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New results on α_s from hadronic τ decay

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We describe recent developments in the determination of the strong coupling from finite energy sum rule (FESR) analyses of non-strange spectral distributions measured in hadronic τ decay, focusing, in particular, on the consequences of a newly discovered redundancy in the "truncated OPE" version of these analyses. Such analyses employ OPE fits to weighted spectral integrals involving multiple weights at a single upper integration limit, s_0, with a number of the non-perturbative contributions in principle present removed by hand. We show that (i) contrary to conventional understanding, α_s in these analyses is obtained from fits which retain only perturbative contributions to FESRs of a subset of spectral integrals involving weights of the highest (rather than lowest) degrees entering the analysis, and (ii) the non-perturbative condensates nominally determined in the full analysis are obtained in a redundant manner, leading to potentially very large systematic uncertainties due to amplifications of any residual non-perturbative contamination of the results for α_s produced by the perturbative tOPE truncation in the FESRs which determine α_s. Finally, we show that alternate multi-weight, multi-s_0 FESR analyses do not suffer from these redundancy issues and, through their use of multiple s_0, provide non-trivial additional constraints on the theory representations employed.

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1. Introduction

Hadronic-τ-decay-based finite-energy-sum-rule (FESR) analyses provide one of the most precise determinations of $\alpha_s(\mu^2)$ and, because of the low scale, $\mu \simeq m_\tau \simeq 1.78$ GeV, one of the most stringent tests of the running predicted by QCD. A further advantage of the low scale, and resulting long running from $\mu = m_\tau$ to M_Z, is the factor of ~ 3 reduction in relative error on the $n_f = 5$ result, $\alpha_s(M_Z^2)$, relative to that on the $n_f = 3$ result $\alpha_s(m_\tau^2)$.

Such a relatively low scale, however, also makes considering possible non-perturbative (NP) effects unavoidable. This can only be done in an approximate manner, and the treatment of such contributions is the main source of systematic uncertainties in τ-based determinations. We begin by briefly reviewing the τ-based analysis framework.

The FESRs employed in τ-based analyses have the form

$$I^{(w)}_{\rm ex}(s_0) \equiv \frac{1}{s_0}\int_0^{s_0} ds\, w(s)\,\rho^{(0+1)}_T(s) \;=\; -\frac{1}{2\pi i\, s_0}\oint_{|z|=s_0} dz\, w(z)\,\Pi^{(0+1)}_T(z) \;\equiv\; I^{(w)}_{\rm th}(s_0)\,, \qquad (1)$$

with s the hadronic invariant mass squared, $\Pi^{(J)}_T(z)$ the spin $J = 0, 1$ vacuum polarization functions of the hadronic flavor ud vector ($T = V$), axial-vector ($T = A$) or vector-plus-axial-vector ($T = V + A$) current-current two-point functions, $\rho^{(J)}_T(s)$ the spectral function of $\Pi^{(J)}_T(z)$, and $w(s)$ any function analytic inside and on the countour $|z| = s_0$. For large enough s_0, the RHS of Eq. (1) is evaluated using the OPE representation of $\Pi^{(J)}_T(z)$, possibly supplemented by duality violating (DV) corrections, while the LHS is evaluated using experimental results for the differential distributions, $dR_{ud;V/A}/ds$, where

$$R_{ud;V/A} = \Gamma[\tau^- \to ({\rm hadrons})_{ud;V/A}\nu_\tau(\gamma)]/\Gamma[\tau^- \to e\bar{\nu}_e\nu_\tau(\gamma)], \qquad (2)$$

with $\Gamma[\tau^- \to ({\rm hadrons})_{ud;V/A}\nu_\tau(\gamma)]$ the flavor ud V/A-current-induced hadronic τ decay width. In the SM, one has [1]

$$dR_{ud;V/A}/ds = [12\pi^2|V_{ud}|^2 S_{EW}/m_\tau^2]\,\left[w_\tau(y_\tau)\,\rho^{(0+1)}_{ud;V/A}(s) - w_L(y_\tau)\,\rho^{(0)}_{ud;V/A}(s)\right], \qquad (3)$$

with $y_\tau = s/m_\tau^2$, $w_\tau(y) = (1-y)^2(1+2y)$, $w_L(y) = 2y(1-y)^2$, V_{ud} the ud CKM matrix element, and S_{EW} a known short-distance electroweak correction. With contributions to $\rho^{(0)}_{ud;V}(s)$ and all non-π-pole contributions to $\rho^{(0)}_{ud;A}(s)$ proportional to $(m_d \pm m_u)^2$, and hence numerically negligible, results for $dR_{ud;V/A}/ds$ provide an experimental determination of $\rho^{(0+1)}_{ud;V/A}(s)$. The OPE representation of $\Pi^{(0+1)}_T(z)$ ($\Pi_T(z)$ in what follows) has the form

$$\Pi^{\rm OPE}_T(z) = \sum_{k=0}^{\infty} C^T_{2k}(z)/(-z)^k\,. \qquad (4)$$

The perturbative ($k = 0$) contribution to the theory side of Eq. (1) can be rewritten, by partial integration, in terms of the perturbative series for the scale-invariant Adler function, which is known to 5-loop order [2]. We use below the FOPT version of the integrated $D = 0$ term, with $\alpha_s(\mu^2)$ evaluated at $\mu^2 = s_0$, since the alternate, CIPT, version, with variable scale $\mu^2 = z$, was recently shown to be inconsistent with the standard OPE [3]. The $D = 2$ term, containing perturbative quark-mass-squared contributions, is numerically negligible for the isovector channel cases. The $C^T_{D\geq 4}$, which contain NP dimension-D condensate contributions, are independent of z up to logarithmic corrections suppressed by additional powers of α_s. Neglecting these corrections, and considering polynomial weights $w(y) = \sum_{m=0}^{N} b_m y^m$, with $y = s/s_0$, the term $b_m y^m$, $m \geq 1$ in $w(y)$ generates a single NP condensate contribution, $(-1)^m b_m C_{2m+2}/s_0^{m+1}$, on the theory side of Eq. (1).

An important feature of the experimental V and $V + A$ spectral functions is the presence of resonance-induced oscillations about perturbative expectations. Such oscillations, evident even in the upper part of the τ kinematic range, are not reproduced by a purely OPE representation and signal the presence of non-negligible DV contributions, connected to OPE breakdown in the vicinity of the Minkowski axis [4]. Even in the $V + A$ channel, where the oscillations are less prominent, they are not small, with ALEPH [5] results for $\rho_{V+A}(s)$ showing ratios of DV to α_s-dependent, dynamical perturbative QCD contributions ~ 1.27 and ~ -0.75 in the bins from 1.450 to 1.475 GeV^2 and 1.95 to 2.00 GeV^2, respectively. While no rigorous result exists for the form of the DV contribution, $\Delta_T(z)$, to $\Pi_T(z)$, the connection between DV contributions and the asymptotic nature of the OPE leads to the expectation that $\Delta_T(z)$ will decay exponentially as $|z| \to \infty$. For weights polynomial in s, the basic FESR relation can then be rewritten, suppressing spin labels, as [6]

$$\frac{1}{s_0}\int_0^{s_0} ds\, w(s)\, \rho_T(s) = -\frac{1}{2\pi i s_0}\oint_{|s|=s_0} dz\, w(z)\, \Pi_{\rm T;OPE}(z) - \frac{1}{s_0}\int_{s_0}^{\infty} ds\, w(s)\, \rho_{T;DV}(s)\,, \qquad (5)$$

with $\rho_{T;DV} \equiv [{\rm Im}\Delta_T]/\pi$ interpretable as the DV part of the channel T spectral function.

The localization of DV contributions to the vicinity of the Minkowski axis and expected exponential damping of $\rho_{T;DV}(s)$ with increasing s means DV contributions to a given FESR can be suppressed by increasing s_0 and/or employing "pinched weights" ($w(s)$ with a zero at $s = s_0$). FESR determinations of α_s in the literature have either (1) chosen to neglect DV contributions [2, 5, 8, 9, 11, 12], working at a single, as-high-as-possible s_0 and employing weights which are at least doubly pinched, or (2) used a Regge- and large-N_c-motivated model for DVs to estimate residual DV effects, employing multi-weight, multi-s_0 FESR fits to constrain the fitted OPE and DV parameters, and test the form of the DV representation, through the differing s_0- and weight-dependences of integrated DV contributions and OPE contributions of different dimensions [13–17, 21]. We refer to these approaches as the "truncated OPE model" (or "tOPE") and "DV model" approaches, respectively.

A complication for the single-$s0$ tOPE approach is that an at-least-doubly-pinched polynomial $w(s)$ used to suppress integrated DV contributions involves at least two positive integer powers of s. The associated FESR, which has a theory side depending on α_s and at least two C_D, thus represents one equation with ≥ 3 unknowns. Adding additional pinch-weighted FESRs does not solve this problem since each new FESR introduces at least one new C_D: the number of unknown OPE parameters always exceeds the number of single-s_0 spectral integrals available to fit them. In the tOPE approach, this problem is "solved" by throwing away the highest-D unsuppressed theory-side condensate contributions in principle present until the number of parameters remaining no longer exceeds the number of single-s_0 spectral integrals, hence the "truncated" in the tOPE name.

The unavoidable, in principle dangerous truncation-in-dimension of the tOPE approach is avoided in the DV model approach by using as experimental input spectral integrals at multiple s_0.

The disadvantage of this solution is that the additional s_0 are necessarily lower than the as-high-as-possible single s_0 used in a tOPE analysis of the same channel. Working at lower s_0 reduces the suppression of DV contributions, making their neglect less safe, and the inclusion of a representation of DV effects more important.

We turn now to a more detailed investigation of these two approaches, beginning with a newly discovered "redundancy" in the tOPE appproach and the re-interpretations of information obtained from such tOPE analyses this redundancy observation requires. In the subsequent section we show that this redundancy is avoided in the multi-weight, multi-s_0 analyses used in the DV model approach, and highlight the advantages of this approach.

2. The truncated OPE model approach

The first implementations of the tOPE approach [2, 8, 9] employed standard χ^2 fits to the $s_0 = m_\tau^2$ FESRs of the $km = 00, 10, 11, 12, 13$ "classic km spectral weights", $w_{km}(y) = w_\tau(y)(1-y)^k y^m$ [10]. The addition of the $km = 10, 11, 12, 13$ FESRs was meant to provide additional input to fix the $D = 6$ and 8 condensates appearing in the $km = 00$ FESR, allowing the remaining parameter in that FESR, α_s, to be determined. Since the 5 weights have degrees up to 7, however, the associated FESRs have theory sides depending on 8 parameters, α_s and the 7 condensates $C_{4,6,8,10,12,14,16}$. The reduction to a 4-parameter set obtainable from a fit to the 5 $s_0 = m_\tau^2$ spectral integrals was accomplished by dropping all contributions proportional to $C_{10,12,14,16}$, More recent tOPE analyses [11, 12], employing 2013 ALEPH $V + A$ data [5], worked at slightly lower $s_0 = 2.8$ GeV^2 to avoid the large errors in the last two ALEPH bins, and considered not only the classic km spectral weights but also alternate 5-FESR analyses based on either the $km = 00, 10, 11, 12, 13$ "modified km spectral weights", $\hat{w}_{km}(y) = (1-y)^{2+k} y^m$ or the $m = 1, 2, 3, 4, 5$ $(2m)$ "optimal weights", $w^{(2m)}(y) = 1 - (m+2)y^{m+1} + (m+1)y^{m+2}$. The weights w_{00}, $\hat{w}_{00}$ and the $w^{(2m)}$ are doubly pinched and the remainder of the w_{km} and $\hat{w}_{km}$ triply pinched. In the modified km spectral weight case, terms involving α_s, C_4, C_6 and C_8 were retained and in-principle-present unsuppressed contributions proportional to $C_{10,12,14}$ dropped. In the optimal weight case, no unsuppressed C_4 contributions appear, and the in-principle-present unsuppressed contributions dropped were those proportional to $C_{12,14,16}$. The conventional understanding of these analyses is that α_s is "basically" determined by the lowest-degree weight FESR (for the classic km spectral weight and optimal weight cases, the τ kinematic weight (w_{00}) FESR) [11] with results for the retained condensates determined mostly from the higher-degree-weight FESRs. As will be shown below, this is not, in fact, the case.

A key result for the discussion below, also discussed in Ref. [12], is the following "redundancy observation". Consider a data set $\{d_k,\ k = 1, \cdots, N\}$, with invertible covariance matrix C, and corresponding theoretical representations $\{t_k\}$, depending on $M < N$ theory parameters $\{p_k\}$ which one fits with a standard χ^2 minimization. Now expand the data set to include one more data point, d_{N+1}, such that the expanded covariance matrix is also invertible and the corresponding theory representation, t_{N+1}, introduces a single new parameter p_{M+1}. If one now performs the standard χ^2 fit of the expanded parameter set $p_1, \cdots, p_M, p_{M+1}$ to the extended data set, the results of that fit are "redundant", with (1) extended-fit results for both the χ^2 and previously fitted parameters $p_1, \cdots, p_M$ *identical* to those of the unextended fit and (2) the single new parameter p_{M+1} serving only to ensure the new theory representation t_{M+1} exactly matches the single new data point d_{N+1}. Adding the new data point has produced no new information on the previously fitted parameters and the single new parameter, p_{M+1}, is tautologically determined and hence physically unconstrained. A proof that this result, which is trivially obvious in the absence of correlations between d_{N+1} and the other data points, is true more generally is provided in Appendix D of Ref. [21].[1]

The consequences of the redundancy observation for some single-s_0, multi-weight tOPE analyses can be made more evident by changing the weight-function basis, taking advantage of the fact that fits employing the χ^2 constructed from the covariance matrix of the spectral integrals of any basis of the space spanned by the original weight set yield identical fit parameter and minimum χ^2 results. Useful alternate bases for the classic and modified classic km spectral weight cases are [21] $\{w'_k(y),\ k = 1, \cdots, 5\}$ with $w'_1(y) = 1 - (15/2)y^4 + 12y^5 - (17/2)y^6 + 3y^7$, $w'_2(y) = 1-9y^4+12y^5-4y^6$, $w'_3(y) = 1+2y^3-9y^4+6y^5$, $w'_4(y) = 1-3y^2+2y^3 = w_{00}(y)$ and $w'_5(y) = 1 + (2/3)y - (23/3)y^4 + 6y^5$ for the former and $\{\hat{w}'_{m+1}(y) = w^{(2m)}(y),\ m = 0, \cdots, 4\}$ for the latter. The w'_k can also be written in terms of the $w^{(2m)}$ with, e.g., $w'_1 = (3/2)w^{(23)} - w^{(24)} + (1/2)w^{(25)}$ and $w'_2 = (9/5)w^{(23)} - (4/5)w^{(24)}$.

For the classic km spectral weight case, consider building up the full 5-FESR fit starting with the 2-weight fit involving the w'_1 and w'_2 FESRs and adding, in order, the w'_3, w'_4 and w'_5 FESRs. With the tOPE $D = 8$ truncation, the theory sides of the w'_1 and w'_2 FESRs contain only perturbative contributions; the 2-weight fit thus returns a value for only the single OPE parameter α_s. The redundancy observation then implies that the 3-FESR fit obtained by adding the w'_3 FESR produces α_s and minimized χ^2 results identical to those of the 2-FESR fit, together with a redundant determination of the single new theory-side parameter, C_8, introduced by the w'_3 FESR. Similarly, (i) the 4-FESR fit obtained by adding the w'_4 FESR produces α_s, minimized χ^2 and C_8 results identical to those of the 3-FESR fit, plus a redundant determination of the new parameter, C_6, and (ii) the full 5-FESR fit, obtained by adding the final, w'_5, FESR produces α_s, minimized χ^2, C_8 and C_6 results identical to those of the 4-FESR fit, plus a redundant determination of the final retained parameter, C_4. The result for α_s is thus produced, not "basically" by the τ kinematic (w'_4) FESR, but by the purely perturbative theory-side 2-FESR fit involving the two highest-degree-weight FESRs in the analysis. All three retained C_D, moreover, are obtained by redundant one-spectral-integral-in-one-new-OPE-parameter-out matching, and have no physically constrained meaning.

A similar argument shows the results for α_s and the minimum χ^2 of the modified km spectral weight analysis are those obtained from a purely perturbative theory-side fit to the FESRs of the two highest-degree-weights ($w^{(23)}$ and $w^{(24)}$) of the alternate basis, with the retained NP condensates C_8, C_6 and C_4 obtained, redundantly, in that order, as the $w^{(2,2)}$, $w^{(2,1)}$ and, finally, $w^{(2,0)}$ FESRs are added to reach the full 5-FESR analysis.

From the discussion above, we see that α_s determinations in which both classic and modified km spectral weight analyses are considered rest on the assumption that NP contributions can be neglected on the theory sides of the FESRs of the three highest-degree members, $w^{(25)}$, $w^{(24)}$ and $w^{(23)}$, of the optimal-weight set. A 5-FESR optimal-weight analysis adhering (for self-consistency)

[1]This proof is not, however, generalizable to the case of non-χ^2 fits, which (see below) are unavoidable for multi-weight, multi-s_0 analyses. This point was apparently missed by the authors of Ref. [12], who claimed, without proof, that the redundancy result also holds for the (unavoidably non-χ^2) fits employed in our multi-weight, multi-s_0 analyses. As shown below, this claim is incorrect.

to this same assumption, thus obtains α_s from a purely perturbative theory-side 3-FESR fit involving the $w^{(25)}$, $w^{(24)}$ and $w^{(23)}$ FESRs, with the remaining retained OPE parameters, C_8 and C_6, obtained redundantly as the $w^{(22)}$ and $w^{(21)}$ FESRs are added. The absence of unsuppressed theory-side contributions proportional to C_4 in the $w^{(2m)}$ FESRs, however, allows for an alternate 5-FESR optimal-weight tOPE analysis with not just α_s, C_6, C_8 but also C_{10} retained; α_s is then determined from a purely perturbative theory-side 2-FESR fit to the $w^{(24)}$ and $w^{(25)}$ FESRs. With C_{10} no longer neglected, this version of the optimal-weight analysis is inconsistent with the classic and modified km spectral weight treatments, and those analyses must be jettisoned if this optimal-weight version is chosen. The redundancy observation is illustrated, for this version of the optimal-weight analysis, in Table 1.

weight labels $(2m)$	χ^2	$\alpha_s(m_\tau^2)$	$10^3 C_{10}$	$10^3 C_8$	$10^3 C_6$
$m = 5, 4$	3.068384	0.31685(0.00253)	—	—	—
$m = 5, 4, 3$	3.068384	0.31685(0.00253)	0.3464(0.1187)	—	—
$m = 5, 4, 3, 2$	3.068384	0.31685(0.00253)	0.3464(0.1187)	−0.8720(0.2107)	—
$m = 5, 4, 3, 2, 1$	3.068384	0.31685(0.00253)	0.3464(0.1187)	−0.8720(0.2107)	1.3771(0.2371)

Table 1: Redundancy of the ALEPH-based, $s_0 = 2.8$ GeV^2, $C_{D>10} = 0$, $V + A$-channel optimal-weight tOPE analysis. C_D in units of GeV^D. Lines 1-4 contain the results of fits to the 2- to 5-FESR subsets involving the $(2m)$ optimal weights specified by the m listed in Column 1.

The redundant, one-new-spectral-function-in-one-new-C_D-out nature of the tOPE NP condensate determinations makes these determinations extremely sensitive to any residual NP contributions actually present, but neglected in the purely perturbative theory-side treatments of the FESRs responsible for determining α_s. Consider, for example, the classic km spectral weight determination of C_8, produced by exactly matching the spectral integral and theory sides of the w_3' FESR. As per the discussion above, C_8 is then proportional to the difference between the w_3' spectral integral and the $D = 0$ w_3' theory-side contribution, with the latter fixed by the α_s obtained from the previous-stage, purely perturbative combined w_1' and w_2' FESR fit. Any NP theory-side contributions to the w_1' and/or w_2' FESRs which are not, in fact, negligible, will contaminate the result for α_s and hence also the result for C_8. With $D = 0$ contributions numerically dominant on the theory sides, a very close cancellation will be unavoidably present in the difference between the w_3' spectral integral and $D = 0$ theory-side terms. This close cancellation will strongly amplify any residual NP contamination of the result for α_s. This amplification is an unavoidable feature of the way in which NP condensates are determined in single-s_0 tOPE analyses. Some specific numerical examples of the very large systematic uncertainties even relatively modest NP contaminations of α_s can produce are provided in Sec. 4 of Ref. [21].

While the redundancy of tOPE NP condensate determinations precludes the existence of internal self-consistency checks of those results, limited internal self-consistency tests *are* available for the associated α_s results. Consider, e.g., the three 5-FESR tOPE analyses above. In those analyses, α_s is obtained from combined fits to either two or three FESRs with purely perturbative theory sides. Each such FESR, however, also provides a single-FESR α_s determination. The set consisting of the two- or three-FESR combined fit result and associated single-FESR determinations can be tested for consistency by checking the differences between pairs of these results for consistency with zero, within properly correlated errors. The set of such self-consistency tests is largest for the version of the optimal-weight analysis with $C_{D>8} = 0$ truncation, where there are three, rather than just two, single-FESR determinations, and the tOPE assumptions are, moreover, compatible with those of the classic and modified km spectral weight analyses.

As an illustration, consider the $s_0 = 2.8$ GeV^2, $V + A$ channel, $C_{D>8} = 0$-truncated, optimal-weight tOPE analysis, using 2013 ALEPH experimental input [5], with updated branching fraction input. The 3-FESR ($w^{(25)}$, $w^{(24)}$ and $w^{(23)}$) fit which determines α_s in this case, produces [21] a χ^2/dof of 11.6/2 and a result, with experimental error, $\alpha_s(m_\tau^2) = 0.3125(23)_{\rm ex}$, which differs, e.g., from the single-FESR, $w^{(25)}$ determination, $\alpha_s(m_\tau^2) = 0.3228(43)_{\rm ex}$, by $0.0103(37)_{\rm ex}(10)_{\rm th} = 0.0103(38)$, where the second error is an estimate of the theory uncertainty induced by the $O(\alpha_s^5)$ $D = 0$ series truncation and the strong correlations have been taken into account in evaluating both error components. These differences represent a systematic effect not quantified in Refs. [11, 12]. See Ref. [21] for further details.

Evidence that this non-trivial discrepancy may result from contamination by residual higher-D NP contributions, removed by the tOPE truncation but in fact still present, in the $w^{(25)}$ and/or $w^{(24)}$ and/or $w^{(23)}$ FESRs, is provided by the results of the same, optimal-weight, analysis at the next two lowest s_0 ($s_0 = 2.6$ and 2.4 GeV^2) accessible with the ALEPH binning. These, like $s_0 = 2.8$ GeV^2, lie in the region of compatibility (within experimental errors) of the measured and perturbative versions of the V+A spectral function, making neglect of DV contributions at these s_0 as plausible as it was at $s_0 = 2.8$ GeV^2. If the discrepancy at $s_0 = 2.8$ GeV^2 is, indeed, due to omitted, but in fact non-negligible, higher D OPE terms, then, since such contributions scale as $1/s_0^{D/2}$, the fit quality should deteriorate and the significance of any discrepancies grow as s_0 is decreased. This is, indeed, what is found; see Ref. [21] for details.

The same internal self-consistency tests can also be performed in the V channel, which has the advantage of significantly reduced experimental errors in the upper part of the τ kinematic range made possible by the use of $e^+e^- \to hadrons$ cross section data, in combination with CVC, for the contributions from higher-multiplicity modes which play a numerically important role in this region [19]. Direct τ-based determinations of these contributions suffer from the low statistics/large statistical errors which are unavoidable near the kinematic endpoint. This improvement, which is possible only in the V channel, helps sharpen the precision of the self-consistency tests. Details of the CVC improvement can be found in Refs. [19, 20]. Since the improved $\rho_V(s)$ [19] and ALEPH $V + A$ binnings differ, we choose the first V-channel s_0, 2.88 GeV^2, greater than the $s_0 = 2.8$ GeV^2 used in $V + A$-channel value study above. The discrepancies between pairs of single-FESR ($w^{(25)}$, $w^{(24)}$ and $w^{(23)}$) α_s determinations are so large ($\sim 6 - 11\,\sigma$) that the corresponding combined 2-FESR and 3-FESR fits have extremely large χ^2/dof, with even the less disastrous combined 2-FESR ($w^{(25)}$ and $w^{(24)}$) fit producing $\chi^2/dof = 43.1$ [21].

3. The DV model approach

A major disadvantage of the single-s_0 tOPE approach is that, with the restriction to a single s_0, the possibility of using the s_0 dependence of the spectral integrals entering the analysis to disentangle theory-side contributions with different s_0 dependences is lost. This is a particular problem for the treatment of NP contributions where the redundant nature of NP condensate determinations precludes any test of whether NP theory-side contributions scale as they should with s_0. It is, therefore, of interest to consider multi-s_0 analyses, bearing in mind the growth of residual DV contributions expected with decreasing s_0. With no rigorous QCD-based result known for $\Delta_T(s)$ or $\rho_{T;DV}(s)$, this possibility can only be investigated using a model for DVs, albeit, in what follows, one motivated by qualitative features widely believed to hold in QCD. In Ref. [7], we developed a theoretical framework for quark-hadron DVs in terms of a generalized Borel–Laplace transform of $\Pi_T(q^2)$ and hyperasymptotics. In the chiral limit, assuming the spectrum becomes Regge-like at large s as $N_c \to \infty$, we showed that the large-s form of $\rho_{T;DV}(s)$ can be parametrized as

$$\rho_{T;DV}(s) = e^{-\delta_T - \gamma_T s} \sin(\alpha_T + \beta_T s) , \tag{6}$$

up to (i) slowly varying logarithmic corrections and (ii) a more rapidly varying, overall correction factor $1 + (c_T/s)$. The form Eq. (6) has been used in the $T = V$ channel studies outlined below, with the parameters $\alpha_V, \beta_V, \gamma_V$ and δ_V to be determined, together with α_s and the condensates C_D, from fits to multi-s_0, typically multi-weight spectral integral sets. With Eq. (6) valid only at sufficiently large s, one assumes the region of validity overlaps with the τ kinematic range. This assumption is investigated, and a range of s_0 for which it is compatible with data identified, by considering fits with s_0 in ranges from $s_0^{\rm min}$ to m_τ^2, studying the quality of the fits and fit parameter stability as a function of $s_0^{\rm min}$.

A second, practical issue also arises for multi-weight versions of such multi-s_0 analyses. Explicitly, to take maximal advantage of the additional s_0-dependence constraints, one typically considers all $s_0 \geq s_0^{\rm min}$ the experimental binning allows. For such s_0, consider starting with a first set, $\{I^{w_1}(s_0^{(k)}), k = 1, \cdots, N\}$, of w_1-weighted spectral integrals, and adding a second, higher-degree w_2-weighted set, $\{I^{w_2}(s_0^{(k)})\}$. The theory side of the w_2 FESR then involves at least one NP condensate not present on the w_1 FESR theory side. The initial spectral integral set is necessarily linearly independent and has an invertible covariance matrix. The same is true of the second set, considered on its own. However, if one considers expanding the initial set by adding members of the second set, it turns out that only one of the w_2-weighted spectral integrals can be added before the expanded set becomes linearly dependent [12, 21]. The covariance matrix of the full combined multi-s_0, two-weight set is thus singular, with $N - 1$ zero eigenvalues, making it impossible to construct the standard χ^2 minimizer and perform a standard χ^2 fit to that expanded set. One is left with two options: to either throw away all but one (say, that with $s_0 = \hat{s}_0$) of the w_2-weighted spectral integrals and perform a standard χ^2 fit to the one-point-expanded initial spectral integral set, or retain the full multi-s_0, two-weight set, but perform a fit using a different miminizing function. The first option suffers from the redundancy problem: by adding the new, w_2, FESR at only the single new value $s_0 = \hat{s}_0$, the χ^2 fit to the one-point-expanded set leaves the χ^2 and fit parameter values obtained from the multi-s_0, w_1 FESR fit unchanged and provides only a physically unconstrained, redundant determination of the nominal $s_0 = \hat{s}_0$ value of whatever new C_D combination is introduced by the w_2 FESR. The whole point of adding the second weight set is thus lost. We, therefore, choose instead the second option, performing a fit to the "block-diagonal Q^2" minimizer

$$Q^2 = \chi^2_{w_1} + \chi^2_{w_2} , \tag{7}$$

with $\chi^2_{w_k}$ the χ^2 of the w_k-weighted spectral integral set. The full covariance matrix of the expanded two-weight, multi-s_0 set is, of course, propagated through the fit when determining fit parameter errors and correlations.

A specific example is useful to illustrate why, starting from a multi-s_0, w_1-weighted FESR set, the addition of a second, w_2-weighted set, at the same s_0, generates new constraints on the theory representations employed. The associated discussion will also serve to address, and demonstrate to be incorrect, the claim made in Ref. [12], based on a suggestive argument which fails to take into account the unavoidably non-χ^2 nature of such fits, that such two-weight, multi-s_0 fits are necessarily redundant, with the w_2 FESR providing no new constraints on parameters fixed previously in the w_1 FESR fit.

Consider a V-channel, multi-s_0, two-FESR, block-diagonal Q^2 fit with $w_1(y) = 1$ and $w_2(y) = 1 - y^2$, employing theory-side representations consisting of the usual NP and perturbative OPE contributions, supplemented by DV contributions modeled using Eq. (6). The theory side of the w_1 FESR then receives only perturbative OPE and DV contributions. Starting with the w_1 FESR, the standard χ^2 fit to the set of w_1-weighted spectral intgrals thus produces results for α_s and the DV parameters α_V, β_V, γ_V and δ_V. Adding the w_2 FESR introduces the single new theory parameter, C_6. If the DV model employed provided a perfect representation of DV effects in the given s_0 fit window, the nominally $D = 6$ OPE w_2 FESR residuals (plural) produced by subtracting from the w_2-weighted spectral integrals the sums of the corresponding w_2 $D = 0$ perturbative and DV contributions evaluated using w_1 fit results as input would be proportional to C_6 and scale with s_0 as $1/s_0^3$. If the DV representation is good, but not perfect, this will be approximately true and the combined, two-weight block-diagonal Q^2 fit will return α_s and DV parameter values close, but not identical, to those obtained from the w_1 fit. If, on the other hand, the DV representation is not good, the residuals will not scale properly with s_0 and, if one nonetheless tries to force them do so by using the QCD form, $-C_6/s_0^3$, for the $D = 6$ OPE, w_2 FESR theory-side contribution, the w_2-only χ^2 part of Q^2 will be non-negligible, since no single choice of C_6 will allow the theory and spectral integral sides of the w_2 FESR to match for all s_0 in the fit window. The two-weight, block-diagonal Q^2 fit will deal with this through adjustments to the α_s and DV parameter values obtained from the w_1 fit, increasing the w_1 χ^2 contribution to Q^2, while decreasing the w_2 χ^2 contribution. The minimized block-diagonal Q^2 will have a w_1 χ^2 component larger than that obtained from the w_1 χ^2 fit. Were the claim made in Ref. [12] to be true, in contrast, regardless of whether the DV model provides a good or poor representation of physical DV effects, the two-weight block-diagonal Q^2 fit would return values for α_s, the DV parameters, and hence also the χ^2 of the w_1 χ^2 part of Q^2, *identical* to those of the single-weight w_1 fit. The results of $s_0^{min} = 1.5863$ GeV^2, w_1 χ^2 and block-diagonal, two-weight (w_1 and w_2), Q^2 fits performed in Ref. [19] are shown in Table 1. While close, they are not identical, unambiguously establishing that the redundancy claim made in Ref. [12] is incorrect.

weight	$\alpha_s(m_\tau^2)$	δ_V	γ_V	α_V	β_V	$10^2\, C_6$
w_1	0.3056(64)	3.61(30)	0.52(18)	−1.62(51)	3.95(26)	—
w_1&w_2	0.3073(69)	3.50(31)	0.58(19)	−1.57(55)	3.92(29)	−0.62(13)

Table 2: Results of $s_0^{min} = 1.5863$ GeV^2, w_1 χ^2 and two-weight w_1 and w_2 block-diagonal Q^2 fits based on $\rho_V(s)$ from Ref. [19]. C_6 is in units of GeV^6, β_V and γ_V in units of GeV^{-2}.

To illustate the practical utility of the additional theory constraints produced by multi-weight, multi-s_0 block-diagonal Q^2 fits, consider a modified version of the above two-weight analysis with the w_2 FESR theory representation deliberately chosen to differ from QCD expectations. Explicitly, in the representation of $\Pi_V(z)$ on the FESR contour, consisting of perturbative, $D \geq 4$ NP condensate and DV contributions, imagine replacing the $D = 6$ OPE contribution expected in QCD, C_6/z^3, with the manifestly non-QCD term

$$\Delta\Pi_{\text{non-QCD}}(z) = C' \left[\log\left(-z/\mu^2\right) - 2\log^2\left(-z/\mu^2\right)\right]/z^5 \,. \tag{8}$$

The FOPT ($\mu^2 = s_0$) versions of the w_1- and w_2-weighted $\Delta\Pi_{\text{non-QCD}}(z)$ contour integrals are easily shown to be 0 and C'/s_0^5, respectively. The non-QCD representation of $\Pi_V(z)$ thus produces the same w_1 theory-side contribution as does the expected QCD form, but an altered NP w_2 theory-side contribution with C'/s_0^5 replacing the expected $-C_6/s_0^3$ QCD term. The single-weight w_1 fit is, in this case, incapable of distinguishing between the non-QCD and expected QCD versions of the theory representation.

We now explore what happens when one adds the two-weight (w_1 and w_2) block-diagonal Q^2 fit, comparing, to be specific, α_s results obtained in Ref. [19] from V-channel w_1 spectral integral fits in windows $s_0^{\text{min}} \leq s_0 \leq 3.06$ GeV^2, with $s_0^{\text{min}} \geq 1.55$ GeV^2, to those obtained from two two-weight (w_1 and w_2) block-diagonal Q^2 fits in the same s_0 windows, using, in one case, the *non-QCD* form, C'/s_0^5, for the NP, non-DV contribution to the theory side of the w_2 FESR, and in the other the expected QCD form, $-C_6/s_0^3$.

According to the argument of Ref. [12], since the second weight w_2 introduces the single new parameter (C_6 or C', depending on which w_2 theory representation is chosen), the combined fits should simply determine the new parameter (C_6 or C'), but otherwise be completely redundant, leaving the result for α_s obtained from the w_1 fit unchanged, regardless of which form of the non-DV, NP contribution is used.

Figure 1 shows the results of this exercise, as a function of s_0^{min}. The blue squares show the results for $\alpha_s(m_\tau^2)$ obtained from the single-weight, w_1, fit, the green crosses those from the combined w_1 and w_2 fit with the unmodified, expected QCD theory representation, and the red circles those from the combined fit with the modified *non*-QCD representation. The single-weight (w_1) and two-weight (w_1 and w_2) fit results produced by the non-QCD representation are far from equal, with a systematic upward shift immediately evident for all red points. This is in contrast to the excellent agreement between the single- and two-weight results (shown by the blue and green points, respectively) obtained when the expected QCD $D = 6$ NP form is used. The first of these observations establishes that, contrary to the claim of Ref. [12], the implementation of the additional multi-s_0 constraints produced by adding the w_2 FESR, using the block-diagonal-Q^2 two-weight fit form, are highly non-trivial, allowing, in this case, the non-QCD theory representation to be ruled out on self-consistency grounds alone. The existence of the inconsistency in this case, moreover, establishes that the agreement between single-weight (w_1) and combined two-weight (w_1 and w_2), block-diagonal-Q^2 fit results produced when the expected QCD non-DV, NP form is used represents a non-trivial self-consistency test on the underlying theory representation.

Analogous tests of the consistency of results from single-weight w_1 χ^2 and two-weight, block-diagonal Q^2 fits employing the DV model theory representation was also performed in Ref. [19],

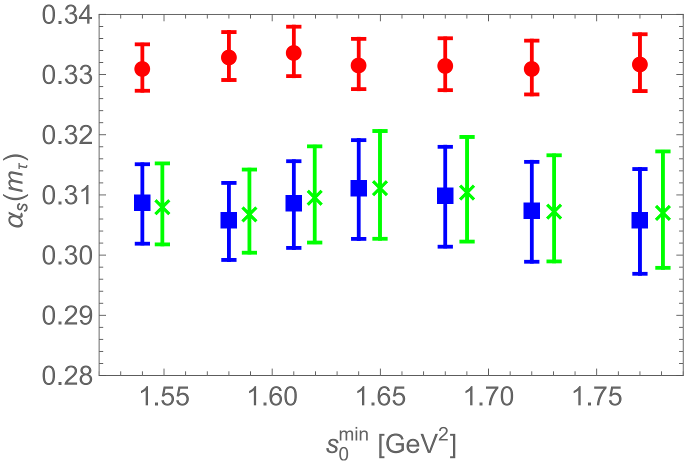

Figure 1: Comparison of $\alpha_s(m_\tau^2)$ results from the multi-s_0, V-channel $w_1 = 1$ χ^2 FESR fits described in the text (blue points) to those of two-weight, w_1 and w_2, block-diagonal Q^2 fits, in the same s_0 fit windows. The green points show the latter for the case of the theory representation that produces the non-DV, NP w_2 FESR theory-side contribution, $-C_6/s_0^3$, expected in QCD, the red points those for the modified representation that produces the alternate, non-QCD form C'/s_0^5. The green points are plotted with a small horizontal offset, for visual clarity.

with w_2 replaced by one of the doubly pinched weights $w_3 = w_{00}$ or $w_4(y) = 1 - 2y^2 + y^4$. Excellent consistency was again observed. The full set of the consistency tests performed in Ref. [19] were also performed in Ref. [20], using a further updated version of $\rho_V(s)$. The same excellent consistency was observed. The Ref. [19] result for $\alpha_s(m_\tau^2)$, 0.3077(75), was obtained from an average over results in a 7-point s_0^{min} stability window, and the analogous, update from Ref. [20], 0.2983(101), from a 10-point stability-region average. The lower central value and larger error of the updated result are due largely to the significant 9% downward shift in central value, and factor of 2.4 increase in error, of the HFLAV22 [22] $\tau \to \pi^- 3\pi^0$ branching fraction, used to normalize the $\pi^- 3\pi^0$ spectral distribution in Ref. [20], relative to the HFLAV19 [23] value used in Ref. [19]. Of interest for future improvements to this analysis is the observation that Pais relation expectations [24] for the two 4π mode τ spectral distributions based on current $e^+e^- \to 4\pi$ cross-section data are not in agreement with currently measured τ results within the $\sim 1\%$ or so expected for isospin-breaking corrections to the Pais relations. The smaller Pais-relation errors for s greater than ~ 2 GeV^2, combined with the sensitivity of α_s to the input 4π branching fraction normalizations, reflected in the difference of the results from Refs. [19] and [20], provides motivation for future improvements to the τ 4π data situation.

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