, m is the mass, and y^μ is the imaginary part of the (complexified) position z^μ, which is defined by the incidence relations,

$$\mu^{\dot\alpha I} = \frac{1}{2}z^{\dot\alpha\beta}\lambda_\beta{}^I\,, \quad \bar\mu_I{}^\alpha = \frac{1}{2}\bar\lambda_{I\dot\beta}\bar z^{\dot\beta\alpha}\,, \quad z^\mu = x^\mu + iy^\mu = (\bar z^\mu)^*\,. \tag{3}$$

x^μ is the real position and y^μ corresponds to the spin-length vector $y^\mu = -a^\mu = -s^\mu/m$.

The worldline couples to the Maxwell field by the Newman-Janis shift, where the position of the particle is shifted towards the helicity-dependent imaginary spin direction $x^\mu \to x^\mu \pm iy^\mu$ [6],

$$S_{\text{int}} = q\int \left[A^+_\mu(z)\dot z^\mu + A^-_\mu(\bar z)\dot{\bar z}^\mu\right] d\sigma\,. \tag{4}$$

Because the interaction term (4) solely depends on the composite variables z^μ and $\bar z^\mu$, computations based on worldline quantum field theory (WQFT) formalism [14] become simpler when organised in terms of composite fluctuation fields δz^μ and $\delta\bar z^\mu$, instead of the fundamental degrees of freedom appearing in (1).[1] Consult ref. [1] for details.

[1]This is due to adopting QED-like treatment of photon propagators. Organisation in terms of twistor variables may become simpler if all degrees of freedom (including the photons) are given in terms of twistor variables.

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3. Spin resummation in two-body dynamics

The main results of ref. [1] are the eikonal and how it encodes scattering dynamics in a compact way. The latter will be explained in section 4.

The leading order (1st post-Lorentzian; 1PL) eikonal is

$$\chi_{(1)} = \frac{q_1 q_2 \gamma}{4\pi\sqrt{\gamma^2-1}} \left[\frac{1}{\epsilon} + \mathrm{Re}\left(\log \frac{(b^\mu + i y_\perp^\mu)^2}{b_0^2} \right) \right.$$
$$\left. - \frac{\epsilon[b, v_1, v_2, y_\perp]}{2\gamma\sqrt{b^2 y_\perp^2 - (b\cdot y_\perp)^2}} \log\left(\frac{b^2 + y_\perp^2 + 2\sqrt{b^2 y_\perp^2 - (b\cdot y_\perp)^2}}{b^2 + y_\perp^2 - 2\sqrt{b^2 y_\perp^2 - (b\cdot y_\perp)^2}} \right) \right] , \tag{5}$$

where $b^\mu \sim x_1^\mu - x_2^\mu$ is the impact parameter, ϵ and b_0 are IR regularisation parameters, $v_i^\mu = p_i^\mu / m_i$ are the velocities, $\gamma = -v_1 \cdot v_2$ is the rapidity factor, and $y_\perp^\mu$ is the projection of $y^\mu = y_1^\mu + y_2^\mu$ onto the impact parameter plane.

The full next-to-leading order (2PL) eikonal is presented as a machine-readable file in ref. [1]. We only report the simplified eikonal in special kinematic configurations.

In the aligned spin configuration, defined by the conditions $y_1^\mu \propto y_2^\mu$ and $y \cdot v_{1,2} = y \cdot b = 0$, the eikonal simplifies to

$$\chi_{(2,\text{aligned})} = \frac{(q_1 q_2)^2 \left(b^2 - \frac{(\zeta-2)\gamma}{(\gamma^2-1)} \epsilon[b, v_1, v_2, y] + \frac{\gamma^2(1-\zeta)+\zeta}{\gamma^2-1} y^2 \right)}{32\pi m_1 \sqrt{\gamma^2-1}\,(b^2-y^2)^{3/2}} + (1 \leftrightarrow 2) , \tag{6}$$

where ζ is the ratio parameter defined by $y_1^\mu = \zeta y^\mu$. This singularity structure is consistent with the singularity structure $\propto (b^2 - y_1^2)^{-3/2}$ reported for spinning-spinless gravitational scattering [7–9].

The axial scattering is defined by the conditions $y_1^\mu \propto y_2^\mu \propto b^\mu$, and the eikonal simplifies to

$$\chi_{(2,\text{axial})} = \frac{(q_1 q_2)^2 \sqrt{b^2}}{16\pi^2 m_1 (\gamma^2-1)^{3/2}} \left[\frac{\gamma^2(\zeta-1) - \zeta}{b^2} K\left(-\frac{y^2}{b^2} \right) - \frac{\gamma^2(\zeta-2) - (\zeta-1)}{b^2+y^2} E\left(-\frac{y^2}{b^2} \right) \right]$$
$$+ (1 \leftrightarrow 2) . \tag{7}$$

4. Eikonal as scattering generator and causality cuts

The eikonal can be identified as the matrix elements of the N-matrix, which is related to the S-matrix by exponentiation $S = \exp(\frac{i}{\hbar} N)$ [10]. The Kosower-Maybee-O'Connell (KMOC) approach [11] relates the S-matrix to (classical) scattering observables by the relation

$$\langle\langle O_{\text{out}} \rangle\rangle = \langle\langle O_{\text{in}} + \Delta O \rangle\rangle = \langle\langle S^\dagger O_{\text{in}} S \rangle\rangle . \tag{8}$$

The double bracket $\langle\langle \bullet \rangle\rangle = \langle \psi_{\text{in}} | \bullet | \psi_{\text{in}} \rangle$ denotes the quantum-mechanical expectation value for the incoming state $|\psi_{\text{in}}\rangle$. As an operator equation, (8) can be recast in terms of the N-matrix as [12]

$$O_{\text{out}} = e^{-\frac{i}{\hbar}N} O_{\text{in}} e^{+\frac{i}{\hbar}N} = (e^{-\frac{i}{\hbar}N})_{\text{adj}}[O_{\text{in}}] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar} \right)^n \underbrace{[N, [N, [N, \cdots, [N}_{n\text{ times}}, O_{\text{in}}]] \cdots]] . \tag{9}$$

The adjoint action is defined by commutators $A_{\text{adj}}[B] := [A, B]$. This equation can be interpreted as a symmetry transformation on the observable O_{in} generated by N.

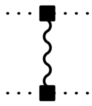
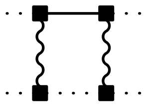
The view of the eikonal χ as the *scattering generator* is motivated by "de-quantising" (9), where commutators are replaced by Poisson brackets $\frac{1}{i\hbar}[\bullet, \bullet] \to \{\bullet, \bullet\}$ and N is replaced by χ [1, 13],

$$O_{\text{out}} = e^{\{\chi, \bullet\}}[O_{\text{in}}] = O_{\text{in}} + \{\chi, O_{\text{in}}\} + \frac{1}{2!}\{\chi, \{\chi, O_{\text{in}}\}\} + \frac{1}{3!}\{\chi, \{\chi, \{\chi, O_{\text{in}}\}\}\} + \cdots . \tag{10}$$

This is a canonical transformation generated by χ.

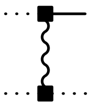
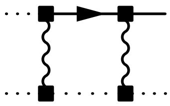
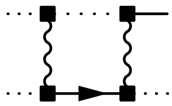
An important observation is that the Poisson brackets are computed by *causality cuts*, which are defined as the difference between retarded and advanced propagators [1].[2] The causality cuts allow diagrammatic representation of (10) using the WQFT formalism [14]. The next-to-leading order impulse is presented as an example.

The schematic representation of the eikonal in a diagrammatic description is

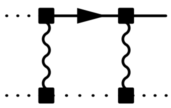
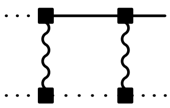
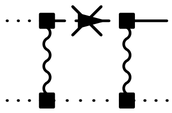
$i\chi_{(1)} =$  , $i\chi_{(2)} =$  $+ \ (1 \leftrightarrow 2)$. (11)

The dotted lines represent the background straight-line trajectory of the scattering particles, solid lines represent propagating fluctuations of the worldline, wavy lines represent exchanged photons, and the time-symmetric prescription is used for the propagators [14].

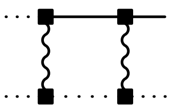
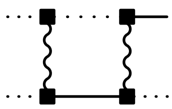
The impulse is represented diagrammatically as

$\Delta_{(1)} p_1^\mu =$  $= \{\chi_{(1)}, p_1^\mu\}$, $\Delta_{(2)} p_1^\mu =$  $+$  . (12)

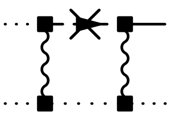
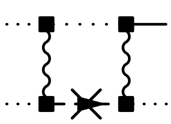
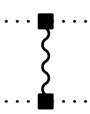
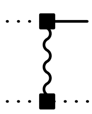
The arrows on the internal lines denote their causality prescriptions, and the external line yields the observable one-point function $\langle\langle \Delta p_1^\mu \rangle\rangle$ [15]. The retarded propagators can be converted to time-symmetric propagators by adding causality cuts, diagrammatically represented as

 $=$  $+ \frac{1}{2} \Big($  $\Big)$. (13)

The first diagram denotes the analogous calculation using time-symmetric propagators, and the second diagram denotes the causality cut contributions. The time-symmetric propagator diagrams can be reorganised as a single Poisson bracket

 $+$  $= \{\chi_{(2)}, p_1^\mu\}$. (14)

The causality cut diagrams can be reorganised as iterated Poisson brackets

 $+$  $= -i \Big\{$  ,  $\Big\} = \{\chi_{(1)}, \{\chi_{(1)}, p_1^\mu\}\}$. (15)

[2]This definition of causality cut is based on the follow-up study [3], which differs from the original definition for worldline propagators (difference between retarded and time-symmetric) given in ref. [1] by a factor of 2.

Combining the two generates all 2PL contributions to the impulse given by (10),

$$\Delta_{(2)} p_1^\mu = \{\chi_{(2)}, p_1^\mu\} + \frac{1}{2}\{\chi_{(1)}, \{\chi_{(1)}, p_1^\mu\}\} . \tag{16}$$

The diagrammatic representation of the spin kick Δy_1^μ reorganises in a similar way, with an additional diagram contributing to $\{\chi_{(2)}, y_1^\mu\}$ originating from the non-vanishing three-point function $\langle\langle \delta z^\mu \delta z^\nu \delta \bar{z}^\lambda \rangle\rangle$. This three-point function contributes to the Poisson brackets between composite field propagators and y_1^μ, e.g. $\{\langle \delta z^\alpha \delta \bar{z}^\beta \rangle, y_1^\mu\}$.

The causality cuts (15) generate the "longitudinal impulse" orthogonal to the impact parameter plane, due to "noncommutativity" of the impact parameter space $\{b^\mu, b^\nu\} \neq 0$ [1]. This contribution can be viewed as the rotation of the impact parameter plane, alluding to the "KMOC cut" generating a frame rotation in the one-loop scattering waveform [16].

5. Outlook

One of the difficulties in studying all-orders-in-spin resummation is evaluation of master integrals, which take the form of typical Feynman integrals deformed by an exponential factor due to Newman-Janis shift. Such integrals can be viewed as *tensor integral generating functions* (TIGF), and can be evaluated efficiently from conventional multiloop integration techniques [9]. It would be interesting to study their mathematical properties for application to various problems, e.g. scattering waveforms [17] or tensor reduction of Feynman integrals [18].

The causality cuts diagrammatically representing Poisson brackets between subdiagrams (15) generalises to arbitrary diagrams. Reverse-engineering the eikonal from extending (16) to higher PL orders, it is found that the diagrams for the eikonal have propagator causality prescriptions that differ from the naïve Feynman propagators. This is because the eikonal is computed by the Magnus series, while the usual Feynman diagrams correspond to computing the Dyson series [3]. Efficient methods for determining eikonal's causality prescription will appear shortly in a forthcoming work [3].

The twistor worldline model [1, 2] provides a valuable toy model for studying binary dynamics where spin effects can be tracked analytically to all orders. Pushing the calculations to higher perturbation orders and studying resummation of spin effects will yield invaluable insights on spinning binary dynamics, which will lead to more accurate modelling of spin effects in waveform models necessitated by the expected precision of future gravitational wave observatories.

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References

[1] J. H. Kim, J. W. Kim and S. Lee, *Massive twistor worldline in electromagnetic fields*, JHEP **08**, 080 (2024) [2405.17056].

[2] J. H. Kim, J. W. Kim and S. Lee, *The relativistic spherical top as a massive twistor*, J. Phys. A **54**, no.33, 335203 (2021) [2102.07063].

[3] J. H. Kim, J. W. Kim, S. Kim and S. Lee, *Classical eikonal from Magnus expansion*, [2410.22988].

[4] B. P. Abbott *et al.* [LIGO Scientific and Virgo], *Observation of Gravitational Waves from a Binary Black Hole Merger*, Phys. Rev. Lett. **116**, no.6, 061102 (2016) [1602.03837].

[5] A. Dhani, S. Völkel, A. Buonanno, H. Estelles, J. Gair, H. P. Pfeiffer, L. Pompili and A. Toubiana, *Systematic Biases in Estimating the Properties of Black Holes Due to Inaccurate Gravitational-Wave Models*, [2404.05811].

[6] E. T. Newman and J. Winicour, *A curiosity concerning angular momentum*, J. Math. Phys. **15**, 1113-1115 (1974)

[7] R. Aoude, K. Haddad and A. Helset, *Classical Gravitational Spinning-Spinless Scattering at $O(G^2S^\infty)$*, Phys. Rev. Lett. **129**, no.14, 141102 (2022) [2205.02809].

[8] P. H. Damgaard, J. Hoogeveen, A. Luna and J. Vines, *Scattering angles in Kerr metrics*, Phys. Rev. D **106**, no.12, 124030 (2022) [2208.11028].

[9] G. Chen, J. W. Kim and T. Wang, *Systematic integral evaluation for spin-resummed binary dynamics*, [2406.17658].

[10] P. H. Damgaard, L. Plante and P. Vanhove, *On an exponential representation of the gravitational S-matrix*, JHEP **11**, 213 (2021) [2107.12891].

[11] D. A. Kosower, B. Maybee and D. O'Connell, *Amplitudes, Observables, and Classical Scattering*, JHEP **02**, 137 (2019) [1811.10950].

[12] P. H. Damgaard, E. R. Hansen, L. Planté and P. Vanhove, *Classical observables from the exponential representation of the gravitational S-matrix*, JHEP **09**, 183 (2023) [2307.04746].

[13] R. Gonzo and C. Shi, *Scattering and bound observables for spinning particles in Kerr spacetime with generic spin orientations*, [2405.09687].

[14] G. Mogull, J. Plefka and J. Steinhoff, *Classical black hole scattering from a worldline quantum field theory*, JHEP **02**, 048 (2021) [2010.02865].

[15] G. U. Jakobsen, G. Mogull, J. Plefka and B. Sauer, *All things retarded: radiation-reaction in worldline quantum field theory*, JHEP **10**, 128 (2022) [2207.00569].

[16] A. Georgoudis, C. Heissenberg and R. Russo, *An eikonal-inspired approach to the gravitational scattering waveform*, JHEP **03**, 089 (2024) [2312.07452].

[17] A. Brandhuber, G. R. Brown, G. Chen, G. Travaglini and P. Vives Matasan, *Spinning waveforms in cubic effective field theories of gravity*, [2408.00587].

[18] B. Feng, *Generation function for one-loop tensor reduction*, Commun. Theor. Phys. **75**, no.2, 025203 (2023) [2209.09517].