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Classical Spin in General Relativity from Amplitudes

N. Emil J. Bjerrum-Bohr[a] **and Gang Chen**[a,*]

[a] *Niels Bohr International Academy, Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*

E-mail: bjbohr@nbi.dk, gang.chen@nbi.dk

This talk reviews recent results with classical spin in general relativity using quantum field theory. We report on a recent breakthrough: modern amplitude technology and bootstrap methods allow the expression of tree-level classical spin amplitude in terms of entire functions manifestly free of spurious poles. This organization enables an all-order spin framework for computations. We exemplify this with a bending angle computation.

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1. Introduction

Computation of gravitational scattering amplitudes complements conventional formalism in general relativity by enabling precision predictions for observables in specific dynamical regimes. For early approaches, see [1] and more recent developments for spinless black hole mergers [2, 3]. For spinning black holes, formalism is more complicated. Still, binary mergers present diverse signal characteristics, making it critical to develop a computational strategy for accurately capturing spin effects in a post-Minkowskian (PM) expansion. For progress with spin using amplitude methods, see [6, 7].

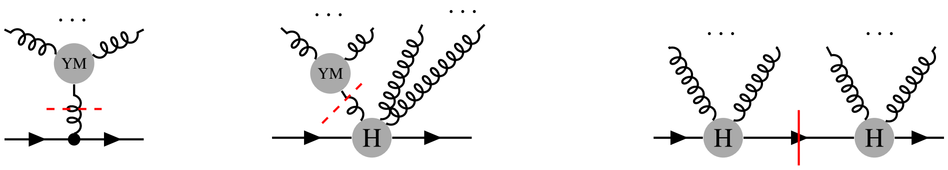
We will here focus on our recent quantum field theoretical framework for classical spin in general relativity [8, 9] that allows a new analysis of gravitational wave measurements from spinning binary mergers. We will first review the recent progress from bootstrapping the tree-level classical spin amplitude in terms of entire functions, which contain factorization information about spin up to infinite order and no spurious poles. Finally, we will outline how to compute the bending angle to the second post-Minkowskian order in this scheme.

2. A new formalism for classical spin in amplitudes

In [8], we found that amplitudes for gluons coupled to a particle with classical spin can inferred from a bootstrap procedure utilizing gauge-invariant numerators expressed only in terms Abelian field strength tensors and entire functions. The starting point for the procedure is the three-point function $R_a^\mu(p, v)$ with one massless vector line and two massive scalar lines with mass m, velocity v, and classical spin vector a, deduced from the work of ref. [6]. (Covariant polarization vector contraction with the $R_a^\mu(p, v)$ function leads to the three-point tree amplitude.)

$$R_a^\mu(p_1, v) = m\,\cosh(p_1\cdot a)v^\mu - i\left(\frac{\sinh(p_1\cdot a)}{p_1\cdot a}\right)(p_1\cdot S)^\mu\,, \tag{1}$$

here $S^{\mu\nu}$ denotes is the spin tensor defined from the Levi-Civita symbol $S^{\mu\nu} \equiv -m\epsilon^{\mu\nu\rho\sigma}v_\rho a_\sigma$. The three-point function is used in a bootstrap procedure where we require consistency for the following types of factorizations of massless and heavy-mass channels sketched below



(2)

The amplitudes above with a capital H are 'heavy mass effective field theory' amplitudes (this is the same notation used as in [4, 10]), but now with an associated spin on the massive line. Employing a particular basis of entire functions is convenient for arriving at a formalism without spurious pole terms. We recursively define these functions from

$$G_r(x_1; x_2, \cdots, x_r) \equiv \frac{1}{x_2\cdots x_r}\times\left(\sum_{\tau_a\cup\tau_b=\tau}\frac{\sinh(x_{1\tau_a})}{x_{1\tau_a}}\prod_{i\in\tau_b}(-\cosh(x_i))\right). \tag{3}$$

The bootstrap features a particular scaling behavior in spin parameter length $|a|$ that empirically restricts the types of entire functions needed. For instance, for an $(r+2)$-point amplitude, it is observed that we only need the entire functions to order G_r. (We refer to these as primary entire functions.) As an example, we can write the classical massive spinning four-point amplitude uniquely as

$$\mathcal{N}_a([1,2],v) = -\frac{w_1 \cdot F_1 \cdot F_2 \cdot w_2}{m\, v \cdot p_1} + \mathcal{N}'_a([1,2],v), \tag{4}$$

where $F_i^{\mu\nu} \equiv p_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu p_i^\nu$ denotes the abelian field strength, and we have $w_i^\mu \equiv m \cosh(x_i) p_4^\mu - i\, G_1(x_i)(p_i \cdot S)^\mu$. At four-point order, we need the entire function

$$G_2(x_1;x_2) \equiv \frac{1}{x_2}\Big(\frac{\sinh(x_{12})}{x_{12}} - \cosh(x_2)\,\frac{\sinh(x_1)}{x_1}\Big) \equiv \frac{1}{x_2}\Big(G_1(x_{12}) - \cosh(x_2) G_1(x_1)\Big), \tag{5}$$

and thus get the numerator

$$\begin{aligned}\mathcal{N}'_a(12,v) = m &\Big((a \cdot F_2 \cdot F_1 \cdot v)\,(a \cdot p_1) - (a \cdot F_2 \cdot v)\,(a \cdot F_1 \cdot p_2)\Big)\, G_1(x_1)\, G_1(x_2) \\ &+ i\,\Big(\big(a \cdot F_1 \cdot F_2 \cdot S \cdot p_2 + a \cdot F_2 \cdot F_1 \cdot S \cdot p_1\big)\, G_2(x_1;x_2) + \mathrm{tr}(S \cdot F_2 \cdot F_1) G_1(x_{12})\Big).\end{aligned} \tag{6}$$

Amplitudes with gravitons can be directly constructed from gluon amplitudes using an adapted double-copy procedure as explained in [9]. We decompose the amplitude in three pieces

$$M(1,2,v) = -\frac{\mathcal{N}_a(1,2,v)\,\mathcal{N}_0(1,2,v)}{2(p_1 \cdot p_2)} + \frac{\mathcal{N}_{\mathrm{r}}(1,2,v)}{4(p \cdot p_1)(p \cdot p_2)} + \mathcal{N}_{\mathrm{c}}(1,2,v). \tag{7}$$

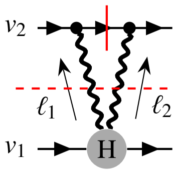
We derive the first term from the double copy. The second term contains a spin-flip correction. The last term is a contact term without physical propagators inferred from the Teukolsky solution [11] since it does not factorize. To account for the last two terms it is necessary to enlarge the basis of entire functions for consistency with factorization behavior on the massive cut lines. We find that the required additional functions are of the type, $G^{r_1,\cdots,r_j}(x_1,\ldots,x_j) \equiv (\prod_{i=1}^{j} \partial_{x_i}^{r_i}) G_j(x_1;x_2,\ldots,x_j)$, and we dub them descendent entire functions since they involve derivatives of the G_j functions we discussed in the gluon case [9]. To constraint the $\mathcal{N}_{\mathrm{r}}, \mathcal{N}_{\mathrm{c}}$, we again consider spin variable scaling and consistency requirements for the far-zone behavior of gravitational scattering in the Teukolsky solution up to fifth-order spin. The final result with the minimal coupling and high energy scaling behavior that fits the classical second-order post-Minkowskian gravitational bending angle up to a^5 order is ($F_{ij} \equiv F_i \cdot F_j$)

$$\begin{aligned}M = &-\Big(\frac{p_4 \cdot F_{12} \cdot p_4}{p_2 \cdot p_4}\Big)\Bigg(\frac{w_1 \cdot F_{12} \cdot w_2}{2(p_1 \cdot p_2)(p_1 \cdot p_4)} - \frac{1}{2(p_1 \cdot p_2)}\Big(iG_2(x_1,x_2)\,(a \cdot F_{12} \cdot S \cdot p_2) + iG_2(x_1,x_2)(a \cdot F_{21} \cdot S \cdot p_1) \\ &+ iG_1(x_{12})\,\mathrm{tr}\,(F_1 \cdot S \cdot F_2) + G_1(x_1)G_1(x_2)\big((a \cdot F_1 \cdot p_4)(a \cdot F_2 \cdot p_1) - (a \cdot F_1 \cdot p_2)(a \cdot F_2 \cdot p_4) + p_1 \cdot p_4 (a \cdot F_{12} \cdot a)\big)\Big)\Bigg) \\ &+ \frac{\Big((\partial_{x_1} - \partial_{x_2}) G_1(x_1) G_1(x_2)\Big)}{4(p_4 \cdot p_1)(p_4 \cdot p_2)}\Big(p_4 \cdot p_2 (p_4^2 (a \cdot F_{12} \cdot a)(a \cdot F_{21} \cdot p_4) + a^2 (p_4 \cdot F_{12} \cdot p_4)(a \cdot F_{12} \cdot p_4)) - (1 \leftrightarrow 2)\Big) \\ &+ \Big(\frac{i(\partial_{x_1} - \partial_{x_2}) G_2(x_1,x_2)}{4(p_4 \cdot p_1)(p_4 \cdot p_2)}\Big)\Big((p_4 \cdot p_2)(a \cdot F_{21} \cdot p_4)((a \cdot F_2 \cdot p_4)(a \cdot \widetilde{F}_1 \cdot p_4) - (a \cdot F_1 \cdot p_4)(a \cdot \widetilde{F}_2 \cdot p_4)) + (1 \leftrightarrow 2)\Big) \\ &+ \Big(\frac{(\partial_{x_1} - \partial_{x_2})^2}{2!} G_1(x_1) G_1(x_2)\Big)\Big((a \cdot F_1 \cdot p_4)(a \cdot F_2 \cdot p_4)(a \cdot F_{12} \cdot a) - (a^2/2)((a \cdot F_{12} \cdot p_4)(a \cdot F_{21} \cdot p_4) \\ &- (a \cdot F_{12} \cdot a)(p_4 \cdot F_{12} \cdot p_4))\Big) + \Big(\frac{i(\partial_{x_1} - \partial_{x_2})^2}{2!} G_2(x_1,x_2)\Big)\Big(-\frac{1}{2}((a \cdot F_{12} \cdot a)(a \cdot F_2 \cdot p_4)(a \cdot \widetilde{F}_1 \cdot p_4) - (1 \leftrightarrow 2))\Big).\end{aligned} \tag{8}$$

3. Binary scattering with classical spin

To facilitate computations, we will consider the formalism employed in [4, 12] as a way to generate observables from scattering theory (we work here only to second post-Minkowkian order).

The starting point in this case is the classical $2 \to 2$ one-loop amplitude,



$$M_{\text{HEFT}}(q, v_1, v_2, a_1, a_2) = \frac{-1}{2!} \int \frac{d^D \ell_1}{(2\pi)^{D-1}} \Big(\frac{\delta(2m_2 \ell_1 \cdot v_2)}{\ell_1^2 \ell_2^2} M_{a_2}(\ell_1, v_2) M_{a_2}(\ell_2, v_2)\, M_{a_1}(\ell_1, \ell_2, v_2) \Big) . \tag{9}$$

Here $\ell_2 = q - \ell_1$. M_{a_2} and M_{a_1} are three-point and four-point heavy mass effective field theory amplitudes. We can, for instance, use this result to compute the scattering angle χ obtained by transforming the result for the spinning Compton amplitude in the previous section to impact parameter space.

$$\chi = \int \frac{d^D q}{(2\pi)^{D-2}} \frac{e^{iq\cdot b} \delta(q\cdot v_1)\delta(q\cdot v_2)}{4 m_1 m_2} M_{\text{HEFT}}(q, v_1, v_2, a_1, a_2) . \tag{10}$$

To carry out this integral, we introduce the integral representations of the entire functions

$$G_1(x_1) G_1(x_2) = \int_0^1 d\sigma_1 d\sigma_2 \cosh(\sigma_1 x_1) \cosh(\sigma_2 x_2) , \tag{11}$$
$$G_2(x_1, x_2) = \int_0^1 d\sigma_1 d\sigma_2 \Big(\sigma_1 \sinh(\sigma_1 x_1 + \sigma_1 \sigma_2 x_2) - \sinh(\sigma_2 x_2) \cosh(\sigma_1 x_1) \Big) ,$$

which removes spurious denominators at the price of auxiliary integrations over σ_i.

Next, we work on the integrals in (10). Using an integration-by-parts reduction, we find 320 sectors, each with three master integrals,

$$\chi = \int_0^1 \prod_{j=1}^{4} d\sigma_j \sum_{\alpha=1}^{320} \Big(c_{1,\alpha}(\mathbf{y}) \widehat{\mathcal{I}}_1^{(\alpha)}[\mathbf{y}] + c_{2,\alpha}(\mathbf{y}) \widehat{\mathcal{I}}_2^{(\alpha)}[\mathbf{y}] + c_{3,\alpha}(\mathbf{y}) \widehat{\mathcal{I}}_3^{(\alpha)}[\mathbf{y}] \Big) . \tag{12}$$

Here the summation is over distinct *sectors* (α) defined by the exponential factor $\exp(iq\cdot\hat{b} + i\ell_1\cdot\hat{a})$

$$(\alpha) \equiv (\hat{b}, \hat{a}) \equiv (b + c_1'(\sigma)\tilde{a}_1 + c_2'(\sigma)\tilde{a}_2 ,\, c_3'(\sigma)\tilde{a}_1 + c_4'(\sigma)\tilde{a}_2) , \tag{13}$$
$$(\mathbf{y}) = (y_1, \hat{y}_2, \hat{y}_3, \hat{y}_4, \sigma_1, \sigma_2, \sigma_3, \sigma_4),\ \ y_1 = v_1\cdot v_2 ,\, y_2 = (\hat{a}\cdot\hat{a})(-\hat{b}\cdot\hat{b}) ,\, y_3 = (\hat{b}\cdot\hat{b}) ,\, y_4 = (\hat{b}\cdot\hat{a})/(-\hat{b}\cdot\hat{b}) .$$

Finally, we solve the differential equations and evaluate the master integrals using

$$\widehat{\mathcal{I}}_1[\mathbf{y}] = -\frac{1}{16\pi^2\sqrt{-\hat{y}_3}\sqrt{y_1^2-1}} \frac{K(\hat{y}_2')}{\pi} , \qquad \widehat{\mathcal{I}}_2[\mathbf{y}] = \frac{\sqrt{-\hat{y}_3}}{16\pi^2\sqrt{y_1^2-1}} \frac{(\hat{y}_2'-1)K(\hat{y}_2')+E(\hat{y}_2')}{\pi} ,$$
$$\widehat{\mathcal{I}}_3[\mathbf{y}] = \frac{C\sqrt{-\hat{y}_3}}{\sqrt{y_1^2-1}} \left[\frac{\sqrt{\hat{y}_2'+\hat{y}_4^2-2\hat{y}_4}}{\pi\sqrt{\hat{y}_2'-2\hat{y}_4}} \Big(\frac{\pi-2K(\hat{y}_2')}{2} - \big(E(\hat{y}_2') - K(\hat{y}_2')\big) \int_0^{\hat{y}_4} \frac{dz}{\mathcal{Y}} - K(\hat{y}_2') \int_0^{\hat{y}_4} \frac{z dz}{\mathcal{Y}} \Big) + \frac{E(\hat{y}_2')+(1-\hat{y}_4)K(\hat{y}_2')}{\pi} \right] ,$$

where $\hat{y}_2' = \hat{y}_2 + 2\hat{y}_4$ and $\mathcal{Y} = \sqrt{(\hat{y}_2' - 2z)(z^2 - 2z + \hat{y}_2')}$ denotes the equation for the elliptic curve.

The advantage of the present analysis is that one can study the eikonal phase (10) and its spin dependence quantitatively by evaluating integrations at high precision. Programs for evaluating the spin-expanded integrand are available in the public repository `KerrEikonal2pm` [13] and ref. [14].

4. Conclusion

The results discussed in this talk contribute a new theoretical understanding and provide a practical framework for computing the gravitational dynamics of spinning black holes (in principle) at any multiplicity and spin order [9]. Using the introduced systematic procedure to simplify and evaluate integrals for integrands we successfully computed the scattering dynamics of a second-order post-Minkowskian binary Kerr system in [14]. It should be noted that the corresponding eikonal phase with spin was presented in a closed form, which enables an improved analytic or numerical analysis.

Advancing our understanding of classical spin effects is a fascinating challenge. A better understanding of the contact term is necessary to obtain results valid beyond the a^5 order without fitting with the Teutolsky solution [11]. Building on this, we have suggested in [8, 9] that scaling constraints could be sufficient to address it, for example, based on the specific cancellation of divergences or by reframing the problem in the context of geodesic motion (for instance in a self-force approach [15]). We leave these and many other exciting questions for future research.

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