PROCEEDINGS OF SCIENCE

Weak decays and finite-volume QED

Matteo Di Carlo,[a,b] **Maxwell T. Hansen,**[b] **Nils Hermansson-Truedsson**[b,c,*] **and Antonin Portelli**[b]

[a] *Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland*

[b] *Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3FD, United Kingdom*

[c] *Division of Particle and Nuclear Physics, Department of Physics, Lund University, Sölvegatan 14A, 223 62 Lund, Sweden*

E-mail: matteo.dicarlo@cern.ch, maxwell.hansen@ed.ac.uk, nils.hermansson-truedsson@ed.ac.uk, antonin.portelli@ed.ac.uk

The precision goal on the Cabibbo-Kobayashi-Maskawa matrix elements $|V_{us}|$ and $|V_{ud}|$ requires the inclusion of isospin-breaking effects. These effects, associated to QED and the light-quark mass difference, can be included in lattice QCD+QED simulations. The long-range nature of QED, however, poses a problem for finite-volume lattice simulations that necessitates the adoption of a prescription for the definition of QED in a spacetime of finite size. In this talk we discuss the novel prescription $\mathrm{QED_r}$, a specific choice within the class of infrared-improved prescriptions from Davoudi et al. 2019. We in particular emphasise how $\mathrm{QED_r}$ circumvents previous bottlenecks in analytical predictions of the finite-volume dependence in hadron masses and leptonic decays. These observables are in particular highly relevant for precision determinations of Cabibbo-Kobayashi-Maskawa elements and associated unitarity tests.

The 11th International Workshop on Chiral Dynamics (CD2024)
26-30 August 2024
Ruhr University Bochum, Germany

*Speaker

PoS(CD2024)031

1. Introduction

Precision tests of the Standard Model offer an indirect way to search for new physics particles. In the flavour physics sector, tests involving the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements can be performed [1]. For instance, the unitarity of the CKM matrix implies the relation

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1\,, \tag{1}$$

which can be probed by studying decays of hadrons. One example is the leptonic decay where a hadron P produces a lepton ℓ and neutrino ν_ℓ. Another is the semi-leptonic decay when there is an additional hadron in the final state. In the following we will consider leptonic decays of pions and kaons, where the ratio $|V_{us}|/|V_{ud}|$ can be determined through the ratio of decay rates,

$$\frac{\Gamma\left[K \to \mu\nu_\mu(\gamma)\right]}{\Gamma\left[\pi \to \mu\nu_\mu(\gamma)\right]} = \frac{|V_{us}|^2}{|V_{ud}|^2}\,\frac{m_\pi}{m_K}\,\frac{m_K^2 - m_\mu^2}{m_\pi^2 - m_\mu^2}\,\frac{f_K^2}{f_\pi^2}\left(1 + \delta R_K - \delta R_\pi\right). \tag{2}$$

On the right-hand side m_P and f_P are the isospin-symmetric mass and decay constant, respectively. The isospin-breaking effects from quantum electrodynamics (QED) and the mass difference between up and down quarks in quantum chromodynamics (QCD) are encoded in δR_P. These are essential to include for (sub-)percent-level precision. We stress that the definition of an isospin-symmetric point is separation-scheme dependent, and comparisons of high-precision determinations of quantities such as δR_P requires control of the scheme dependence [2].

From the above relation we see that $|V_{us}|^2/|V_{ud}|^2$ can be obtained by combining experimental information of masses and decay rates with theoretical predictions of the decay constants and their isospin-breaking corrections. The decay constant is inherently non-perturbative, and can be predicted in a systematically improvable fashion with lattice QCD [2]. In lattice QCD, the underlying correlation functions for the processes of interest are calculated non-perturbatively using numerical simulations in discretised, finite-volume Euclidean spacetimes. To make reliable physical predictions the simulation artifacts have to be controlled and removed. Isospin-breaking effects to δR_P can also be included in lattice QCD [3, 4], but complications arise due to the long-range nature of QED.

In Ref. [4] the final precision on the difference $\delta R_K - \delta R_\pi$ was severely limited due to systematic uncertainties from the finite-volume approximation,

$$\delta R_K - \delta R_\pi = -0.0086(13)_{\text{stat.+sys.}}(39)_{\text{vol.}}\,, \tag{3}$$

The second error corresponds to the uncertainty associated to finite-volume effects. In this talk we discuss the origin of this precision bottleneck and a possible solution. We recently presented the proposed method in Refs. [5, 6] based on previous work [7]. The solution relies on the freedom to modify the finite-volume photon action used in the lattice QCD+QED simulation while still retaining the correct physical, infinite-volume prediction.

2. QED in a finite volume

Periodic boundary conditions of a finite-volume spacetime imply, from Gauss' law, that the net charge inside the volume must be zero [8]. This problem is related to the absence of a mass gap in QED and zero-momentum modes of the physical photons. While problematic for simulations of finite-volume lattice QCD+QED, dedicated prescriptions for electromagnetism in the finite volume can be used to circumvent the issue. In $\mathrm{QED_M}$ [9, 10] a small photon mass is introduced to produce an artificial mass gap. In $\mathrm{QED_C}$ [11, 12] the zero-momentum modes of photons are forbidden from the use of charge-conjugated boundary conditions in the simulation. In $\mathrm{QED_L}$ [8] and associated infrared-improvement schemes [7] the zero-momentum modes are instead quenched by removing them from the underlying action. The systematic finite-volume effects in all of these prescriptions have to be well under control in order to make reliable and physical infinite-volume predictions. It should be noted that an alternative approach is to introduce and control a separation scale such that the effects of QED can be kept in infinite volume and combined with finite-volume QCD matrix elements, see e.g. Refs. [13, 14].

Here we consider the recently proposed improvement scheme $\mathrm{QED_r}$ [5, 6], corresponding to a specific choice of photon action in Ref. [7]. This prescription is designed to reduce finite-volume effects in hadron masses and leptonic decays, as needed to test the CKM unitarity relation in Eq. (1). Previously, finite-size corrections in $\mathrm{QED_L}$ were systematically studied using analytical effective field theory techniques for the hadron masses [7, 15–19] and leptonic decays [4, 17, 18, 20]. There is an inherent, systematic counting in terms of the finite-volume extent L. Ref. [3] numerically observed sizable volume effects for leptonic decays, later shown in Refs. [4, 18] to arise from enhanced higher-order corrections in the large-volume expansion. Such sub-leading terms in the expansion are naively expected to be numerically suppressed, but are difficult to determine in practice due to non-trivial dependence on the internal structure of hadrons. It was specifically the ignorance of higher-order corrections in leptonic decays and the hadron masses that led to the roughly 50% relative uncertainty in Eq. (3). We next introduce $\mathrm{QED_r}$ to remove the leading higher-order contributions in order to allow for future high-precision determinations of a range of hadronic observables in lattice QCD+QED.

2.1 The infrared-improved action $\mathrm{QED_r}$ and its relation to $\mathrm{QED_L}$

In the following we consider a Euclidean spacetime of geometry $\mathbb{R} \times L^3$, where the spatial extent in each dimension is L and the time extent infinite. In this case, the spatial photon momentum $\mathbf{k} = (k_x, k_y, k_z)$ is quantised in integer units of $2\pi/L$, i.e. $\mathbf{k} = 2\pi\,\mathbf{n}/L$ where $\mathbf{n} \in \mathbb{Z}^3$ is a vector of integers. The problematic zero-momentum mode $\mathbf{k} = \mathbf{0}$ is removed in $\mathrm{QED_L}$ and $\mathrm{QED_r}$ to circumvent the issue with Gauss' law. In $\mathrm{QED_r}$ and the larger class of prescriptions of Ref. [7], however, a finite set of momentum modes are modified in addition. The specific choice of $\mathrm{QED_r}$ is to modify modes $\mathbf{k}$ on spherical shells at distance R from the origin, i.e. with $|\mathbf{k}| = (2\pi/L)\,R$. The minimal choice employed here is to modify the innermost shell of radius $R = 1$, as depicted in Fig. 1. The associated kinetic photon action in Feynman gauge takes the form

$$S_{\mathrm{r}}[\hat{A}_\mu] = \frac{1}{2L^3} \sum_{\mathbf{k}\neq\mathbf{0}} \int \frac{dk_0}{2\pi}\, \hat{A}_\mu(k)^* \left[\frac{\delta^{\mu\nu} k^2}{1 + h(\mathbf{n})} \right] \hat{A}_\nu(k), \tag{4}$$

where $\hat{A}_\nu(k)$ is the momentum space photon field and $h(\mathbf{n}) = \delta_{\mathbf{n}^2,1}/6$ is the weight function acting on the six modes on the shell of radius $R = 1$. A crucial property of the weight functions is that it is normalised according to $\sum_{\mathbf{n}} h(\mathbf{n}) = 1$. The $\mathrm{QED_L}$ action corresponds to putting the weight function to zero.

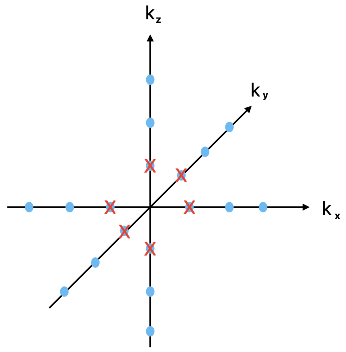

Figure 1: A pictorial representation of the discretised photon momentum modes $\mathbf{k} = (k_x, k_y, k_z) = 2\pi\,\mathbf{n}/L$ along the axes (blue circles). Highlighted are also the six modified modes in $\mathrm{QED_r}$ (red crosses).

The photon propagator can be obtained from the underlying action. In $\mathrm{QED_r}$ and $\mathrm{QED_L}$ they are respectively given by

$$D^{\mathrm{r}}_{\mu\nu}(x) = \frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}}\int\frac{dk_0}{2\pi}\,e^{ikx}\,\frac{\delta_{\mu\nu}}{k_0^2+\mathbf{k}^2}\left[1+h(\mathbf{n})\right]\,. \tag{5}$$

$$D^{\mathrm{L}}_{\mu\nu}(x) = \frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}}\int\frac{dk_0}{2\pi}\,e^{ikx}\,\frac{\delta_{\mu\nu}}{k_0^2+\mathbf{k}^2}\,. \tag{6}$$

As can be seen, the propagator in $\mathrm{QED_r}$ is the same as that in $\mathrm{QED_L}$ up to an additional term depending on the weight function over the six innermost modes.

2.2 Finite-volume effects

Finite-volume effects associated to the photon momentum $\mathbf{k}$ can be calculated as the difference between finite-volume sums and infinite-volume integrals. For a function $f(\mathbf{k})$ we have the finite-volume effects in $\mathrm{QED_r}$ and $\mathrm{QED_L}$ as

$$\Delta^{\mathrm{r}}_{\mathbf{k}}\,f(\mathbf{k}) = \left[\frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}} - \int\frac{d^3\mathbf{k}}{(2\pi)^3}\right]\big(1+h(\mathbf{n})\big)\,f(\mathbf{k})\,, \tag{7}$$

$$\Delta^{\mathrm{L}}_{\mathbf{k}}\,f(\mathbf{k}) = \left[\frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}} - \int\frac{d^3\mathbf{k}}{(2\pi)^3}\right]f(\mathbf{k})\,. \tag{8}$$

We here see that the only difference between $\mathrm{QED_r}$ and $\mathrm{QED_L}$ is in the additional term depending on the weight function.

In practical calculations of finite-volume effects the function $f(\mathbf{k})$ is written in terms of the dimensionless vector $\mathbf{n}$ as $f(2\pi\,\mathbf{n}/L)$. By systematically expanding for large volumes a polynomial series expansion in inverse powers of L is obtained, as done for a range of observables in Refs. [7, 15, 16, 18, 21]. For the hadron mass m_P^2 with X = L, r, the expansion takes the form

$$\frac{\Delta^{\mathrm{X}} m_P^2(L)}{e^2\,m_P^2} = \frac{\delta m_1^{\mathrm{X}}}{L} + \frac{\delta m_2^{\mathrm{X}}}{L^2} + \frac{\delta m_3^{\mathrm{X}}}{L^3} + O\left[\frac{1}{(m_P L)^4}, e^{-m_P L}\right]\,. \tag{9}$$

The δm_j^{X} above are prescription-dependent coefficients. For leptonic decays the relevant function is denoted $Y^{\mathrm{X}}(L)$, related to the virtually corrected decay rate $\Gamma_0^{\mathrm{X}}(L)$ at order e^2 through

$$\Gamma_0^{\mathrm{X}}(L) = \Gamma_P^{\mathrm{tree}} \left[1 + 2\,\frac{e^2}{16\pi^2}\, Y^{\mathrm{X}}(L) \right] . \tag{10}$$

Here Γ_P^{tree} is the tree-level decay rate. The large-volume expansion now takes the form

$$Y^{\mathrm{X}}(L) = \tilde{Y}(L) + Y_0^{\mathrm{X}} + \frac{Y_1^{\mathrm{X}}}{L} + \frac{Y_2^{\mathrm{X}}}{L^2} + \frac{Y_3^{\mathrm{X}}}{L^3} + O\left[\frac{1}{L^4}, e^{-m_\pi L} \right] , \tag{11}$$

where the prescription-independent $\tilde{Y}(L)$ is logarithmically divergent with the volume L and the Y_j^{X} are prescription-dependent coefficients.

The prescription-dependent coefficients appearing in the large-volume expansions depend on dimensionless sum-integral differences that must be known in order to make predictions. The so-called finite-volume coefficients relevant here are given by

$$\bar{c}_j = \Delta_{\mathbf{n}}^{\mathrm{r}} \frac{1}{|\mathbf{n}|^j} , \qquad c_j = \Delta_{\mathbf{n}}^{\mathrm{L}} \frac{1}{|\mathbf{n}|^j} . \tag{12}$$

Above we introduced the dimensionless sum-integral difference operators

$$\Delta_{\mathbf{n}}^{\mathrm{r}} = \left[\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n} \right] \big(1 + h(\mathbf{n})\big) , \qquad \Delta_{\mathbf{n}}^{\mathrm{L}} = \left[\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n} \right] . \tag{13}$$

It thus follows that the coefficients for a given j are related as

$$\bar{c}_j = c_j + 1 . \tag{14}$$

The coefficient $\bar{c}_0$ is of particular interest as it vanishes by construction, $\bar{c}_0 = 0$. As will be shown in the next section, it is precisely this property that allows to remove the unknown finite-volume effects leading to the large systematic uncertainty in Eq. (3).

For processes like leptonic decays where an external momentum is present, one also has to introduce coefficients depending on an external velocity vector $\mathbf{v}$,

$$\bar{c}_j(\mathbf{v}) = \Delta_{\mathbf{n}}^{\mathrm{r}} \frac{1}{|\mathbf{n}|^j} \frac{1}{(1 - \mathbf{v}\cdot\hat{\mathbf{n}})} , \qquad c_j(\mathbf{v}) = \Delta_{\mathbf{n}}^{\mathrm{L}} \frac{1}{|\mathbf{n}|^j} \frac{1}{(1 - \mathbf{v}\cdot\hat{\mathbf{n}})} . \tag{15}$$

It should be stressed that these depend on the particular orientation of the velocity. The relation between the two coefficients for a given j is

$$\bar{c}_j(\mathbf{v}) = c_j(\mathbf{v}) + \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|^j} \frac{h(\mathbf{n})}{(1 - \mathbf{v}\cdot\hat{\mathbf{n}})} . \tag{16}$$

In Table 1 we show values of finite-volume coefficients relevant for the hadron masses and leptonic decays. These were computed using the numerical algorithm presented in Refs. [7, 18]. The velocity was chosen to be $\mathbf{v} = |\mathbf{v}|(1,1,1)/\sqrt{3}$, where $|\mathbf{v}| = 0.912401$ corresponds to the leptonic decay of a kaon into a muon. As can be seen, the QED$_{\mathrm{r}}$ coefficients for $j = 0, 1, 2$ are in magnitude smaller than the corresponding ones in QED$_{\mathrm{L}}$.

j	$c_j(\mathbf{v})$	$\bar{c}_j(\mathbf{v})$	c_j	$\bar{c}_j$
3	4.9451	6.3292	3.8219	4.8219
2	-16.3454	-14.9613	-8.9136	-7.9136
1	-5.7302	-4.3461	-2.8373	-1.8373
0	-2.1237	-0.7396	-1	0

Table 1: Finite-volume coefficients $c_j(\mathbf{v})$ and $\bar{c}_j(\mathbf{v})$ as well as their velocity-independent counterparts in $\mathrm{QED_L}$ and $\mathrm{QED_r}$, respectively.

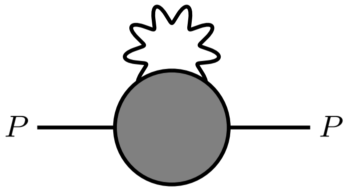

Figure 2: Diagrammatic representation of the self energy of a meson P at order e^2. The grey blob corresponds to the Compton scattering amplitude, and the wiggly line a photon.

3. $\mathrm{QED_r}$ for hadron masses and leptonic decays

The finite-volume effects of a specific observable in general depend on the properties of the particles involved in the process. These properties are charges, masses and internal structure in terms of form factors and their derivatives, appearing in coefficients in large-volume expansions such as Eqs. (9) and (11). The structure dependence can be highly non-trivial. In Ref. [18] it was shown that branch-cuts in underlying correlation functions (and hence form factors) must be known numerically in order to predict finite-volume effects. Below, we introduce $\mathrm{QED_r}$ to remove the leading contributions from these branch-cuts in the hadron masses and leptonic decays, and compare to $\mathrm{QED_L}$.

3.1 Hadron masses

We start by considering the mass of a hadron P at rest. The corresponding electromagnetic finite-volume effects in its mass m_P^2 are given by [18]

$$\Delta^{\mathrm{X}} m_P^2(L) = -\frac{e^2}{2} \lim_{p^2 \to -m_P^2} \Delta_{\mathbf{k}}^{\mathrm{X}} \int \frac{dk_0}{2\pi} \frac{C_{\mu\mu}(p,k,-k)}{k^2} \,. \tag{17}$$

Here the hadron momentum $p = (p_0, \mathbf{0}) = (im_P, \mathbf{0})$, the photon momentum $k = (k_0, \mathbf{k})$ and $C_{\mu\mu}(p,k,-k)$ is the Compton scattering amplitude in forward kinematics. A diagrammatic representation of the electromagnetic correction to the mass is shown in Fig. 2.

The first step to derive the finite-volume effects is to perform the k_0 integral, picking up all the analytical structure in the integrand within the integration contour. The analytical properties of the Compton amplitude can be determined by e.g. studying it in a spectral decomposition [5, 6, 18] or in an effective skeleton expansion of the underlying correlation function [7, 16–18]. For the integrand here there are three kinds of singularities, one pole corresponding to the propagating photon going on shell, another pole corresponding to an intermediately propagating hadron and also a branch-cut

PoS(CD2024)031

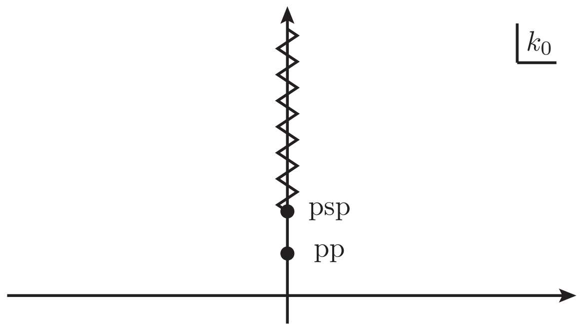

Figure 3: The upper half of the k_0-plane and the analytical structure of the integrand in Eq. (17). There are two poles corresponding to the photon (pp) and the pseudoscalar (psp) as well as a branch-cut.

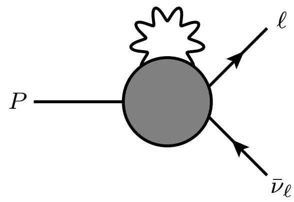

Figure 4: The leptonic decay $P^- \to \ell^- \bar{\nu}_\ell$. The grey blob contains the relevant CKM matrix element mediating the decay.

along the imaginary axis. This is shown in Fig. 3. After the k_0 integral, the resulting expression takes the form of Eq. (7) and the large-volume expansion can be performed. The result is

$$\Delta^{\mathrm{X}} m_P^2(L) = e^2 m_P^2 \left\{ \frac{c_2^{\mathrm{X}}}{4\pi^2\, m_P L} + \frac{c_1^{\mathrm{X}}}{2\pi\, (m_P L)^2} - \frac{c_0^{\mathrm{X}}}{(m_P L)^3} \left(\frac{\langle r_P^2 \rangle m_P^2}{3} + C \right) + O\left[\frac{1}{(m_P L)^4} \right] \right\} \tag{18}$$

Here we used the short-hand notation $c_j^{\mathrm{r}} = \bar{c}_j$ and $c_j^{\mathrm{L}} = c_j$, and $\langle r_P^2 \rangle$ is the electromagnetic charge radius that can be determined experimentally, with dispersion theory, effective field theory or lattice QCD [1]. The quantity C corresponds to the branch-cut in the forward Compton scattering amplitude integrated along the whole cut, which is highly non-trivial and unknown how to estimate. Its exact definition is given in Refs. [5, 6, 18]. Since $\bar{c}_0 = 0$ the whole contribution at order $\bar{c}_0/(m_P L)^3$ is identically removed in $\mathrm{QED_r}$ from the modification of the underlying action, thus allowing us to circumvent the need to determine non-trivial structure-dependent quantities as would be the case in $\mathrm{QED_L}$.

3.2 Leptonic decays

Next we consider the decay $P^- \to \ell^- \bar{\nu}_\ell$ in the rest frame of the decaying meson, depicted in Fig. 4. In this case there will be a non-zero spatial lepton momentum $\mathbf{p}_\ell$ entering the calculation, and thus finite-volume coefficients $\bar{c}_j(\mathbf{v}_\ell)$ and $c_j(\mathbf{v}_\ell)$ where $\mathbf{v}_\ell = \mathbf{p}_\ell / \sqrt{m_\ell^2 + \mathbf{p}_\ell^2}$. As already noted before, these coefficients depend on the orientation of the momentum. The function of interest is $Y^{\mathrm{X}}(L)$ appearing in the decay rate in Eq. (10). In the large-volume expansion $Y^{\mathrm{X}}(L)$ takes the form in Eq. (11). The coefficients in the expansion can be determined in the same fashion as for the mass, but the underlying correlation function and its associated analytical structure in k_0 are significantly more complicated. There are single and double poles corresponding to photons, pseudoscalars as well as the lepton, and two different branch-cuts. The interested reader is referred to Ref. [6].

The large-volume expansion coefficients in Eq. (11) are found to be [6]

$$Y_0^{\rm X} = \frac{c_3^{\rm X} - 2\,(c_3^{\rm X}(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi} + 2\,(1-\log 2)\,, \tag{19}$$

$$Y_1^{\rm X} = -\frac{(1+r_\ell^2)^2 c_2^{\rm X} - 4\,r_\ell^2 c_2^{\rm X}(\mathbf{v}_\ell)}{m_P(1-r_\ell^4)}\,, \tag{20}$$

$$Y_2^{\rm X} = -\frac{F_A^P}{f_P}\frac{4\pi\left[(1+r_\ell^2)^2 c_1^{\rm X} - 4\,r_\ell^2 c_1^{\rm X}(\mathbf{v}_\ell)\right]}{m_P(1-r_\ell^4)} + \frac{8\pi\left[(1+r_\ell^2)c_1^{\rm X} - 2\,c_1^{\rm X}(\mathbf{v}_\ell)\right]}{m_P^2(1-r_\ell^4)}\,, \tag{21}$$

$$Y_3^{\rm X} = \frac{c_0^{\rm X}}{m_P^3}\frac{32\pi^2\,(2+r_\ell^2)}{(1+r_\ell^2)^3} + \frac{32\pi^2\,c_0^{\rm X}(\mathbf{v}_\ell)}{f_P\,m_P^2\,(1-r_\ell^4)}\left[F_V^P - F_A^P + 2\,r_\ell^2\frac{\partial F_A^P}{\partial x_\gamma}\right] + c_0^{\rm X}\,C_\ell^{(2)}\,, \tag{22}$$

Here $r_\ell = m_\ell/m_P$ is the ratio of lepton to meson mass, $B_1(\mathbf{v}_\ell)$ is a known function from Ref. [18], and $F_{A,V}^P$ are structure-dependent form factors related to the real radiative leptonic decay $P \to \ell\bar{\nu}_\ell\gamma$ where $x_\gamma = 2p\cdot k/m_P^2$. These form factors are known from experiments, chiral perturbation theory as well as lattice QCD [1, 2]. The quantity $C_\ell^{(2)}$ contains highly non-trivial branch-cuts which are completely unknown. In QED$_{\rm r}$ this contribution is removed by construction due to $\bar{c}_0 = 0$, which means that the full $Y_3^{\rm r}$ can be predicted. At the time of Ref. [4] only the first term on the right-hand side of $Y_3^{\rm L}$ was known, and it turned out to be numerically large. This contribution, which also vanishes in QED$_{\rm r}$, was the origin of the large uncertainty in $\delta R_K - \delta R_\pi$ in Eq. (3). In other words, if the analysis were performed in the same way as in Ref. [4] the final uncertainty would be much reduced.

To conclude this section, we note that it is even possible to remove the full coefficient $Y_3^{\rm r}$. This follows from the freedom to choose the orientation of the velocity $\mathbf{v}_\ell$, since there always exist directions such that $\bar{c}_0(\mathbf{v}_\ell) = 0$. With such a lepton velocity $Y_3^{\rm r} = 0$, thus also removing the propagation of uncertainty from determinations of the form factors $F_{A,V}^P$.

4. Conclusions and outlook

In this talk we have presented the novel finite-volume prescription QED$_{\rm r}$ [6], which corresponds to a specific choice of infrared improvement introduced in Ref. [7] of the widely used QED$_{\rm L}$ [8]. The prescription in general removes velocity independent finite-volume effects at order $1/L^3$ proportional to $\bar{c}_0 = 0$, and is therefore useful for hadron masses, leptonic decays and also the muon magnetic moment [21, 22]. We expect it will be of use to a wide variety of different processes in lattice QCD+QED, needed for systematically improvable precision tests of the Standard Model.

Acknowledgements

M.T.H. and A.P. are supported in part by UK STFC grants ST/T000600/1 and ST/X000494/1, and M.D.C by UK STFC grant ST/P000630/1. M.T.H. is also supported by the UKRI Future Leader Fellowship MR/T019956/1. M.D.C. and A.P. received funding from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme under grant agreement No 757646, and A.P. additionally under grant agreement No 813942. M.D.C. has received funding from the European Unions Horizon Europe research and innovation programme, under the Marie Skłodowska-Curie grant agreement No 101108006. N. H.-T. was previously funded by the Swedish Research Council, grant number 2021-06638, and now by the UK Research and Innovation, Engineering and Physical Sciences Research Council, grant number EP/X021971/1.

References

[1] Particle Data Group collaboration, *Review of particle physics*, *Phys. Rev. D* **110** (2024) 030001.

[2] Flavour Lattice Averaging Group (FLAG) collaboration, *FLAG Review 2024*, 2411.04268.

[3] M. Di Carlo, D. Giusti, V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo et al., *Light-meson leptonic decay rates in lattice QCD+QED*, *Phys. Rev. D* **100** (2019) 034514 [1904.08731].

[4] P. Boyle et al., *Isospin-breaking corrections to light-meson leptonic decays from lattice simulations at physical quark masses*, *JHEP* **02** (2023) 242 [2211.12865].

[5] M. Di Carlo, *Isospin-breaking corrections to weak decays: the current status and a new infrared improvement*, *PoS* **LATTICE2023** (2024) 120 [2401.07666].

[6] M. Di Carlo, M.T. Hansen, N. Hermansson-Truedsson and A. Portelli, *QED_r: a finite-volume QED action with redistributed spatial zero-momentum modes*, 2501.07936.

[7] Z. Davoudi, J. Harrison, A. Jüttner, A. Portelli and M.J. Savage, *Theoretical aspects of quantum electrodynamics in a finite volume with periodic boundary conditions*, *Phys. Rev. D* **99** (2019) 034510 [1810.05923].

[8] M. Hayakawa and S. Uno, *QED in finite volume and finite size scaling effect on electromagnetic properties of hadrons*, *Prog. Theor. Phys.* **120** (2008) 413 [0804.2044].

[9] M.G. Endres, A. Shindler, B.C. Tiburzi and A. Walker-Loud, *Massive photons: an infrared regularization scheme for lattice QCD+QED*, *Phys. Rev. Lett.* **117** (2016) 072002 [1507.08916].

[10] A. Bussone, M. Della Morte and T. Janowski, *Electromagnetic corrections to the hadronic vacuum polarization of the photon within $\mathrm{QED_L}$ and $\mathrm{QED_M}$*, *EPJ Web Conf.* **175** (2018) 06005 [1710.06024].

[11] A.S. Kronfeld and U.J. Wiese, *SU(N) gauge theories with C periodic boundary conditions. 1. Topological structure*, *Nucl. Phys. B* **357** (1991) 521.

[12] B. Lucini, A. Patella, A. Ramos and N. Tantalo, *Charged hadrons in local finite-volume QED+QCD with C^* boundary conditions*, *JHEP* **02** (2016) 076 [1509.01636].

[13] N. Asmussen, J. Green, H.B. Meyer and A. Nyffeler, *Position-space approach to hadronic light-by-light scattering in the muon $g-2$ on the lattice*, *PoS* **LATTICE2016** (2016) 164 [1609.08454].

[14] N.H. Christ, X. Feng, L.-C. Jin, C.T. Sachrajda and T. Wang, *Radiative corrections to leptonic decays using infinite-volume reconstruction*, *Phys. Rev. D* **108** (2023) 014501 [2304.08026].

[15] Z. Davoudi and M.J. Savage, *Finite-Volume Electromagnetic Corrections to the Masses of Mesons, Baryons and Nuclei*, *Phys. Rev. D* **90** (2014) 054503 [1402.6741].

[16] BMW collaboration, *Ab initio calculation of the neutron-proton mass difference*, *Science* **347** (2015) 1452 [1406.4088].

[17] V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo, S. Simula and N. Tantalo, *Finite-Volume QED Corrections to Decay Amplitudes in Lattice QCD*, *Phys. Rev. D* **95** (2017) 034504 [1611.08497].

[18] M. Di Carlo, M.T. Hansen, A. Portelli and N. Hermansson-Truedsson, *Relativistic, model-independent determination of electromagnetic finite-size effects beyond the pointlike approximation*, *Phys. Rev. D* **105** (2022) 074509 [2109.05002].

[19] N. Hermansson-Truedsson, M. Di Carlo, M.T. Hansen and A. Portelli, *Structure-dependent electromagnetic finite-volume effects through order* $1/L^3$, *PoS* **LATTICE2023** (2024) 265 [2310.13358].

[20] N. Tantalo, V. Lubicz, G. Martinelli, C.T. Sachrajda, F. Sanfilippo and S. Simula, *Electromagnetic corrections to leptonic decay rates of charged pseudoscalar mesons: finite-volume effects*, 1612.00199.

[21] J. Bijnens, J. Harrison, N. Hermansson-Truedsson, T. Janowski, A. Jüttner and A. Portelli, *Electromagnetic finite-size effects to the hadronic vacuum polarization*, *Phys. Rev. D* **100** (2019) 014508 [https://arxiv.org/abs/1903.10591].

[22] N. Hermansson-Truedsson, *Structure-dependent electromagnetic finite-volume effects to the hadronic vacuum polarisation*, in *41st International Symposium on Lattice Field Theory*, 8, 2024 [2408.08042].