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The Fate of Chiral Symmetry in the Quark-Gluon Plasma

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We propose an instanton-based random matrix model of the lowest part of the spectrum of the Dirac operator in the high temperature phase of quantum chromodynamics. Our model differs in two important ways from previously considered similar instanton models. Firstly, it is extremely simple, having only two parameters, one of which is the topological susceptibility. Secondly, it provides an excellent description of the distribution of the lowest eigenvalues of the Dirac operator, obtained from quenched lattice simulations with the overlap operator. We argue that the singular spike in the Dirac spectrum at zero is due to a gas of free instantons, and we show that for two chiral flavors, the $U(1)_\mathrm{A}$ symmetry breaking pion minus delta susceptibility remains nonzero in the chiral limit.

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1. Introduction

Quantum chromodynamics, the theory of strongly interacting quarks and gluons, is a peculiar theory, having several global symmetries, albeit only approximate ones. Notwithstanding the approximate nature of these symmetries, the way they are manifested under different circumstances has a profound influence on the physics of strongly interacting systems. The symmetry that we are concerned with here is the $SU(2)_{\rm L} \times SU(2)_{\rm R} \times U(1)_{\rm V} \times U(1)_{\rm A}$ chiral symmetry that rotates the left- and right-handed components of the quark fields independently. This symmetry holds approximately because the two light quarks the u and the d are almost massless, their masses being much smaller than the typical energy scales of low-energy QCD. The symmetry would be exact if the two light quarks were exactly massless.

At low temperature the $SU(2)_{\rm L} \times SU(2)_{\rm R}$ part of the symmetry breaks spontaneously to its diagonal subgroup, giving rise to the constituent quark masses and the rich phenomenology of low-energy hadron physics. Above the crossover temperature of about 155 MeV, the spontaneously broken symmetry gets restored and quarks are liberated from hadrons, strongly interacting matter crosses over into the quark-gluon plasma state.

The fate of the flavor singlet, $U(1)_{\rm A}$ part of the symmetry is more complicated, as it is a symmetry of the classical theory, but upon quantization, it is broken by the axial anomaly. Its breaking is closely related to topologically nontrivial configurations of the gauge field, configurations with nonzero topological charge. The topological charge fluctuates strongly at low temperature, but above the crossover temperature to the quark-gluon plasma, its fluctuations rapidly decrease. It was therefore speculated that at high enough temperature the $U(1)_{\rm A}$ symmetry could also be "effectively" restored, meaning that its anomalous breaking would not affect some physical quantities [1, 2]. Subsequently, this effect was also supported by lattice simulations, claiming that above a certain temperature the susceptibility difference $\chi_\pi - \chi_\delta$ vanishes, showing that in this quantity the $U(1)_{\rm A}$ breaking does not show up [3–5].

In the present paper we argue that even though above the critical temperature the symmetry breaking is small and rapidly decreasing, it never becomes zero since at any finite temperature, topological fluctuations are present in the system. Our key observation is that the spike in the spectrum of the Dirac operator at zero virtuality, first seen in Ref. [6], and recently confirmed by several other studies, is caused by a noninteracting gas of unit charge topological lumps. Using a random matrix model of the Dirac operator, a model that exactly reproduces the statistical properties of the low-end of the lattice overlap Dirac spectrum, we show that the spectral spike is singular, and its influence on chiral quantities can be nontrivial in the chiral limit [7].

2. Random matrix model of the quenched Dirac spectrum

The order parameter of the spontaneously broken flavor non-singlet $SU(2)_{\rm A}$ symmetry is the chiral condensate $\langle\bar{\psi}\psi\rangle$. In the low temperature, symmetry broken phase, it is nonzero, whereas above T_c it vanishes. This quantity can be identically written in terms of the spectrum of the Dirac operator as

$$\langle\bar{\psi}\psi\rangle = \propto \frac{1}{V}\sum_k \frac{1}{i\lambda_k + m} \propto \int_{-\Lambda}^{\Lambda} d\lambda \frac{m}{\lambda^2+m^2}\rho(\lambda) \xrightarrow[m\to 0]{} \rho(0), \tag{1}$$

where $i\lambda_k$ are the eigenvalues of the Dirac operator, and the sum over the eigenvalues is expressed in terms of the spectral density of the Dirac operator, also making use of its symmetry $i\lambda \leftrightarrow -i\lambda$ of the spectrum. The chiral limit of this expression, connecting the spectral density at zero virtuality with the condensate is the Banks-Casher relation [8]. Based on this, one would expect that in the symmetry restored phase, the spectral density at zero should vanish.

This was the accepted view until the appearance of lattice quark discretizations respecting chiral symmetry exactly, and making it possible to properly resolve the lowest part of the Dirac spectrum. Using the overlap Dirac operator [9], it was found that the Dirac spectral density, instead of vanishing at zero, develops a sharp spike there [6]. Based on the Banks-Casher relation, this behavior calls into question even the restoration of the flavor nonsinglet chiral symmetry. For some time, this results did not attract much attention, probably because it was thought to be a coarse lattice or a quenched artifact.

More recently the presence of the spike was confirmed by several independent studies, demonstrating that the spike is neither a quenched, nor a coarse lattice artifact, but stays in the continuum limit [10–14]. The spike was also seen with dynamical overlap quarks [15], demonstrating that its presence is not a result of the improper suppression of small Dirac modes by staggered sea quarks, used in other simulations. Unfortunately, dynamical overlap simulations are too expensive to be performed on large enough volumes needed to find out the exact nature of the spectral spike, and its influence on chiral quantities.

Already in the work first reporting the spectral spike, an explanation in terms of a gas of free instantons, or more generally a collection of noninteracting lumps of unit topological charge, “instantons”[1] was put forward [6]. More recently, the noninteracting nature of this topological gas was verified more precisely in quenched QCD [16–18].

It is thus natural to describe the lowest part of the Dirac spectrum, the eigenmodes in the spectral spike, using an instanton model. The most natural way to do this is through a random matrix model of the so called zero mode zone (ZMZ) of the Dirac operator, the subspace spanned by the zero modes of the instantons and anti-instantons [19]. Let us first consider the ZMZ in the simplest nontrivial configuration, where an instanton and an anti-instanton is present in a trivial gauge field background. In this case the ZMZ is a two-dimensional subspace, and the Dirac operator in the zero mode basis has the form

$$D(A)_{\text{zmz}} = \begin{pmatrix} 0 & iw \\ iw & 0 \end{pmatrix}, \tag{2}$$

where

$$w \propto \mathrm{e}^{-\pi T r} \tag{3}$$

represents the mixing between the two opposite chirality zero modes. Since the high temperature zero modes fall-off spatially exponentially with the exponent πT (T is the temperature) [20], we assume that the mixing also has this dependence on the distance r between the instanton and the anti-instanton. Generalizing this idea, we can also write down the matrix of the ZMZ of the Dirac operator for a general configuration of n_i instantons and n_a anti-instantons with fixed locations as

[1]From now on, for simplicity we will use this terminology, calling lumps of unit topological charge *instantons*, even though most likely these objects are very far from exact solutions of the field equations, i.e. real instantons. This property, however, will not be important for our purposes.

$$D_{\text{zmz}} = \left(\begin{array}{c|c} \overbrace{\quad 0 \quad}^{n_i} & \overbrace{\quad iW \quad}^{n_a} \\ \hline iW^{\dagger} & 0 \end{array}\right), \tag{4}$$

where the zeros represent two diagonal blocks of zeros. This matrix has all the important properties of a chirally symmetric Dirac operator. In particular, it has $|n_i - n_a|$ exact zero eigenvalues, and anticommutes with γ_5, represented in the basis of zero modes by a diagonal matrix with n_i entries of -1 and n_a entries of $+1$.

To complete the model and to be able to generate an ensemble of such matrices, we have to specify how the instanton configurations are chosen randomly. Since the instanton ensemble represents a free instanton gas, the number of instantons n_i and ant-instantons n_a are chosen from independent and identical Poisson distributions with parameters $\chi V/2$, where $\chi = \langle Q^2 \rangle$, V is the four-volume of the system, and $Q = n_i - n_a$ is the topological charge, χ here is the topological susceptibility.

The model has two parameters that we have not specified so far, the susceptibility χ and the prefactor (we call it A) of the exponential fall-off in Eq. (3). We determine these parameters from overlap Dirac spectra of a quenched lattice simulation. For that we used a set of 20k $32^3 \times 8$ lattice configurations generated with the Wilson action at a temperature of $T = 1.1T_c$. We computed the lowest part of the overlap spectrum on these configurations, and determined the topological susceptibility from the index theorem by counting the number of zero eigenvalues. We determined the distribution of the lowest Dirac eigenvalue on these configurations, and fitted the parameter A to reproduce that in a simulation of the random matrix model.

In Fig. 1 we show the lattice data as well as the simulation results for the best fit parameter in a spatial box of linear size $L = 2.4$ fm, corresponding to the $32^3 \times 8$ ensemble. We can see that this one-parameter fit describes the lattice data perfectly. As a further check, we repeat the comparison between the lattice and the model given distribution, but now in a larger volume, resulting again in good agreement. We emphasize that this second comparison involves no more fitting, for the simulation of the random matrix model we used the parameters determined on the smaller volume. After performing several other tests using other volumes, the distribution of the second, third lowest eigenvalues, as well as distributions separately in different charge sectors, we conclude that our instanton-based random matrix model gives a satisfactory description of the zero mode zone of the lattice overlap Dirac operator on quenched configurations.

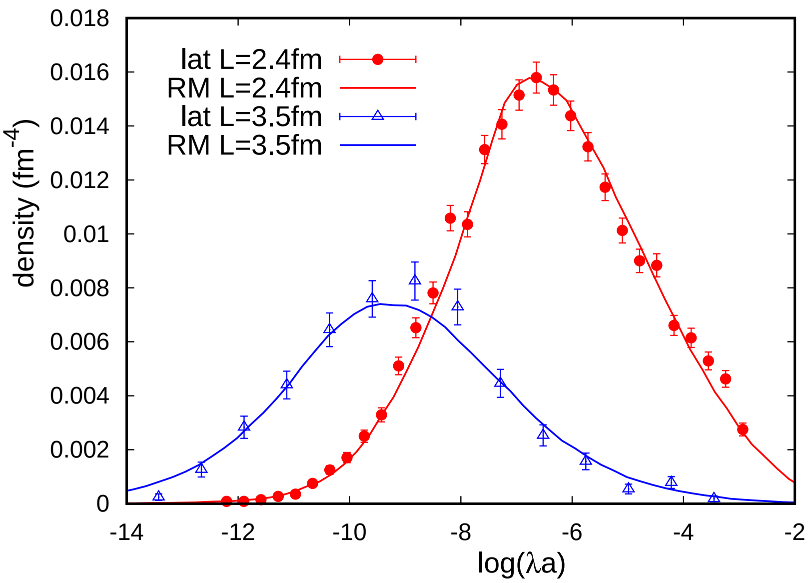

Figure 1: The distribution of the smallest eigenvalue of the overlap Dirac operator on quenched gauge backgrounds in two different volumes. In both cases the temperature is set to $T = 1.1T_c$, and for a better resolution of the small eigenvalues we plotted the distribution of the natural logarithm of the eigenvalues. The symbols represent lattice data, the continuous lines are data calculated from the random matrix model. The smaller volume was used for fitting the parameter A, the larger volume involves no further fitting, it is a prediction of the model.

3. Including dynamical quarks

Our study so far involved only the quenched case, however, we would also like to include dynamical quarks in our model. This amounts to supplementing the probabilities of the different instanton configurations of the quenched model with the determinant of the Dirac operator, resulting from integrating out the dynamical quarks. To facilitate this, we split the determinant into two parts as

$$\det(D+m) = \prod_{\text{zmz}}(\lambda_i + m) \times \prod_{\text{bulk}}(\lambda_i + m), \tag{5}$$

i.e. the contribution of the eigenvalues in the zero mode zone and the contribution of the rest of the eigenvalues that we refer to as the bulk. The important observation here is that as can be seen in Fig. 2, depicting the quenched spectral density, the spike and the bulk are well separated by a substantially depleted region in the spectrum. For this reason we do not expect correlations between the ZMZ and the bulk of the spectrum. Here we are interested only in chiral quantities that in the chiral limit – by an argument similar to the one leading to the Banks-Casher relation – are completely determined by the lowest part of the spectrum, the asymptotic behavior of $\rho(\lambda)$ as $\lambda \to 0$. For these quantities it is enough to include in the reweighting only the ZMZ part of the determinant, the bulk is not correlated with the physical quantities we want to calculate, and only gives a constant factor that cancels in expectations.

The ZMZ part of the spectrum is exactly the part that we have access to in the random matrix model, so we can just diagonalize the random matrices in the ensemble and reweight each

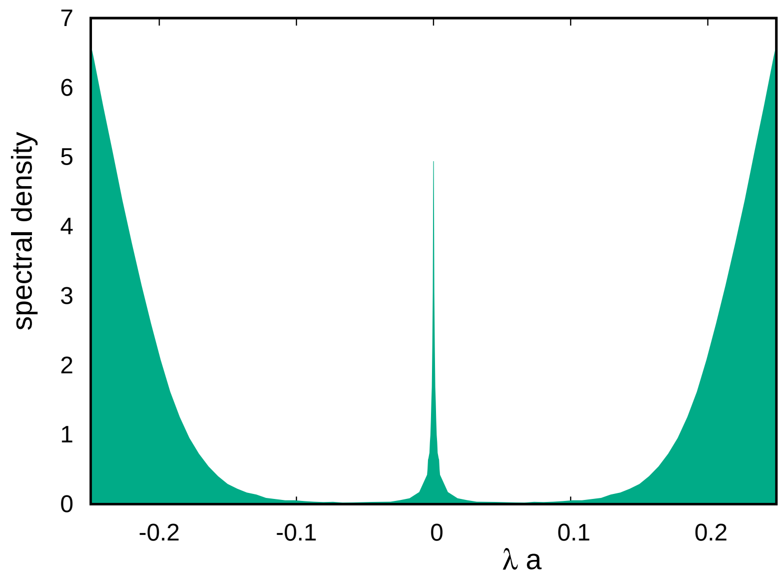

Figure 2: The spectral density of the overlap Dirac operator on quenched gauge backgrounds with temporal size $N_t = 8$ and temperature $T = 1.05T_c$. We removed the exact zero eigenvalues, they would produce a Dirac delta at zero in the spectral density.

configuration with the product of the corresponding eigenvalues. This completely determines the ensemble of random matrices in the model with dynamical quarks. So the instanton configurations are generated with weights proportional to

$$P(n_i, n_a) \propto \underbrace{\mathrm{e}^{-\chi_0 V} \frac{1}{n_i!} \frac{1}{n_a!} \left(\frac{\chi_0 V}{2}\right)^{n_i+n_a}}_{\text{free instanton gas with random locations}} \times \det(D+m)^{N_\mathrm{f}}, \tag{6}$$

where the determinant part of the weight depends on the instanton and anti-instanton locations.

We can now simulate this model, and compute different physical quantities as a function of the quark mass. For illustration, in Fig. 3 we show the topological susceptibility as a function of the quark mass with two flavors of degenerate quarks. The continuous line is not a fit to the simulation data, but the function $\chi(m) = \chi_0 m^2$, where χ_0 is the quenched susceptibility, that is the susceptibility that we would obtain without reweighting with the determinant.

This peculiar result clearly calls for an explanation. To that end, let us rewrite the determinant as

$$\det(D+m)^{N_\mathrm{f}} \quad = \quad m^{N_f\,(n_i+n_a)} \times \prod_i \left(1 + \frac{|\lambda_i|^2}{m^2}\right), \tag{7}$$

where the product runs through the nonzero pairs of eigenvalues of the random matrix. Let us first assume that the quark mass is so large that for all the eigenvalues $|\lambda_i| \ll m$. In this case, the reweighting factor, the determinant is just $m^{N_f\,(n_i+n_a)}$, and remarkably, this does not depend on the location of the instantons and anti-instantons. So even after the reweighting, the gas of instantons remains noninteracting. Indeed, the reweighting factor can be absorbed into the Poisson

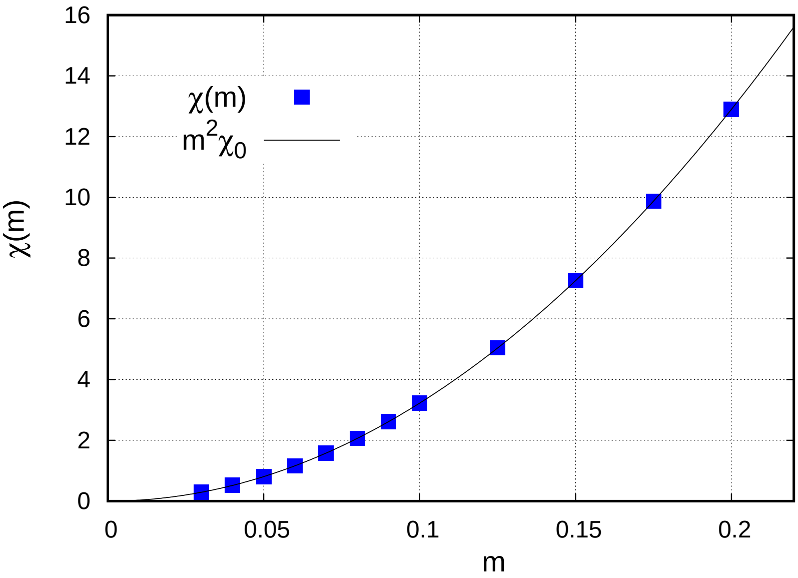

Figure 3: The quark mass dependence of the topological susceptibility from a simulation of the random matrix model with two degenerate flavors of light quarks. The symbols correspond to the simulation data, and the continuous line is not a fit, but the function $m^2\chi_0$, where χ_0 is the quenched topological susceptibility, the parameter of the Poisson distribution used in the model, as shown in Eq. (6). In the absence of dynamical quarks, the model would yield this topological susceptibility.

distributions of Eq. (6), and the reweighting amounts to a rescaling of the topological susceptibility as $\chi_0 \to m^{N_f}\,\chi_0$. This is exactly the behavior that we see in the simulation data.

However, to derive this result, we had to assume that the quark mass is large enough. What happens in the chiral limit? In that case, we can imagine that we do the reweighting in two steps: first we reweight each configuration with the factor $m^{N_f\,(n_i+n_a)}$, which reduces the topological susceptibility substantially, meaning that after this reweighting, the dominant configurations will have much fewer instantons, typically farther apart. Since the matrix elements of the Dirac operator are exponentially small in the instanton–anti-instanton distance, a more dilute instanton gas will produce much smaller eigenvalues. Indeed, looking at the simulation data in detail, reveals that due to this mechanism, in the chiral limit the typical eigenvalues decrease much faster in magnitude than the quark mass, and $|\lambda_i| \ll m$ is maintained for arbitrarily small quark masses. Physically this means that even though the quark determinant introduces interactions among the instantons and anti-instantons, this is negligible, and all the way to the chiral limit the instanton gas remains noninteracting. Another consequence of the abundance of small Dirac eigenvalues is that the spectral spike is in fact singular at zero [7, 21, 22].

4. Banks-Casher type sums and chiral symmetry

The singular nature of the spectral density means that above the critical temperature, the Banks-Casher relation cannot be used, and we have to reconsider its derivation, and generalize it to the case of the singular spectral density. This is, however, rather simple if we use the property of the spectrum that even in the chiral limit $|\lambda_i| \ll m$. The chiral condensate can be written as

$$\langle\bar{\psi}\psi\rangle \quad \propto \quad \langle\sum_i \frac{m}{m^2+|\lambda_i|^2}\rangle \quad \approx \quad \underbrace{\binom{\text{avg. number of in-}}{\text{stantons in free gas}}}_{m^{N_\mathrm{f}}\chi_0 V} \times \frac{1}{m} \quad = \quad m^{N_\mathrm{f}-1}\chi_0 V, \tag{8}$$

where the number of terms in the sum is equal to the number of instantons and anti-instantons, and if $|\lambda_i| \ll m$, then each term gives the same, $1/m$ contribution. In the chiral limit, the condensate goes as m^{N_f-1} which means that for two and more light flavors it vanishes, as expected, and the spontaneously broken flavor nonsinglet part of the chiral symmetry is restored. For one flavor, the condensate is finite in the chiral limit, but in this case there is no symmetry, and the condensate is not an order parameter.

Using the same argument we can also compute the $U(1)_\mathrm{A}$ breaking susceptibility $\chi_\pi - \chi_\delta$ which can also be rewritten in terms of the spectral sum, in this case as

$$\chi_\pi - \chi_\delta \quad = \quad \langle\sum_i \frac{m^2}{(m^2+\lambda_i^2)^2}\rangle \quad \approx \quad \underbrace{\binom{\text{avg. number of in-}}{\text{stantons in free gas}}}_{m^{N_\mathrm{f}}\chi_0 V} \cdot \frac{1}{m^2} \quad = \quad m^{N_\mathrm{f}-2}\chi_0 V. \tag{9}$$

Contrary to some expectations, this quantity vanishes in the chiral limit only if the number of light flavors is at least three.

5. Conclusions

We argued that at temperatures above the crossover to the quark-gluon plasma, the chiral properties of QCD are governed by a noninteracting gas of unit charge topological lumps. This picture is consistent with the constraints on the Dirac spectrum obtained in Ref. [23, 24], and also with the expansion of the QCD free energy around the chiral point [25, 26]. The would be zero modes of the instantons form a sharp spike in the spectral density of the Dirac operator, the spike, containing the eigenvalues that can result in chiral symmetry breaking in the chiral limit. As this spectral spike is singular at zero, a straightforward application of the Banks-Casher relation is not possible. However, due to the exponentially small mixing of instanton and anti-instanton zero modes, typical eigenvalues in the zero mode zone of the Dirac operator always remain much smaller than the quark mass. This makes it possible to evaluate Banks-Casher type integrals even in the chiral limit. We demonstrated the validity of this picture using an instanton-based random matrix model that could reproduce the distribution of the lattice overlap Dirac operator eigenvalues.

We emphasize that this picture applies to arbitrarily high (but finite) temperatures. In fact, our instanton-based random matrix model describes QCD more and more precisely as the temperature gets higher, since for higher temperatures the decoupling of the bulk and the zero mode zone in the determinant is more complete. In the other direction, going down in temperature toward the crossover, we cannot tell how far our model gives a good description. To establish that, we would need dynamical simulations with exactly chiral quarks.

We have seen that for two light flavors the $U(1)_\mathrm{A}$ breaking susceptibility is nonzero even in the chiral limit, and certainly nonzero for physical quark masses. This is due to the singular spike in the spectrum, created by the instantons. This chirality breaking, however, might be small. The relative contribution of the spike and the bulk of the spectrum to certain physical quantities is a dynamical question, depending on the quark mass and the temperature. To quantitatively assess this, one would again need to perform simulations with chiral dynamical quarks that are capable of resolving the small eigenvalue in the spectral spike.

Acknowledgments

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