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An Almost Complete Yang-Mills Calculation

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I present an analytic framework for effective $SU(N)$ Yang-Mills theories in the four-dimensional continuum. I use Background and effective field theory techniques to include non-perturbative contributions from diagonal elements of the quartic interaction terms. This approach is inspired by Savvidy, who claims select first-order contributions from quartic interactions stabilize IR divergence found at one-loop order, making IR finite Yang-Mills calculations possible. I assess the potential of this claim in extending Savvidy's analysis to include all diagonal quartic interaction terms.

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1. Introduction

In this article, I calculate an effective Yang-Mills Lagrangian by incorporating linear first-order contributions from cubic and quartic interactions with the use of the BGFM and a modified Hubbard-Stratonovich transformation, similar to those found in large-N calculations [1–3]. Off diagonal terms from quartic and cubic interactions are *not* included in this calculation; thus, this result fails to be a full Yang-Mills calculation and is far from a complete and unique transseries representation of the theory. Nonetheless, having a toy model for finding IR finite calculations is progress toward a full resurgent analysis of YM theories in $3 + 1d$.

Section 2 presents an outline of the background field method. Section 3 uses a Hubbard-Strotanovitch transformation to set up an effective Lagrangian that includes cubic and quartic interactions at first order. Section 4 presents detailed calculations of the effective theory and its associated gap equations and compares these calculations with existing lattice data.

2. The Background Field Set up for Yang Mills

Consider the Yang-Mills Lagrangian density for general SU(N) given by

$$L = \frac{1}{4}(\mathcal{F}^a_{\mu\nu})^2. \tag{1}$$

The adjoint representation defines the field strength tensor and covariant derivative as:

$$\begin{aligned} \mathcal{F}^a_{\mu\nu} &= \partial_\mu \delta^{ac} A^c_\nu(x) - \partial_\nu \delta^{ac} A^c_\mu(x) + g_0 f^{abc} A^b_\mu(x) A^c_\nu(x) \\ D^{ac}_\mu &= \partial_\mu \delta^{ac} + g_0 f^{abc} A^b_\mu(x), \end{aligned} \tag{2}$$

where f^{abc} are the structure constants of some SU(N) algebra, and g_0 is the bare Yang-Mills coupling constant. In Euclidean space with the temporal direction compactified on the thermal cylinder, the partition function is [4]

$$Z = \int \mathcal{D}A e^{-\frac{1}{4}\int_x (\mathcal{F}^a_{\mu\nu})^2}. \tag{3}$$

Equation (3) demands that gauge fields $A^a_\mu(x)$ are periodic in the temporal direction giving $A^a_\mu(0,\vec{x}) = A^a_\mu(\beta,\vec{x})$ [5]. Next the gauge fields are separated into a background-field $B^a_\mu(x)$ and fluctuations $a^a_\mu(x)$ such that $A^a_\mu(x) = \frac{B^a_\mu(x)}{g_0} + a^a_\mu(x)$ where $B^a_\mu(x)$ act as the zero modes of the fields in momentum space. With this, integrals over linear contributions of the fluctuations vanish as do terms that go like $B^a_\mu(k) a^b_\mu(k)$ due to orthogonality. This yields the field strength tensor as [6, 7]:

$$\begin{aligned} \mathcal{F}^a_{\mu\nu} &= \frac{1}{g_0} F^a_{\mu\nu} + D^{ac}_\mu\, a^c_\nu(x) - D^{ac}_\nu\, a^c_\mu(x) + g_0 f^{abc} a^b_\mu(x) a^c_\nu(x) \\ D^{ac}_\mu &= \partial_\mu \delta^{ac} + f^{abc} B^b_\mu(x) \\ F^a_{\mu\nu} &= \partial_\mu \delta^{ac} B^c_\nu(x) - \partial_\nu \delta^{ac} B^c_\mu(x) + f^{abc} B^b_\mu(x) B^c_\mu(x) \end{aligned} \tag{4}$$

Choosing a covariantly constant self-dual field strength tensor [5],

$$F^a_{\mu\nu} = \begin{pmatrix} 0 & B^a & 0 & 0 \\ -B^a & 0 & 0 & 0 \\ 0 & 0 & 0 & B^a \\ 0 & 0 & -B^a & 0 \end{pmatrix}, \tag{5}$$

and a background-field configuration

$$B^a_\mu(x) = -\frac{1}{2} F^a_{\mu\nu} x_\nu, \tag{6}$$

satisfies the classical source-free Yang-Mills equations of [8, 9]

$$D^{ab}_\mu F^b_{\mu\nu} = 0. \tag{7}$$

The gauge split action remains invariant to local gauge transformations of the background field [6]

$$\begin{aligned} a^a_\mu(x) &\to a^a_\mu(x) - f^{abc}\beta^b(x) a^c_\mu(x) \\ B^a_\mu(x) &\to B^a_\mu(x) + B^a_\mu(x) D_\mu \beta^a(x) \\ c^a(x) &\to c^a(x) - f^{abc}\beta^b(x) c^c(x). \end{aligned} \tag{8}$$

Although generic manipulations of terms containing a^a_μ destroy local gauge invariance with respect to a^a_μ, it is possible for the theory to remain invariant under local transformations of the background field [10]. Considering kinetic terms in the action such as $(D^{ac}_\mu a^c_\nu)^2$, integration by parts can be performed over the gauge fluctuations a^a_ν as they now transform as matter fields in the adjoint representation [6]. Equation (3) augmented with ghost fields $c^a(x)$ now reads

$$Z = \int B \int \mathcal{D}a\mathcal{D}\bar{c}\mathcal{D}c e^{-\int_x \frac{1}{4g_0^2}(F^a_{\mu\nu})^2 - S_0 - S_I} \tag{9}$$

where [9, 11]

$$\begin{aligned} S_0 &= \int_x \frac{1}{2} a^a_\mu \Big[-(D^2)^{ac}\delta_{\mu\nu} - 2F^b_{\mu\nu} f^{abc} \Big] a^c_\nu + \bar{c}^a(-D^2)^{ac} c^c \\ &\& \\ S_I &= \int_x g_0 (D_\mu a^a_\nu) f^{abc} a^b_\mu a^c_\nu - \bar{c}^a (f^{abc} D^{gb}_\mu) a^g_\mu c^c + \frac{g_0^2}{4} (f^{abc} a^b_\mu a^c_\nu)^2. \end{aligned} \tag{10}$$

3. Vacuum Calculations with the Use of an Auxiliary Field

Starting with the gauge fixed partition function and dropping linear couplings in a^a_μ to ghost fields gives

$$Z = \int dB\mathcal{D}a\mathcal{D}c\mathcal{D}\bar{c} e^{-\int_x (\frac{1}{4g_0^2} F^a_{\mu\nu})^2 - S_0 - S_I} \tag{11}$$

where

$$\begin{aligned} S_0 &= \int_x \frac{1}{2} a^a_\mu [\theta_{\text{Glue}}] a^c_\nu + \bar{c}^a [\theta_{\text{Ghost}}] c^c \\ S_I &= \int_x -g_0 (D_\mu a^a_\nu) f^{abc} a^b_\mu a^c_\nu + \frac{g_0^2}{4} (f^{abc} a^b_\mu a^c_\nu)^2. \end{aligned} \tag{12}$$

To integrate the interaction terms at R_0 level [12] a Hubbard Stratonovich transformation is applied to (11) of the form [2, 13],

$$\int_{-\infty}^{\infty} \mathcal{D}\sigma \int_{0}^{\infty} \mathcal{D}\xi \mathrm{Re}[e^{i\int_x \xi^a_{\mu\nu}(\sigma^a_{\mu\nu} - f^{abc}a^b_\mu a^c_\nu)}] = 1. \tag{13}$$

I will make the ansatz that $\xi^a_{\mu\nu}$ is diagonal in Lorentz space, thus (13) acts only on terms diagonal in Lorentz space. Terms off-diagonal in Lorentz space in S_I are discarded. It can be shown that deep in the IR, diagonal contributions to $\det[\theta_{\mathrm{Glue}}]$ are order ϵ, and the tensor contributions are of order ϵ^2 for $\epsilon << 1$. Further, lattice data reports 0^{++} as the lowest lying nontrivial gluonic bound state, implying that tensor self-energies are Boltzmann suppressed in relation to 0^{++}. After applying (13) to (11) the effective action of the gluon contribution is

$$S^{\mathrm{glue}}_{\mathrm{eff}} = -\int_x \frac{1}{2} a^a_\mu [\theta^{ac}_{\mathrm{Glue}} - i\xi^b_{\mu\nu} f^{abc}] a^c_\nu - g_0 (D_\mu a^a_\nu)\sigma^a_{\mu\nu} + (\frac{g_0}{2}\sigma^a_{\mu\nu})^2 - i\xi^a_{\mu\nu}\sigma^a_{\mu\nu}. \tag{14}$$

fThe cubic term $-g_0(D_\mu a^a_\nu)\sigma^a_{\mu\nu}$ now vanishes under the the gauge fixing condition $D_\mu a^a_\mu = 0$. Savvidy likewise claims cubic interactions do not contribute to regulating zero mode divergence at first order [9], and some, including Feynman, speculate that quartic terms are responsible for the mass gap in YM theories [14]. Integrating out $\sigma^a_{\mu\nu}$, letting $\xi^a_{\mu\nu} = \bar{\xi}^a_{\mu\nu} + (\xi')^a_{\mu\nu}(x)$, and discarding fluctuations $(\xi')^a_{\mu\nu}(x)$ results in

$$S^{\mathrm{glue}}_{\mathrm{eff}} = -\int_x \frac{1}{2} a^a_\mu [-(D^2)^{ac}\delta_{\mu\nu} - 2A^{ac}F_{\mu\nu} + \bar{\xi}_{\mu\nu}A^{ac}] a^c_\nu + \frac{(\bar{\xi}^a_{\mu\nu})^2}{g_0^2}. \tag{15}$$

The variable substitution $\bar{\xi}^a_{\mu\nu} \to B^a \Delta \delta_{\mu\nu}$ gives the effective action of

$$S^{\mathrm{Glue}}_{\mathrm{eff}} = -\int_x \frac{1}{2} a^a_\mu [-(D^2)^{ac}\delta_{\mu\nu} - 2A^{ac}F_{\mu\nu} + \Delta B A^{ac}\delta_{\mu\nu}] a^c_\nu + \frac{(B^a\Delta\delta_{\mu\nu})^2}{g_0^2}, \tag{16}$$

and allows for a decoupling of gap equations with respect to B and Δ under certain renormalization schemes. Taking the saddle point approximation for B and Δ, the partition function is now

$$Z = \int \mathcal{D}a\mathcal{D}c\mathcal{D}\bar{c}\ \mathrm{Re}\Big[e^{-\int_x \frac{1}{4g_0^2}(\bar{F}^a_{\mu\nu})^2 + \frac{(B^a\Delta\delta_{\mu\nu})^2}{g_0^2} + a^a_\mu\left[-(D^2)^{ac}\delta_{\mu\nu} - 2A^{ac}F_{\mu\nu} + \Delta B A^{ac}\delta_{\mu\nu}\right]a^c_\nu + \bar{c}^a\left[(-(D^2)^{ac}\right]c^c}\Big]_{\bar{B},\bar{\Delta}}. \tag{17}$$

where $\bar{\Delta}$ and $\bar{B}$ are saddle points of their respective functions in (17). The form of (13) constrains the saddle equations of Δ to the positive real axis, likewise constraining the eigenspectrum of $\theta^{R_0}_{\mathrm{Glue}}$ to be semi-positive definite. This must be the case as the original form of (3) is semi-positive definite. The constant $\frac{(\bar{B}^a\bar{\Delta}\delta_{\mu\nu})^2}{g_0^2}$ can be removed with a counter term giving an effective Z as

$$Z = e^{-\frac{\beta V}{g_0^2}(F^a_{\mu\nu})^2 - \frac{1}{2}\ln\det\left[\theta^{R_0}_{\mathrm{Glue}}\right] + \ln\det\left[\theta_{\mathrm{Ghost}}\right]}\Big|_{\bar{B}\bar{\Delta}} \tag{18}$$

where $\theta^{R_0}_{\mathrm{Glue}} = -(D^2)^{ac}\delta_{\mu\nu} - 2A^{ac}F_{\mu\nu} + \Delta B A^{ac}\delta_{\mu\nu}$. Similar to the one-loop theory the effective action in 18 remains gauge invariant with respect to local transformations of $B^a_\mu(x)$ but not $a^a_\mu(x)$ as all functions of $B^a_\mu(x)$ in 18 are powers of D^{ac}_μ and $F^a_{\mu\nu}$ and the term $a^a_\mu(x)\Delta B A^{ac}\delta_{\mu\nu}a^c_\nu(x)$ only transforms locally with respect to $a^a_\mu(x)$.

4. Calculating the Effective Action

To calculate $\ln\det\theta_{\text{Ghost}}$ I will use zeta function regularization of the form [15]

$$\ln\det\theta = -\Big[\frac{d}{ds}\frac{1}{\Gamma[s]}\int_0^\infty d\tau\tau^{s-1}K_\theta\Big]_{s=0}, \tag{19}$$

where

$$K_\theta = \mathrm{Tr}^{ab}_{\mu\nu}\sum_{n,m} e^{-\tau\theta/\mu^2} \tag{20}$$

is the heat kernel of the operator θ and μ is an arbitrary renormalization scale. The heat kernels are given as [9, 11]

$$\begin{aligned} K_{\text{Ghost}} &= \sum_a \beta V\frac{(B\lambda^a)^2}{16\pi^2}\left[\frac{1}{\sinh^2(\frac{B\lambda^a\tau}{\mu^2})}\right] \\ K^{R_0}_{\text{Glue}} &= \sum_a \beta V\frac{(B\lambda^a)^2}{4\pi^2}\left(e^{-\frac{\tau B\lambda^a\Delta}{\mu^2}}\right)\left[2+\frac{1}{\sinh^2(\frac{B\lambda^a\tau}{\mu^2})}\right]. \end{aligned} \tag{21}$$

The ghost field contribution gives

$$\ln\det\theta_{\text{Ghost}} = -\sum_a \beta V\frac{(B\lambda^a)^2}{48\pi^2}\left[\ln\left(\frac{B\lambda^a}{\mu^2}\right)+\ln\left(\frac{2e}{A^{12}}\right)\right]. \tag{22}$$

where A is the Glaisher constant[1]. For the gluon contribution, the exponential dependence on Δ is expanded, giving

$$\begin{aligned} -\frac{1}{2}\ln\det[\theta^{R_0}_{\text{Glue}}] = \sum_a \frac{\beta V(B\lambda^a)^2}{8\pi^2}\times\frac{d}{ds}\Bigg[\frac{1}{\Gamma(s)}\Bigg(2\int_0^\infty d\tau\tau^{s-1}e^{-\tau\frac{B\lambda^a\Delta}{\mu^2}} \\ +\int_0^\infty d\tau\sum_{n=0}^{2}\frac{\tau^{s+n-1}(-\frac{B\lambda^a\Delta}{\mu^2})^n}{n!}\frac{1}{\sinh^2\left(\frac{B\lambda^a\tau}{\mu^2}\right)}+\int_0^\infty d\tau\sum_{n=3}^{\infty}\frac{\tau^{s+n-1}(-\frac{B\lambda^a\Delta}{\mu^2})^n}{n!}\frac{1}{\sinh^2\left(\frac{B\lambda^a\tau}{\mu^2}\right)}\Bigg)\Bigg]_{s=0}. \end{aligned} \tag{23}$$

The first integral in (23) vanishes in the IR, curing the one-loop IR divergence. The second integral containing $n<3$ contributions gives two UV divergences, one from the $n=0$ term and one from the $n=2$ term. Further, the $n=1$ and $n=2$ terms give the only non-vanishing IR contributions. The second sum in (23) contains a series of UV finite contributions and is real and absolutely and uniformly convergent for $\Delta\geq 0$, collectively yielding,

$$\frac{\ln Z}{\beta V} = -\frac{(B\lambda^a)^2}{g_0^2} - \frac{(B\lambda^a)^2}{48\pi^2}\left[11\ln\left(\frac{B\lambda^a}{\mu^2}\right)+C+f(\Delta)+g(\Delta,B,\mu)\right]\Bigg|_{\bar{B}\bar{\Delta}} \tag{24}$$

[1] For the remainder of this paper the color matrix **A** does not appear, only components λ^a are relevant. In the following calculations, any use of the symbol A is the Glaisher constant.

where the sum over color space is implied, and

$$\begin{aligned} C &= \ln\left(\frac{e}{2A^{12}}\right) \\ g(\Delta, B, \mu) &= 3\Delta^2 \ln\left(\frac{B\lambda^a}{\mu^2}\right) + 12\ln(\Delta) \\ f(\Delta) &= -\Delta\left(\ln\left(64\pi^6\right) - 3\Delta\ln(2)\right) + 12\Delta\ln\left(\Gamma\left[\frac{\Delta+2}{2}\right]\right) - 24\zeta^{(1,0)}\left(-1, \frac{\Delta+2}{2}\right). \end{aligned} \tag{25}$$

Equation (6.88) of [9] likewise claims the effective IR finite Lagrangian of a self-dual YM theory in which single interaction terms have been included to regulate the IR divergences is

$$\mathcal{L}_E^{eff} = \frac{H^2}{2} + \frac{11H^2g^2}{48}\left[\ln\left(\frac{2gH}{\mu^2}\right) - \frac{1}{2}\right] \tag{26}$$

with $H = B\lambda^a$ (in this paper's notation). After a change of variable in g and subtracting terms generated from the first order calculation of diagonal interaction (26) matches (24) up to the factor of $\frac{1}{2}$ in the first term which is insignificant once a non-perturbative renormalization scheme is applied, i.e., the logarithmic term matches exactly. Interestingly we have arrived at consistent answers via entirely unique calculation methods. This consistency verifies Savvidy's claim that IR finite YM calculations are possible by including quartic interaction terms and acts as a consistency check for this work. Evaluating (24) at $\lambda^a = 0$ results in $\frac{\ln Z}{\beta V} = 0$. This leaves no gap equations for B in the $\lambda^a = 0$ color channels. However, if $\bar{B} \to \infty$ when $\lambda^a = 0$ then $\frac{\ln Z}{\beta V}$ goes to an undetermined constant. As this is the case the gap equations for $\bar{B}$ determine the value of $\frac{\ln Z}{\beta V}$ when $\lambda^a = 0$.

There are a variety of ways to regulate the divergent gluon contributions, and the self-energy of the gauge fields is numerically sensitive up to the scale of the Landau pole for each regulated Lagrangian [16]. To maintain all IR physics and remove the UV divergent and scale-dependent physics associated with Δ, I will absorb $g(\Delta, B, \mu)$ into the running coupling such that

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} + \frac{1}{48\pi^2}g(\Delta, B, \mu), \tag{27}$$

producing an effective action with all scale dependence coupled to the background field, giving a beta-function $\beta(\mu)$ of

$$\frac{dg(\mu)}{d\ln(\mu)} = \frac{-11g^3(\mu)}{48\pi^2}, \tag{28}$$

which matches the perturbative beta function at first loop order times a factor of $\frac{1}{3}$. Figure 1 includes the full function of (28). Including the full functionality of running coupling implies that the coupling does not go to infinity at the scale of the Landau pole. Instead, the limit of $\beta(\mu)$ as $\mu \to \Lambda_{\text{YM}}$ does not exist, and this region contains a simple pole with a vanishing Cauchy principle value, indicating the possibility of a phase transition. A similar analysis of the complete running coupling is shown in [17–20]. With the complete function of running coupling the theory is described by a negative and weakly coupled theory deep in the IR. In this way, the semi-classical expansion could be well-defined asymptotically to the left of the Landau pole through the appropriate integrating contour. However, the asymptotic transseries generated by trivial and nontrivial saddles will still retain a zero radius of convergence. Nonetheless, if the transseries that capture these nontrivial saddles is resurgent, the theory maintains the possibility of lateral Borel resummation [21, 22]. Further, the complete running coupling, as presented in 1, is qualitatively analogous to scattering length as a function of the magnetic field in Feshbach resonance. With Feshbach resonance, the pole in the scattering length generates a cross-over to a Bose-Einstein condensate of strongly coupled Cooper pairs [23]. Although this analogy is currently only qualitative, I believe it holds promise in explaining confinement, and future work on understanding the complete beta-function should be explored.

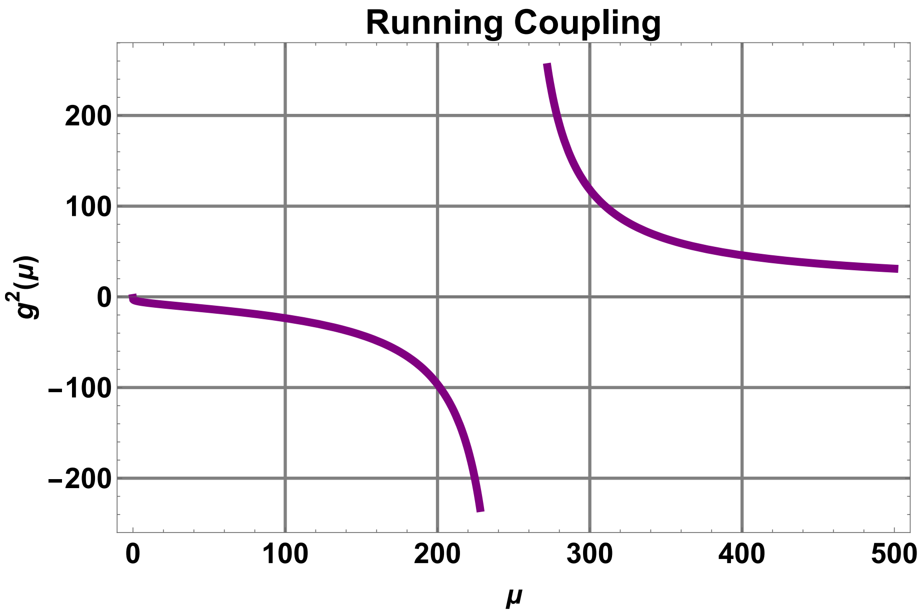

Figure 1: The running coupling is plotted as a function of μ with the a set Landau pole scale of $\Lambda_{\rm YM}$ = 250MeV.

4.1 Results

Solving (28) for $g(\mu)$ gives a renormalized pressure is of

$$\frac{\ln Z}{\beta V} = -\sum_a \frac{(B\lambda^a)^2}{48\pi^2}\left[11\ln\left(\frac{B\lambda^a}{\Lambda_{\rm YM}^2}\right) + C + f(\Delta)\right]_{\bar{B},\bar{\Delta}}. \tag{29}$$

The function $f(\Delta)$ can be plotted to show the existence of a stable minimum. It can be numerically shown that $f(\Delta)$ is monotonically increasing many orders of magnitude beyond the stable minimum shown in 2. The gap equation for Δ is

$$2\zeta^{(1,1)}\left(-1, \frac{\Delta+2}{2}\right) + \gamma_E\Delta - 2\log\left(\Gamma\left(\frac{\Delta+2}{2}\right)\right) - \left(\Delta\left(H_{\frac{\Delta}{2}} + \log(2)\right)\right) + \log(2\pi) = 0, \tag{30}$$

where H is the harmonic number function. As the gap equation for Δ decouples from λ^a its saddle is identical for all color indices giving

$$\bar{\Delta} \to 2.82898. \tag{31}$$

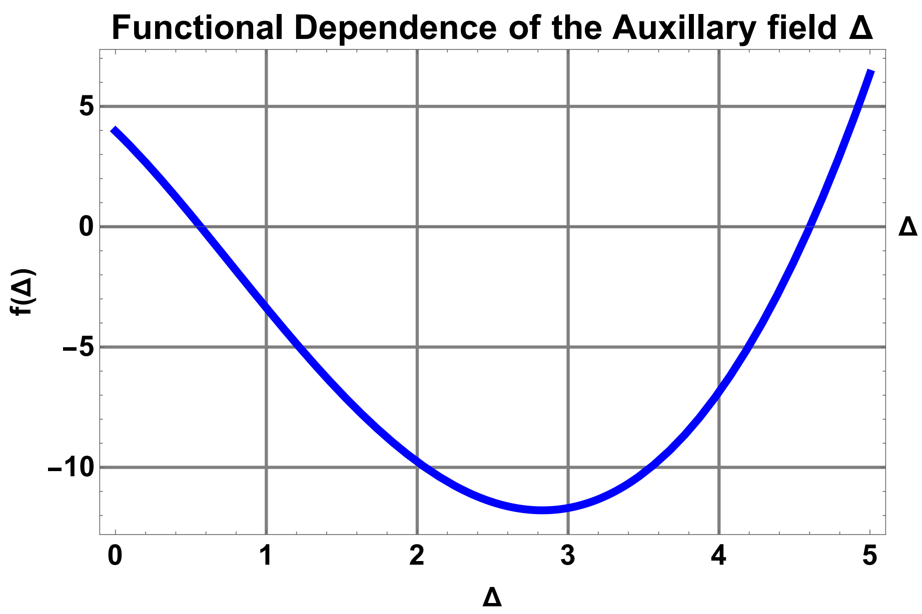

Figure 2: The functional dependence of $\ln Z$ on Δ after renormalization shows a finite stable minimum.

Plugging (31) into (29) gives

$$-\frac{\ln Z}{\beta V} = -\sum_a \frac{(B\lambda^a)^2}{48\pi^2}\left[\left(14.4634 - 11\log\left(\frac{B\lambda^a}{\Lambda_{\rm YM}^2}\right)\right)\right]. \tag{32}$$

The non-zero free energy density contribution $(-\frac{\ln Z}{\beta V})^a$ is plotted, showing the existence of stable minimum and the unstable perturbative saddle at $\bar{B} = 0$, and is likewise monotonically increasing beyond the stable minimum, structurally matching results of [24]. The gap equation for B is

$$\left(14.4634 - 11\log\left(\frac{B\lambda^a}{\Lambda_{\rm YM}^2}\right)\right) - \frac{11}{2} = 0 \tag{33}$$

giving

$$\bar{B} \to \frac{2.25885\Lambda_{\rm YM}^2}{\lambda^a}\,,\ \bar{B} \to 0, \tag{34}$$

where the trivial solution gives an unstable vacuum pressure associated with the perturbative minimum. The following results are given under the assumption $\Lambda_{YM} \sim 250$ MeV. Summing over the $SU(3)$ color index for the stable solution of B gives

$$\frac{\ln Z}{\beta V} = 8 \times 0.0592377\Lambda_{\rm YM}^4 \sim 0.00185\ \mathrm{GeV}^4. \tag{35}$$

Taking the fourth root of the vacuum pressure gives

$$P^{\frac{1}{4}} \sim 207\ \mathrm{MeV} \tag{36}$$

As the perturbative pressure is zero, the difference between non-perturbative and perturbative regimes is trivial and (36) matches upper limits of the MIT bag model [25–27]. As negative pressure

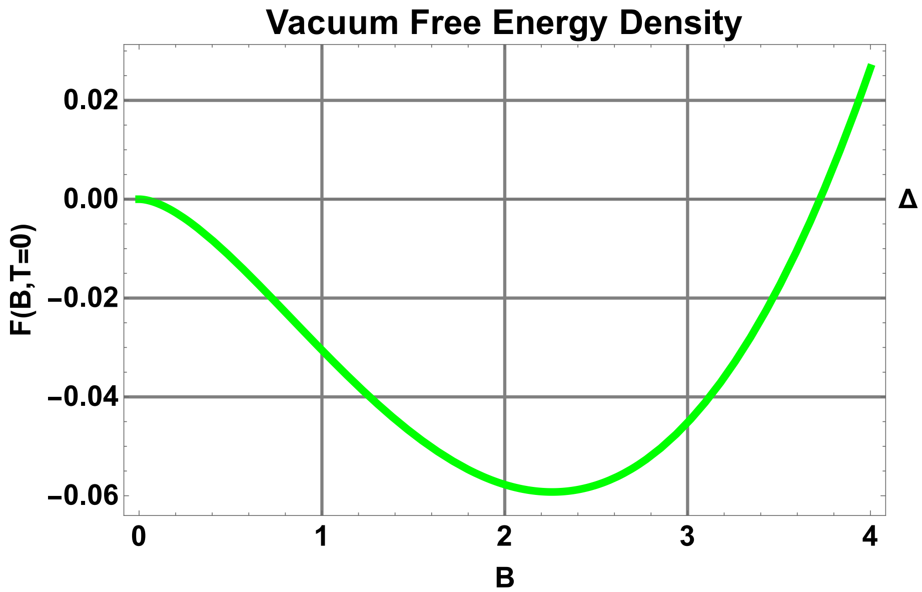

Figure 3: The non-zero free energy density contributions per component are shown with $\lambda^a = 1$ and $\Lambda_{YM} = 1$.

generates vacuum expansion, this result is phenomenologically consistent with confinement. The self-energy of the gauge fields comes in two forms; a Lorentz scalar and a gauge singlet Π_S,and an antisymmetric Lorentz tensor Π_T, Single components of the gluon self-energy are given as

$$(\Pi_S)^{ac}_{\mu\nu} = \bar{B}\lambda^a\bar{\Delta} = 6.39025\Lambda^2_{\text{YM}} \sim 0.399\ \text{GeV}^2$$
$$(\Pi_T)^{ac}_{\mu\nu} = \pm 2\bar{B}\lambda^a\bar{\Delta} = \pm 4.51771\Lambda^2_{\text{YM}} \sim \pm 0.282\ \text{GeV}^2. \tag{37}$$

From (37) tensor and scalar terms, the effective gluon mass stemming from self energies of the gauge fluctuations propagator can be read off as

$$M^s_g = (\bar{B}\lambda^a\bar{\Delta})^{\frac{1}{2}} \sim 2.53\Lambda_{\text{YM}} \sim 632\ \text{MeV} \tag{38}$$

M^s_g is squarely in the range of the dynamical gluon mass first presented by Cornwall as 500 ± 200 MeV [28], and is in range of [29] who claims a constant infrared effective gluon mass of 648(7) MeV in the decoupled scenario of the gluon propagator, and series of other lattice and SDE methods that put the effective mass in the range around 300 – 650 MeV [30–35] at scales around .05-1 fm. The off-diagonal component has been estimated at around 1 GeV in the range of .4-1 fm and falls off in the IR [36]. In [37], similar analytic methods which implement a Hubbard-Stratonovich transformation in the Landau gauge generate a constant effective gluon mass of $M_g = 2.031\Lambda_{\overline{\text{MS}}}$. The gluon condensate can be evaluated as

$$\langle \bar{F}^a_{\mu\nu}\bar{F}^a_{\mu\nu}\rangle = 32\bar{B}^2 \sim 0.799\ \text{GeV}^4. \tag{39}$$

It has been previously reported that numerical integration applied to the background field gauge give $\langle F^2\rangle = 0.93\ \text{GeV}^4$ [38] evaluated at $\Lambda = 283$ MeV, giving a close match to (39). While earlier models show a condensate value of roughly $\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu}G^{a\,\mu\nu}\rangle \sim .2\ \text{GeV}^4$ [39] and QCD sum rules have given $\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu}G^{a\,\mu\nu}\rangle \sim .01\ \text{GeV}^4$ where $\alpha_s = \frac{g^2}{4\pi}$ [40, 41].

5. Final Considerations

Although this calculation fails to capture all interaction terms of a YM theory, it provides a framework for including quartic and cubic terms in a non-perturbative framework. The eigen-spectrum analysis and computational methods presented in this work can be extended to include off-diagonal interaction terms, matter fields, finite temperature, and chemical potentials in QCD and YM theories alike. Objects such as propagators, Wilson loops, and other gauge invariant quantities could be calculated with respect to the background gauge. Further, saddles found in this framework could be used to explore the potential of YM as a resurgent theory. The complete picture of the running coupling offers a path forward for understanding YM theories as IR and UV complete and framing confinement of QCD in terms of superconductivity and Cooper pairs. Landau pole methods using the full running coupling are important as postulates from a growing literature of work on $\mathcal{PT}$-symmetric and negatively coupled field theories. More analysis and experimental verification [42] of these methods will bolster the claims in this article.

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