PROCEEDINGS OF SCIENCE

Maximally forward-divergent diagrams in $\lambda\phi^4$ thermal theory

Pietro Baratella
Jožef Stefan Institute,
Jamova 39, 1000 Ljubljana, Slovenia
E-mail: pietro.baratella@ijs.si

Working with the S-matrix formulation of thermal physics, I resum a series of diagrams that contribute to the free energy of $\lambda\phi^4$ theory in 3+1 dimensions. The diagrams correspond to tree level amplitudes that have, at fixed order in λ, the maximum number of singular propagators in the forward limit. It was recently argued that this set of diagrams saturates the free energy of a certain integrable 1+1 dimensional theory, and it might play a special role in thermal physics in general.

9th Symposium on Prospects in the Physics of Discrete Symmetries (DISCRETE2024)
2–6 Dec 2024
Ljubljana, Slovenia

https://pos.sissa.it/

PoS(DISCRETE2024)090

1. Introduction

Dashen, Ma and Bernstein (DMB) instructed us on how to compute thermal physics quantities — I will focus here on the free energy — with the sole knowledge of the scattering amplitudes among the microscopic constituents of the system [1]. In particular, the DMB formula for the free energy F requires, as an input, S-matrix elements evaluated in the forward limit, that is of the form $S_{\alpha\alpha} \equiv \langle\alpha|S|\alpha\rangle$, where α is an asymptotic state and S is the scattering operator.

Taking the forward limit generically means, with enough particles partaking the scattering, encountering divergences of the form $\lim_{p^2\to 0}(p^2+i\epsilon)^{-1}$, coming from propagators going on shell. For definiteness, let us consider a renormalisable theory of scalars interacting through a quartic coupling. For a $n \to n$ amplitude, at tree level, there is a *maximum* of $n-2$ propagators that can go on shell in the forward limit. In this Proceeding, I will show how the contribution to the free energy of these 'maximally forward-divergent' diagrams can be resummed into a compact formula. There are at least three reasons why it makes sense to sum together these contributions.

1. First of all, a technical one. Once it is understood how to treat the dangerous propagators, these singular contributions are probably the simplest to compute, as we will see.

2. Second, Ref. [2] strongly hinted at the fact that, for the integrable version of the long string effective theory in $1+1$ dimensions, an analogous set of diagrams gives the whole free energy of the system (the free energy had previously been computed non-perturbatively with the Thermodynamic Bethe Ansatz (TBA) [3]). From this observation, it could be argued that these contributions are somehow special to thermal physics in general.

3. Finally, for $n > 3$, after the singular propagators are cured, the contribution to the free energy of the individual diagrams is divergent in the infrared (IR). Summing together all of them, or a properly chosen subset thereof, gives a IR finite object.

The aim of these notes is to elaborate on the first and third point. The second point should be taken as a deeper motivation to study this problem.

2. Evaluation of diagrams

The amplitudes we are going to consider have a rather simple interpretation: they correspond to histories where a bunch of particles just proceeds straight; when two particles meet, they pay a price quantified by λ, but exchange no momentum. Throughout their history, particles are labelled by the initial momenta $\vec{k}_1, \ldots, \vec{k}_n$. At tree level, to make a connected amplitude, $n-1$ interactions are required, and no more, so $T_{n\to n} \propto \lambda^{n-1}$.

A mathematical fact that we are going to use later on is that there is a one-to-one correspondence between these histories and Cayley trees with n vertices and $n-1$ ordered edges (a Cayley tree is just a connected tree with labeled vertices). The graph is constructed as follows: (i) to each particle there corresponds a vertex; (ii) to each interaction, say between i and j, there is an edge connecting vertex i with j; (iii) edges are ordered according to the the time ordering of the events 'i meets j'. For each Cayley graph there are $(n-1)!$ distinct orderings of edges. See Fig. 1.

PoS(DISCRETE2024)090

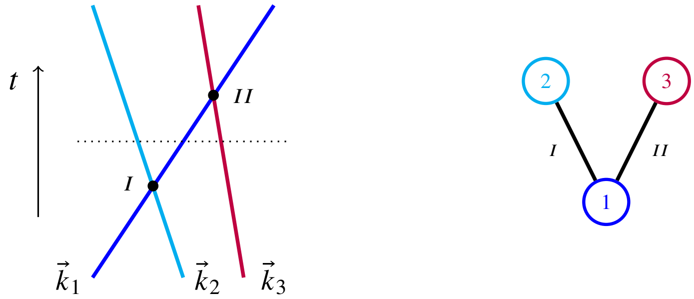

Figure 1: *Left*: example of maximally forward-divergent history with $n = 3$. It is entirely specified in terms of the time ordering of the interactions among the particles. *Right*: Cayley tree with ordered edges associated to the history on the left. Nodes stand for particles, while edges represent their interaction.

Let us now move to the evaluation of amplitudes. A crucial point is to compute *not* the usual matrix elements $S_{\beta\alpha}$, but instead $S_{\beta\alpha}(E) \equiv \delta_{\alpha\beta} - 2\pi i\delta(E - E_\beta)T_{\beta\alpha}(E)$, where the E-dependent T-matrix is defined via

$$T_{\beta\alpha}(E) \equiv V_{\beta\alpha} + \int \mathrm{d}\gamma \frac{V_{\beta\gamma}V_{\gamma\alpha}}{E - E_\gamma + i\epsilon} + \int \mathrm{d}\gamma \mathrm{d}\gamma' \frac{V_{\beta\gamma'}V_{\gamma'\gamma}V_{\gamma\alpha}}{(E - E_\gamma + i\epsilon)(E - E'_\gamma + i\epsilon)} + \ldots \tag{1}$$

When $E = E_\alpha$, Eq. (1) reduces to the Lippmann-Schwinger equation for T, with $V = H - H_0$ being the interaction Hamiltonian of the theory. For an operator O, we are using the notation $O_{\beta\alpha} \equiv \langle\beta|O|\alpha\rangle$, while $\int \mathrm{d}\gamma$ stands for a phase space integration over state γ.

A key property of the special histories we consider is that they are completely captured by contributions to Eq. (1) where the intermediate states γ are identical to α. To be more precise, this condition *defines* what we are going to sum over. Since the amplitude is forward, $\beta \equiv \alpha$ too. We can anticipate that T-matrix elements will take the form $T_{\alpha\alpha}(E) \sim \lambda^{n-1}(E - E_\alpha + i\epsilon)^{2-n}$, where λ characterises the $2 \to 2$ forward scattering, and n is the number of particles of state α. Notice how the limit $E \to E_\alpha$ is singular for $n > 2$, due precisely to the fact that $\gamma \equiv \alpha$ for all intermediate states.

The first amplitude of the series is $2 \to 2$ and acts as a building block for all amplitudes with $n > 2$. It comes from the first term of the expansion in (1), so it is independent of E and reads

$$T_{2\to2} = \langle 1, 2|V|1, 2\rangle = (2\pi)^3\delta^{(3)}(0)\,\lambda = L^3\,\lambda\,, \tag{2}$$

where the appearance of a volume factor L^3 is due to the forward limit. When computing the free energy with the DMB formula, this factor will take care of the extensivity of the thermodynamic function. There is only one Cayley tree with ordered edges, and therefore only one history.

Moving to $n = 3$, we need to consider the second term in the Lippmann-Schwinger expansion, and compute

$$T^G_{3\to3}(E) = \frac{1}{3!}\prod_{i=1}^{3}\int \frac{\mathrm{d}^3k'_i}{(2\pi)^3\,2E'_i}\frac{\langle 1, 2, 3|V|1', 2', 3'\rangle\langle 1', 2', 3'|V|1, 2, 3\rangle}{E - (E'_1 + E'_2 + E'_3) + i\epsilon}\,. \tag{3}$$

where the index G (graph) means that we concentrate on only one Cayley tree with ordered edges, and later on will count them. In particular, the history we compute is the one where $\vec{k}_1$ meets $\vec{k}_2$

first and then $\vec{k}_3$, as depicted in Fig. 1. We get

$$
\begin{aligned}
T^G_{3\to 3}(E) &= \prod_{i=1}^{3}\int \frac{\mathrm{d}^3k_i'}{(2\pi)^3 2E_i'}\langle 3|3'\rangle\langle 2'|2\rangle \frac{\langle 1,2|V|1',2'\rangle\langle 1',3'|V|1,3\rangle}{E-(E_1'+E_2'+E_3')+i\epsilon} \\
&= L^3 \int \frac{\mathrm{d}^3k_1'}{(2\pi)^3 2E_1'} \frac{\lambda^2(2\pi)^3\delta^{(3)}(\vec{k}_1'-\vec{k}_1)}{E-(E_1+E_2+E_3)+i\epsilon} \\
&= L^3 \frac{\lambda^2}{2E_1}\frac{1}{E-(E_1+E_2+E_3)+i\epsilon}\,.
\end{aligned} \tag{4}
$$

The first equality reflects Fig. 1, with particles 3 and 2 travelling freely in the first and second half of their history, respectively. The $1/3!$ that comes with the measure is completely reabsorbed by the permutations among the internal particles $1', 2', 3'$, which give an identical yield. The $2 \to 2$ amplitudes give two factors of $(2\pi)^3\delta^{(3)}(\vec{k}_1 - \vec{k}_1')$ (times λ), and one them is interpreted as a volume factor. The end result is pretty simple. Apart from expected factors, there is a $(2E_1)^{-1}$ that singles out particle 1, which is the one that would have a singular propagator.

Generalising to an arbitrary history is straightforward. Given the associated Cayley tree, we can associate to it the list of 'valences' $\{d_1, \ldots, d_n\}$, *i.e.* the number of edges emanating from each vertex. The amplitude associated to that history is

$$
T^{\{d_1,\ldots,d_n\}}_{\alpha\alpha}(E) = L^3\lambda^{n-1}\left(\frac{1}{E-E_\alpha+i\epsilon}\right)^{n-2}\prod_{i=1}^{n}\left(\frac{1}{2E_i}\right)^{d_i-1}. \tag{5}
$$

3. Contribution to the free energy

The DMB formula expresses the free energy of a system as

$$
F = F_0 - \frac{1}{2\pi i}\int \mathrm{d}E\, e^{-\beta E}\,\mathrm{Tr}_c \ln S(E)\,, \tag{6}
$$

where F_0 is the free theory contribution, and $\ln S(E) = -\sum_{k=1}^{\infty}\left(2\pi i\delta(E-H_0)T(E)\right)^k/k$ is defined via its Taylor expansion about the identity. One needs to take the trace of the operator $\ln S(E)$ in the Hilbert space of the theory, keeping only those contributions that are connected after taking the trace, *i.e.* when the initial and final states are identified. This condition allows for histories that are not connected in the usual sense of amplitudes [2].

Using Eq. (6), the goal is now to evaluate the contribution of a history with given $\{d_1, \ldots, d_n\}$. Expanding (6), we get

$$
F^{\{d_1,\ldots,d_n\}} = \int \mathrm{d}\alpha \int \mathrm{d}E\, e^{-\beta E}\delta(E-E_\alpha)T^{\{d_1,\ldots,d_n\}}_{\alpha\alpha}(E) + \ldots\,, \tag{7}
$$

with $\int \mathrm{d}\alpha$ the phase space integral over the n-particle initial (and final) state. The ellipses in (7) stand for a series of contributions that we always consider together with $T^{\{d_1,\ldots,d_n\}}_{\alpha\alpha}$, when evaluating the free energy. They are of two kinds.

1. On one side we have contributions that have the same exact form as $T_{\alpha\alpha}^{\{d_1,\dots,d_n\}}$, except for having factors of $2\pi i\delta(E-E_\alpha)$ instead of $(E-E_\alpha+i\epsilon)^{-1}$. These come from the expansion of the logarithm in (6). For example, to the process in Fig. 1 is associated another contribution

$$\frac{1}{2}\times L^3\frac{\lambda^2}{2E_1}\,2\pi i\,\delta\big(E-(E_1+E_2+E_3)\big)\,, \tag{8}$$

   where the $1/2$ in front comes from the expansion of $\ln S(E)$. For $n>3$ there are also mixed terms, with a factor of $\big(2\pi i\delta(E-E_\alpha)\big)^{k-1}(E-E_\alpha+i\epsilon)^{n-k-1}$ and coefficients dictated by the logarithm expansion, times λ^{n-1} and a product of $(2E_i)^{1-d_i}$.

2. As mentioned before, there are disconnected histories that become connected once initial and final states are identified. Together with $T_{\alpha\alpha}^{\{d_1,\dots,d_n\}}$ we include also all histories with an arbitrary number of freely propagating particles, provided their lines become all topologically connected — together and with the interacting cluster —, once we do the trace identification. This can be seen in many ways: either as the addition of windings or as the inclusion of exchange effects of identical particles.

Once we put everything together (see [2] for all the algebraic details that, for reasons of space, cannot be discussed here), we find

$$F^{\{d_1,\dots,d_n\}} = L^3\frac{\lambda^{n-1}}{(n-1)!n!}\prod_{i=1}^{n}\int\frac{\mathrm{d}^3k_i}{(2\pi)^3}n_B^{(d_i-1)}(E_i)\left(\frac{1}{2E_i}\right)^{d_i}. \tag{9}$$

where $n_B(E)=(e^{\beta E}-1)^{-1}$ is the Bose-Einstein density, and $n_B^{(m)}$ is its mth derivative with respect to energy. Eq. (9) gives the contribution to the free energy of *one given history*, included the associated higher orders of $\ln S(E)$ and windings [2]. It is important to notice that, to a given $\{d_1,\dots,d_n\}$, there correspond many histories (however the yield to the free energy only depends on the list of valences of the Cayley tree associated to that history). In particular, it is immediate to see that the time ordering of the interaction events is immaterial for (9), so the $(n-1)!$ always goes away.

We can express the contribution to the free energy of the whole set of maximally forward-divergent diagrams as

$$F = L^3\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{n!}\sum_{\{d_1,\dots,d_n\}}N_{\{d_1,\dots,d_n\}}\prod_{i=1}^{n}\int\frac{\mathrm{d}^3k_i}{(2\pi)^3}n_B^{(d_i-1)}(E_i)\left(\frac{1}{2E_i}\right)^{d_i}, \tag{10}$$

where $N_{\{d_1,\dots,d_n\}}$ is the number of distinct Cayley trees that have the same list of valences. To give a few examples, for $n=2$ there is only one tree, with list $\{1,1\}$; for $n=3$ there are 3 trees with valences $\{2,1,1\}$; at $n=4$ there are 4 graphs with $\{3,1,1,1\}$ valences and 12 with $\{2,2,1,1\}$. Starting from $n=6$, the list of valences does not uniquely determine the topology of the tree; however, for the sake of (10), the different topologies have to be counted together. Notice that (10) includes also the free theory contribution, corresponding to $n=1$, a single particle with valence list $\{0\}$, and taking $n_B^{(-1)}(E)=\beta^{-1}\ln(1-e^{-\beta E})$.

4. Resummation with recursive expression

The structure of (10) hints at a nested structure similar to the TBA equations [2, 3]. The fact that histories are counted by Cayley trees also points to something like that. After inspection of the equation, inspired by the TBA equations, we consider

$$f = L^3\beta^{-1}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\ln\left(1-e^{-\beta\varepsilon_{\vec{k}}}\right), \tag{11}$$

$$\varepsilon_{\vec{k}} = E_{\vec{k}} + \frac{\lambda}{2E_{\vec{k}}}\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}}\,n_B(\varepsilon_{\vec{p}})\,. \tag{12}$$

The second equation is recursive, and can be expanded order by order in λ. At every step, one can go to higher orders in the Taylor expansion, or deeper in the recursion. Once the expansion of $\varepsilon_{\vec{k}}$ is put into the equation for f, we get a series of terms with precisely the same form as in Eq. (10). However, if F, as computed before and given by (10), is expressed as $F = \sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{n!}F_{n-1}$ (the index F_{n-1} is chosen in a way that F_0 gives the free energy of the free theory), we find that $f = \sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}F_{n-1}$. This means that f does not give the free energy, but it is close enough so we can reconstruct it. In fact we have $f = \frac{\partial}{\partial\lambda}(\lambda F)$, or

$$F = \frac{1}{\lambda}\int_0^{\lambda}\mathrm{d}\lambda' f(\lambda')\,. \tag{13}$$

Remarkably, Eq. (12) can be solved analytically. The reason is that, for a given λ, the d^3p integral is just a number $c(\lambda)$, times β^{-2} to set dimensions.[1] Therefore it must be $\varepsilon_{\vec{k}} = E_{\vec{k}} + \lambda c(\lambda)/(2\beta^2 E_{\vec{k}})$, with $c(\lambda)$ fixed by the consistency condition

$$c(\lambda) = \frac{1}{4\pi^2}\int_0^{\infty}\mathrm{d}x\,\frac{x}{e^{x+\frac{\lambda c(\lambda)}{2x}}-1}\,. \tag{14}$$

This equation can be solved numerically to arbitrary precision. For consistency with the free theory limit, it must satisfy $\lim_{\lambda\to 0}c(\lambda) = 1/24$. After solving for $c(\lambda)$ in an interval $[0,\lambda_*]$, the free energy can then be obtained by integrating

$$F(\lambda_*) = \frac{1}{2\pi^2}L^3\beta^{-4}\frac{1}{\lambda_*}\int_0^{\lambda_*}\mathrm{d}\lambda\int_0^{\infty}\mathrm{d}x\,x^2\ln\left(1-e^{-\left(x+\frac{\lambda c(\lambda)}{2x}\right)}\right). \tag{15}$$

Eq. (15), together with (14), provides an explicit integral for the series in Eq. (10), which was derived diagrammatically. Remarkably, it is IR finite. At small coupling, for momenta of order $k\sim\beta^{-1}\sqrt{\lambda/48}$, the two terms at the exponent become comparable. This is the regime that is usually characterised by the Debye mass.

[1]More in general, it can be argued that (12) is a special case of

$$\varepsilon_{\vec{k}} = E_{\vec{k}} - \frac{1}{2E_{\vec{k}}}\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\vec{p}}}\,n_B(\varepsilon_{\vec{p}})M(\vec{k},\vec{p})\,,$$

with $M(\vec{k},\vec{p})$ the relevant $2\to 2$ scattering amplitude in the forward limit (cf. with [4]). If M depends on the kinematics, the d^3p integral is a function $c_{\vec{k}}(\lambda)$ and the solution of the recursive equation becomes more challenging.

5. Conclusions

I have studied, within the DMB formalism, an infinite set of thermal diagrams in $\lambda\phi^4$ theory. They are characterised, at given order in λ, by having the maximal number of singular propagators in the forward limit, and their evaluation only requires knowledge of the $2 \to 2$ forward amplitude. Individually, for $n > 2$, they are IR divergent.

It is possible to give a closed expression for the sum of these diagrams, Eq. (15), which is the main result presented here.

The following questions are left open for future work. First of all, the mathematical structure of (11) and (12) definitely requires further scrutiny. Second, the criterion followed here for resumming thermal diagrams is not obviously equivalent to other, previously considered IR safe sums. Therefore, a comparison with previous work on the topic is necessary. Finally, and related to this, an obvious question is how to generalise these results to other theories, and specifically to QCD. It is known that the perturbative series of thermal QCD is not well behaved. The scheme presented here might provide a fresh look into an interesting problem like the stabilisation of the perturbative series for, say, the pressure $p(T)$ of the quark-gluon plasma.

Acknowledgements

I am grateful to Alex Pomarol, Davide Facoetti and Giovanni Villadoro for useful discussions, and to Joan Elias Miró for many useful discussions and the suggestion to work on the topic. This work is supported by the Slovenian Research Agency under the research core funding No. P1-0035, and in part by the research grants N1-0253 and J1-4389.

References

[1] R. Dashen, S. K. Ma and H. J. Bernstein, "S Matrix formulation of statistical mechanics," Phys. Rev. **187** (1969), 345-370

[2] P. Baratella, J. Elias Miro and E. Gendy, "Thermodynamics from the S-matrix reloaded, with applications to QCD and the confining Flux Tube," JHEP **12** (2024), 001

[3] S. Dubovsky, R. Flauger and V. Gorbenko, "Solving the Simplest Theory of Quantum Gravity," JHEP **09** (2012), 133

[4] A. Schenk, "Absorption and dispersion of pions at finite temperature," Nucl. Phys. B **363** (1991), 97-113