PROCEEDINGS OF SCIENCE

New analysis of subthreshold parameters of $\pi\pi$ scattering

Marián Kolesár[a,*] **and Jaroslav Říha**[a]

[a] *Institute of Particle Physics and Nuclear Physics, Charles University*
Prague, Czech Republic

E-mail: marian.kolesar@matfyz.cuni.cz, jaroslav.riha@matfyz.cuni.cz

Using recent experimental data and lattice calculations of scattering lengths of $\pi\pi$ scattering and employing dispersive representation of the amplitude based on Roy equations, we compute the subthreshold parameters of this process. We use Monte Carlo sampling to numerically model the probability distribution of the results based on all uncertainties in the inputs. In the second part of the analysis, employing Bayesian inference, we use the new results for the subthreshold parameters to obtain preliminary constraints on the leading order low energy constant B_0 in the context of three-flavour chiral perturbation theory.

The 11th International Workshop on Chiral Dynamics (CD2024)
26-30 August 2024
Ruhr University Bochum, Germany

*Speaker

https://pos.sissa.it/

PoS(CD2024)041

1. Introduction

Elastic scattering of $\pi\pi$ in the isospin limit, with the EM interactions neglected, can be described by an amplitude $A_{\pi\pi}(s,t,u)$ (for more detail see [1])

$$\begin{aligned}< \pi_4^d(p_4)\pi_3^c(p_3)|\pi_1^a(p_1)\pi_2^b(p_2) > = \ & \delta_{fi} + (2\pi)^4 i\delta^{(4)}(P_f - P_i)\cdot \\ & \cdot[\delta^{ab}\delta^{cd}A_{\pi\pi}(s,t,u) + \delta^{ac}\delta^{bd}A_{\pi\pi}(t,u,s) + \delta^{ad}\delta^{bc}A_{\pi\pi}(u,s,t)]. \end{aligned} \tag{1}$$

Based on the generic properties of analyticity, unitarity and crossing symmetry, the amplitude can be written in terms of subthreshold parameters $\alpha_{\pi\pi}$, $\beta_{\pi\pi}$ and λ_i [2, 3]:

$$\begin{aligned} A_{\pi\pi}(s,t,u) = \ & \frac{\alpha_{\pi\pi}}{3F_\pi^2}M_\pi^2 + \frac{\beta_{\pi\pi}}{3F_\pi^2}(3s - 4M_\pi^2) + \frac{\lambda_1}{F_\pi^4}\left(s - 2M_\pi^2\right)^2 + \frac{\lambda_2}{F_\pi^4}\left[(t-2M_\pi)^2 + (u-2M_\pi)^2\right] \\ & + \frac{\lambda_3}{F_\pi^6}\left(s - 2M_\pi^2\right)^2 + \frac{\lambda_4}{F_\pi^6}\left[(t-2M_\pi)^3 + (u-2M_\pi)^3\right] + \overline{K}(s,t,u) + O\left(p^8\right), \end{aligned} \tag{2}$$

where $\overline{K}(s,t,u)$ contains unitarity corrections.

Such a representation of the amplitude is useful because good convergence of $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ is expected within the framework of 3-flavour chiral perturbation theory (χPT) [4]. As shown in [5], the sum of next-to-next-leading order (NNLO) and all higher orders, written in a convenient form of the chiral expansion of these parameters, is proportional to

$$\delta_{\alpha_{\pi\pi}} \sim O(m_{ud}m_s), \quad \delta_{\beta_{\pi\pi}} \sim O(m_{ud}m_s), \tag{3}$$

instead of $O(m_s^2)$, as is generally expected in $SU(3)$ χPT. Here, m_{ud} denotes the average mass of the u- and d-quarks.

In this work, we take advantage of the experimental data collected by the NA48/2 collaboration [6], which extracted $\pi\pi$ scattering lengths from the K_{e4} decay and K$\to 3\pi$ cusp measurements. In addition, the scattering lengths have also been calculated by several lattice QCD groups, e.g. ETM [7, 8], RBC/UKQCD [9], Mai et al. [10], Fu, Wang [11] and others [12–14]. In the following, we will utilize the results by the ETM collaboration.

In the first part of the analysis we extract the coefficients of the $\pi\pi$ scattering amplitude using inputs for the scattering lengths from experimental data [6] or lattice QCD [7, 8]. The outline of the procedure is:

a_0^0 and a_0^2 from exp./lattice data + solutions to the Roy equations

$\downarrow$

phase shifts and imaginary parts of partial wave amplitudes

$\downarrow$

phenomenological representation

$\downarrow$

$\overline{b}_i$'s

$\downarrow$

$\alpha_{\pi\pi}$, $\beta_{\pi\pi}$ and λ_i

The procedure is implemented numerically, with all uncertainties modelled by using Monte Carlo sampling. Here, the phenomenological representation [1, 15] is based on the application of Roy equations [16] and contains two free parameters - the scattering lengths a_0^0 and a_0^2. As an intermediate results, we also obtain the coefficients $\bar{b}_i$, which were introduced in the context of two-loop calculation of $\pi\pi$ scattering within two-flavour χPT [17].

In the second part, employing Bayesian inference, we use the obtained results for $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ to extract constraints on the leading-order low-energy constant (LEC) B_0 in the context of three-flavour chiral perturbation theory. We implement the Bayesian statistical analysis numerically, using Monte Carlo sampling to obtain the probability distribution functions.

2. Representations of the $\pi\pi$ scattering amplitude

Derived by [16], the fixed-t dispersion relations for isospin amplitudes of $\pi\pi$ scattering can be expressed in terms of imaginary parts in the physical region of the s-channel. This representation for the amplitude contains two subtraction constants, which we may identify with scattering lengths a_0^0 and a_0^2, the coefficients of partial wave decomposition. The amplitude can then be written as:

$$A_{\pi\pi}(s,t,u) = 16\pi a_0^2 + \frac{4\pi s}{3M_\pi^2}\left(2a_0^0 - 5a_0^2\right) + P(s,t,u) + \overline{W}(s,t,u) + O(p^8), \tag{4}$$

where $P(s,t,u)$ is the polynomial part and $\overline{W}(s,t,u)$ contains the unitarity corrections (see ACGL [1] for details). Both of these can be expressed as integral functions of the partial wave amplitudes depending on phase shifts $\delta_l^I(s)$

$$t_l^I(s) = \frac{1}{\sigma(s)} e^{i\delta_l^I(s)} \sin(\delta_l^I(s)), \tag{5}$$

which can then be in-turn written in terms of the Schenk paramaterization [18]

$$\tan(\delta_l^I) = \sqrt{1 - \frac{4M_\pi^2}{s}}\, q^{2l}\left(A_l^I + B_l^I q^2 + C_l^I q^4 + D_l^I q^4\right)\frac{4M_\pi^2 - s_l^I}{s - s_l^I}, \tag{6}$$

where X_l^I are second order polynomials in a_0^0 and a_0^2, while $q^2 = \frac{1}{4}(s - 4M_\pi^2)$. The numerical solution for the Schenk parameters from Roy equations were calculated in the seminal paper [1], which was then extended by DFGS [15] to include the uncertainty of phase shifts at the matching point ($\sqrt{s_0}$=800MeV). Using such a solution allows us to compute phase shifts and subsequently the amplitude in the phenomenological representation (4) given inputs for the scattering lengths.

Introduced in the context of two-loop calculation in [17], another alternative representation can be formulated as follows:

$$A_{\pi\pi}(s,t,u) = \frac{M_\pi^2}{F_\pi^2}(s-1) + \frac{M_\pi^4}{F_\pi^4}\left(\bar{b}_1 + \bar{b}_2 s + \bar{b}_3 s^2 + \bar{b}_4 (t-u)^2\right) + \\ + \frac{M_\pi^6}{F_\pi^6}\left(\bar{b}_5 s^3 + \bar{b}_6 s (t-u)^2\right) + F(s,t,u) + O(p^8), \tag{7}$$

where $F(s,t,u)$ contains the unitarity corrections in this case. Both the representation expressed in terms of the subthreshold parameters (2) and the one containing coefficients $\overline{b}_i$ (7) can be matched to the phenomenological representation (4) (see CGL [19]) and thus the procedure outlined in the introduction can be undertaken in order to extract the values of these parameters from available data on the scattering lengths.

As noted in the Introduction, we use the experimental data published by the NA48/2 collaboration [6]. Their main result (model C) relies on a correlation between a_0^0 and a_0^2, which is based on the relation between the scalar pion radius and the scattering lengths [19]. Our input thus is

$$a_0^0 = 0.2196 \pm 0.0034 \tag{8}$$
$$a_0^2 = -0.0444 + 0.236(a_0^0 - 0.22) - 0.61(a_0^0 - 0.22)^2 - 9.9(a_0^0 - 0.22)^3 \pm 0.0008. \tag{9}$$

In addition, we take advantage of the lattice QCD results by the ETM collaboration [7, 8], expressed in the form:

$$a_0^0 = 0.198 \pm 0.011, \qquad a_0^2 = -0.0442 \pm 0.0005. \tag{10}$$

3. Chiral perturbation theory

$\pi\pi$ scattering can be calculated in the framework of three-flavour chiral perturbation theory [3, 5, 20]. The leading order effective Lagrangian has two couplings B_0 and F_0 [4]

$$\mathcal{L}^{(2)}_{eff} = \frac{F_0^2}{4}\mathrm{Tr}[D_\mu U D^\mu U^+ + 2B_0\mathcal{M}(U^+ + U)], \tag{11}$$

where the matrix field $U = U(\pi, K, \eta)$ contains the pseudoscalar meson fields, while $\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)$.

We utilize the so-called resummed approach to χPT [5], which can be summarized in the following way:

- standard χPT Lagrangian and power counting [4, 21]
- irregularities in the expansion allowed
- only expansions related linearly to Green functions of the QCD currents trusted (safe observables)
- explicitly to NLO, higher orders implicit in remainders
- remainders not neglected, treated as sources of error
- all manipulations in non-perturbative algebraic way

In this framework, the chiral expansion of the subthreshold parameters $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ takes the form (see [5] for explicit formulae):

$$\alpha_{\pi\pi}^{(\mathrm{th})}(F_0, B_0) = \alpha_{\pi\pi}^{(2)} + \alpha_{\pi\pi}^{(4)} + \delta\alpha_{\pi\pi}, \quad \beta_{\pi\pi}^{(\mathrm{th})}(F_0, B_0) = \beta_{\pi\pi}^{(2)} + \beta_{\pi\pi}^{(4)} + \delta\beta_{\pi\pi}, \tag{12}$$

where $\delta\alpha_{\pi\pi}$ and $\delta\beta_{\pi\pi}$ are the higher-order remainders. The NLO terms contain the χPT low-energy coupling constants L_i, which we algebraically reparametrize using chiral expansions of F_P and M_P, following [5]. Indirect remainders $\delta_{F_P}, \delta_{M_P}$ are generated. NNLO LEC's C_i are implicit in the higher order remainders δ_k. After the procedure, we have 10 free parameters:

$$m_{ud}, m_s, B_0, F_0, \delta\alpha_{\pi\pi}, \delta\beta_{\pi\pi}, \delta_{F_\pi}, \delta_{M_\pi}, \delta_{F_K}, \delta_{M_K} \tag{13}$$

4. Statistical analysis and assumptions

We employ Bayesian inference in order to extract constraints on the LO LEC B_0 from obtained results for the subthreshold parameters $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ in the first part of the analysis. For our case, the Bayes' theorem can be written as

$$P(B_0|\text{data}) = \frac{P(\text{data}|B_0)P(B_0)}{\int \mathrm{d}B_0\, P(\text{data}|B_0)P(B_0)}, \tag{14}$$

where $P(B_0|\text{data})$ is the probability density of B_0 being true given input data.

$P(\text{data}|B_0)$ is the known probability density of obtaining observed data for a particular value of B_0 being true. We have two correlated observables, $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$, an thus

$$P(\text{data}|B_0) = \frac{\sqrt{C}}{2\pi}\exp\left(-\frac{1}{2}V^T C V\right), \quad V = \begin{pmatrix} \alpha_{\pi\pi}^{(\text{th})} - \alpha_{\pi\pi}^{(\text{exp})} \\ \beta_{\pi\pi}^{(\text{th})} - \beta_{\pi\pi}^{(\text{exp})} \end{pmatrix} \tag{15}$$

Here C is the covariance matrix.

$P(B_0)$ is the prior probability distribution of B_0, which we use to implement available experimental and theoretical information. It should be also noted, that there is an implicit dependence of $\alpha_{\pi\pi}^{th}$ and $\beta_{\pi\pi}^{th}$ on F_0, the quark masses, the higher order remainders as well as uncertainties of all the inputs. The prior $P(B_0)$ can thus be understood to include all uncertainties and assumptions.

We treat the free parameters (13) in the following way:

m_{ud}, m_s: lattice QCD values from FLAG [22]

B_0, F_0: priors based on paramagnetic inequalities [23]

The assumptions for the remainders are [5, 23]:

$$\delta\alpha_{\pi\pi} = \delta\beta_{\pi\pi} = 0.0 \pm 0.03, \quad \delta_{F_\pi} = \delta_{M_\pi} = \delta_{M_K} = 0.0 \pm 0.1, \quad \delta_{F_K} = 0.1 \pm 0.07. \tag{16}$$

Here we take advantage of the expected good convergence in the case of $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$, as discussed in the Introduction. The assumptions (16) are implemented as normal distributions, not limited by any bound.

Finally, it is convenient to express B_0 in terms of the parameter Y:

$$Y = \frac{2m_{ud}B_0}{M_\pi^2} = \frac{\left(M_\pi^{LO}\right)^2}{M_\pi^2}, \tag{17}$$

the square of the ratio of the pion mass at leading order and the physical pion mass.

5. Results

Following the procedure outlined in the Introduction, we have prepared a numerical ensemble of $5 \cdot 10^4$ entries for all our inputs. First, for comparison purposes, we have tried to reproduce the results from CGL [19] and DFGS [15], using our methodology with their published inputs. In order to take the uncertainty in the phase shifts at the matching point s_0 into account, we used the extended solution by DFGS, as discussed in Section 2. While we were unable to get an exact match, our results were in general agreement with a few exceptions - we obtained somewhat larger error bars for $\bar{b}_1$ in the case of CGL and for $\bar{b}_4$ and λ_2 in the case of DFGS.

Our main results for the inputs (8-10) can be found in Table 1. As can be seen, the obtained values are compatible with the two-flavour χPT predictions of CGL and, in the case of NA48/2 and ETM, the uncertainties are competitive. The error bars are significantly reduced for some of the parameters compared to the DFGS extended fit. In particular, this is the case for $\bar{b}_1$ and $\alpha_{\pi\pi}$, for which we can also exclude the large mean values suggested by DFGS. As will be discussed later, this has significant consequences for the values of Y and B_0, extracted in the second part of the analysis.

	NA48/2 input	ETM input	**DFGS [15]**	**CGL [19]**
$\bar{b}_1$	-12.32±1.89	-11.17±0.93	-1.51±7.01	-12.4±1.6
$\bar{b}_2$	11.24±0.67	7.07±2.08	8.93±1.62	11.8±0.6
$\bar{b}_3$	-0.27±0.06	-0.35±0.07	-0.36±0.07	-0.33±0.07
$\bar{b}_4$	0.73±0.01	0.69±0.02	0.71±0.01	0.74±0.01
$\bar{b}_5$	3.65±0.41	2.89±0.49	3.21±0.44	3.58±0.37
$\bar{b}_6$	2.31±0.03	2.14±0.08	2.2±0.08	2.35±0.02
$\alpha_{\pi\pi}$	1.054±0.071	0.860±0.108	1.384±0.267	1.08±0.07
$\beta_{\pi\pi}$	1.115±0.008	1.051±0.032	1.077±0.025	1.12±0.01
$\rho_{\alpha\beta}$	-0.17	0.94	-0.23	
$10^{-3}\lambda_1$	-3.58±0.65	-4.19±0.66	-4.18±0.63	
$10^{-3}\lambda_2$	9.22±0.12	8.71±0.25	8.96±0.12	
$10^{-4}\lambda_3$	2.37±0.16	2.09±0.19	2.22±0.16	
$10^{-4}\lambda_4$	-1.44±0.02	-1.35±0.04	-1.38±0.04	

Table 1: Comparison of our results (red) with literature

For the second part, our goal has been the extraction of the leading order LEC B_0 from the values of the subthreshold parameters obtained in the previous section from experimental and lattice QCD inputs. Figure 1 illustrates the ensemble of theoretical predictions in comparison with some of the discussed values obtained from experimental data on the scattering lengths (2.5×10^7 samples). As can be seen, both parameters show a significant dependence on Y, although the theoretical uncertainties are quite substantial in the case of $\beta_{\pi\pi}$.

With regard to the dependence of $\alpha_{\pi\pi}$ on Y, we can observe a difference in compatibility with the NA48/2 and DFGS data. As discussed in Kolesár, Novotný [24], for DFGS the central

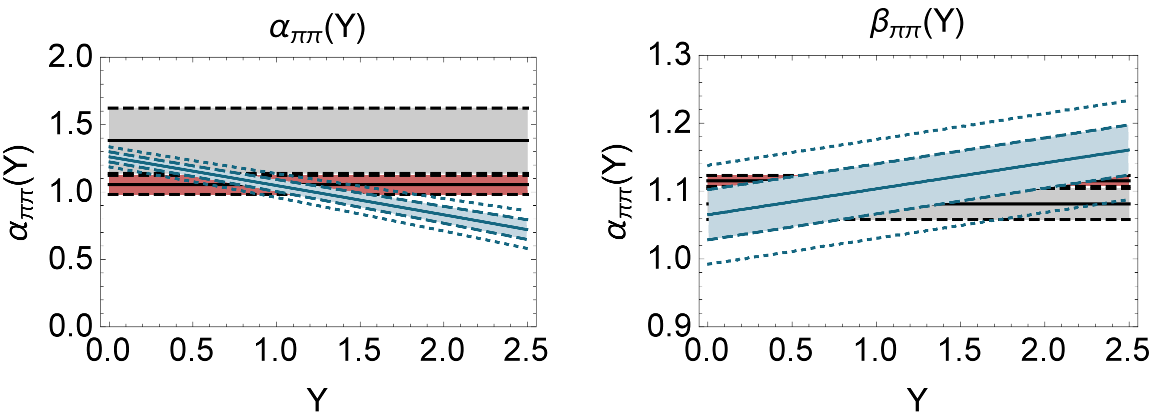
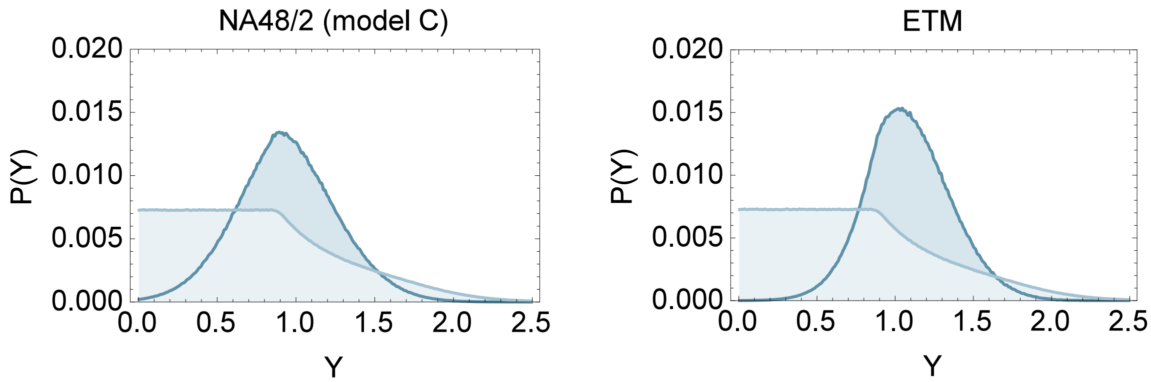


Figure 1: Comparison of theoretical predictions (blue) with DFGS [15] (global fit) (gray) and our result using NA48/2 (model C) data (red). 1σ CL (shaded+dashed) and 2σ CL (dotted) contours depicted.

experimental value is outside the 2σ CL band of the theoretical distribution in the whole range of the free parameters. Clearly, such a high value of $\alpha_{\pi\pi}$ would lead to a very small value of Y and thus a vanishing chiral condensate. However, this would be quite inconsistent with current expectations (see Table 2). In contrast, our results extracted from the NA48/2 data do not confirm such a scenario and a value $Y \sim 1$ can be expected.

Using the Bayesian procedure described in Section 4, we have extracted the probability density functions (PDFs) for Y. Figure 2 depicts PDFs for the $\alpha_{\pi\pi}$, $\beta_{\pi\pi}$ inputs obtained from the NA48/2 (model C) data [6] and lattice QCD calculations by the ETM collaboration [7, 8].



$Y = 0.94 \pm 0.32$ $\quad$ $Y = 1.08 \pm 0.27$

$B_0 = 2.67 \pm 0.91$ GeV $\quad$ $B_0 = 3.08 \pm 0.77$ GeV

Figure 2: PDFs obtained from input data by NA48/2 (model C) and ETM. Semitransparent - priors.

Table 2 compares our main results with values of Y derived directly from selected published sources. As can be seen, our main results are broadly compatible with the values derived from literature. We find that our $\pi\pi$ scattering analysis suggests $Y \sim 1$, which confirms the two-loop χPT fits [25]. A high value of Y, as suggested by $\eta \to 3\pi$ data [24], seems less probable.

Phenomenology	**Y**
Bijnens, Ecker (main fit) [25]	1.07
Bijnens, Ecker (free fit) [25]	0.94
Kolesár, Novotný ($\eta \to 3\pi$) [24]	1.44 ± 0.32
Lattice QCD	
MILC 09 [26]	$0.86\ \binom{+0.21}{-0.20}$
Bernard et al. [27]	0.71 ± 0.10
χQCD 24 [28]	1.32 ± 0.63
Our results	
NA48/2 input	0.94 ± 0.32
ETM input	1.08 ± 0.27

Table 2: Comparison of our results with literature

6. Summary

Let us summarize the presented work:

- Using recent experimental and lattice QCD data on $\pi\pi$ scattering lengths, we extracted the values of the coefficients $\bar{b}_i$ and the subthreshold parameters $\alpha_{\pi\pi}$, $\beta_{\pi\pi}$ and λ_i of the $\pi\pi$ scattering amplitude.
- We excluded large values of $\bar{b}_1$ and $\alpha_{\pi\pi}$ suggested by analysis based on BNL-E865 data [15].
- Using the obtained values for $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$, we applied the Bayesian statistical framework to extract the value of the leading order 3-flavour χPT coupling constant B_0.
- The results confirm that the dominant contribution to the pion mass is probably at the leading order.

Acknowledgments: The work presented in this paper was supported by the Czech Science Foundation grant 23-06770S and by the Charles University grant PRIMUS 23/SCI/025.

References

[1] B. Ananthanarayan, G. Colangelo, J. Gasser, H. Leutwyler, Phys. Rept. 353 (2001) 207–279. arXiv:hep-ph/0005297.

[2] J. Stern, H. Sazdjian, N. H. Fuchs, Physical Review D 47 (9) (1993) 3814–3838. arXiv:hep-ph/9301244.

[3] M. Knecht, B. Moussallam, J. Stern, N. H. Fuchs, Nucl. Phys. B 457 (1995) 513–576. arXiv:hep-ph/9507319.

[4] J. Gasser, H. Leutwyler, Nucl. Phys. B250 (1985) 465–516.

[5] S. Descotes-Genon, N. H. Fuchs, L. Girlanda, J. Stern, Eur. Phys. J. C 34 (2004) 201–227. arXiv:hep-ph/0311120.

[6] J. R. Batley, et al. (NA48/2), Eur. Phys. J. C 70 (2010) 635–657.

[7] L. Liu, et al. (ETM), Phys. Rev. D 96 (5) (2017) 054516. arXiv:1612.02061.

[8] C. Helmes, C. Jost, B. Knippschild, C. Liu, J. Liu, L. Liu, C. Urbach, M. Ueding, Z. Wang, M. Werner (ETM), JHEP 09 (2015) 109. arXiv:1506.00408.

[9] T. Blum, et al. (RBC/UKQCD), Phys. Rev. D 107 (9) (2023) 094512, [Erratum: Phys.Rev.D 108, 039902 (2023)]. arXiv:2301.09286.

[10] M. Mai, C. Culver, A. Alexandru, M. Döring, F. X. Lee, Phys. Rev. D 100 (11) (2019) 114514. arXiv:1908.01847.

[11] Z. Fu, J. Wang, Phys. Rev. D 110 (7) (2024) 074513. arXiv:2409.13197.

[12] S. R. Beane, E. Chang, W. Detmold, H. W. Lin, T. C. Luu, K. Orginos, A. Parreno, M. J. Savage, A. Torok, A. Walker-Loud (NPLQCD), Phys. Rev. D 85 (2012) 034505. arXiv:1107.5023.

[13] K. Sasaki, N. Ishizuka, M. Oka, T. Yamazaki (PACS-CS), Phys. Rev. D 89 (5) (2014) 054502, [Erratum: Phys.Rev.D 105, 019901 (2022)]. arXiv:1311.7226.

[14] S. M. Dawid, Z. T. Draper, A. D. Hanlon, B. Hörz, C. Morningstar, F. Romero-López, S. R. Sharpe, S. Skinner (2025). arXiv:2502.17976.

[15] S. Descotes-Genon, N. H. Fuchs, L. Girlanda, J. Stern, Eur. Phys. J. C 24 (2002) 469–483. arXiv:hep-ph/0112088.

[16] S. M. Roy, Phys. Lett. B 36 (1971) 353–356.

[17] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, M. E. Sainio, Nucl. Phys. B 508 (1997) 263–310, [Erratum: Nucl.Phys.B 517, 639–639 (1998)]. arXiv:hep-ph/9707291.

[18] A. Schenk, Nuclear Physics B 363 (1) (1991) 97–113.

[19] G. Colangelo, J. Gasser, H. Leutwyler, Nucl. Phys. B 603 (2001) 125–179. arXiv:hep-ph/0103088.

[20] J. Bijnens, P. Dhonte, P. Talavera, JHEP 01 (2004) 050. arXiv:hep-ph/0401039.

[21] S. Weinberg, Physica A96 (1-2) (1979) 327–340.

[22] Y. Aoki, et al. (Flavour Lattice Averaging Group (FLAG))arXiv:2411.04268.

[23] M. Kolesár, J. Říha, Eur. Phys. J. Plus 139 (1) (2024) 31. arXiv:2301.08027.

[24] M. Kolesar, J. Novotny, Eur. Phys. J. C 78 (3) (2018) 264. arXiv:1709.08543.

[25] J. Bijnens, G. Ecker, Ann. Rev. Nucl. Part. Sci. 64 (2014) 149–174. arXiv:1405.6488.

[26] A. Bazavov, et al. (MILC), Rev. Mod. Phys. 82 (2010) 1349–1417. arXiv:0903.3598.

[27] V. Bernard, S. Descotes-Genon, G. Toucas, JHEP 12 (2012) 080. arXiv:1209.4367.

[28] J. Liang, A. Alexandru, Y.-J. Bi, T. Draper, K.-F. Liu, Y.-B. Yang (χQCD), Phys. Rev. D 110 (9) (2024) 094513. arXiv:2102.05380.