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Two-loop QCD corrections to pion electromagnetic form factors

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We calculate the next-to-next-to-leading order (NNLO) QCD radiative correction to the pion electromagnetic form factor with large momentum transfer for the first time. We explicitly verify the validity of the collinear factorization to two-loop order for this observable, and obtain the respective IR-finite two-loop hard-scattering kernel in the closed form. The NNLO QCD correction turns to be positive and significant. Incorporating this new ingredient of correction, we then make a comprehensive comparison between the finest theoretical predictions and numerous data for both space-like and time-like pion form factors. Our phenomenological analysis provides strong constraint on the second Gegenbauer moment of the pion light-cone distribution amplitude (LCDA) obtained from recent lattice QCD studies.

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1. Introduction

Originally proposed by Yukawa as the strong nuclear force carrier in 1935 [1], subsequently discovered in the cosmic rays in 1947 [2], the π mesons have always occupied the central stage throughout the historic advancement of the strong interaction. A classic example of probing the internal structure of the charged π is the pion electromagnetic (EM) form factor:

$$\langle \pi^+(P')|J^\mu_{\rm em}|\pi^+(P)\rangle = F_\pi(Q^2)(P^\mu + P'^\mu), \tag{1}$$

with $Q^2 \equiv -(P-P')^2$. The electromagnetic current is defined by $J^\mu_{\rm em} = \sum_f e_f \bar{f}\gamma^\mu f$, with $e_u = 2/3$ and $e_d = -1/3$ indicating the electric charges of the u and d quarks.

At the lowest order in $1/Q$, the pion EM form factor can be expressed in the following form:

$$F_\pi(Q^2) = \iint dx\, dy\, \Phi^*_\pi(x,\mu_F) T(x,y,\frac{\mu_R^2}{Q^2},\frac{\mu_F^2}{Q^2})\Phi_\pi(y,\mu_F), \tag{2}$$

where $T(x,y)$ signifies the perturbatively calculable hard-scattering kernel, and $\Phi_\pi(x,\mu_F)$ represents the nonperturbative yet universal leading-twist pion light-cone distribution amplitude (LCDA), *i.e.*, the probability amplitude of finding the valence u and $\bar{d}$ quark inside π^+ carrying the fractional momenta x and $\bar{x} \equiv 1-x$, respectively. The leading-twist pion LCDA assumes the following operator definition:

$$\Phi_\pi(x,\mu_F) = \int \frac{dz^-}{2\pi i} e^{iz^- xP^+} \left\langle 0 \left| \bar{d}(0)\gamma^+\gamma_5 \mathcal{W}(0,z^-) u(z^-) \right| \pi^+(P) \right\rangle, \tag{3}$$

with $\mathcal{W}$ signifies the light-like gauge link to ensure the gauge invariance. Conducting the UV renormalization for (3), one is led to the celebrated Efremov-Radyushkin-Brodsky-Lepage (ERBL) evolution equation [3, 4]:

$$\frac{d\Phi_\pi(x,\mu_F)}{d\ln\mu_F^2} = \int_0^1 dy\, V(x,y)\, \Phi_\pi(y,\mu_F), \tag{4}$$

with $V(x,y)$ referring to the perturbatively calculable ERBL kernel.

Eq. (2) is expected to hold to all orders in perturbative expansion. The hard-scattering kernel can thus be expanded in the power series:

$$T = \frac{16 C_F \pi\alpha_s}{Q^2}\left\{T^{(0)} + \frac{\alpha_s}{\pi}T^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 T^{(2)} + \cdots\right\}, \tag{5}$$

with $C_F = \frac{N_c^2-1}{2N_c}$, and $N_c = 3$ is the number of colors.

2. Description of the calculation

In the left-hand side of (2), we extract the scalar form factor $F(u,v)$ through the partonic reaction $\gamma^* + u(uP)\bar{d}(\bar{u}P) \to u(vP')\bar{d}(\bar{v}P')$. Some typical Feynman diagrams through two-loop order are depicted in Fig. 1. It is subject to a perturbative expansion:

$$F(u,v) = \frac{16 C_F \pi\alpha_s}{Q^2}\left[F^{(0)}(u,v) + \frac{\alpha_s}{\pi}F^{(1)}(u,v) + \left(\frac{\alpha_s}{\pi}\right)^2 F^{(2)}(u,v) + \cdots\right]. \tag{6}$$

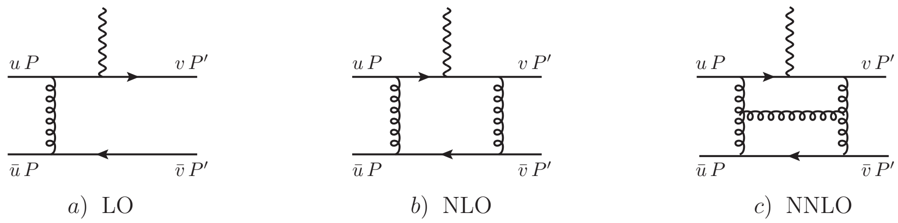

Figure 1: Sample parton-level Feynman diagrams for the reaction $\gamma^*\pi(P) \to \pi(P')$ at various perturbative orders.

In the right-hand side of (2), one can expand the renormalized “pion" LCDA as

$$\Phi(x|u) = \Phi^{(0)}(x|u) + \frac{\alpha_s}{\pi}\Phi^{(1)}(x|u) + \left(\frac{\alpha_s}{\pi}\right)^2 \Phi^{(2)}(x|u) + \cdots . \tag{7}$$

At tree level, the fictitious pion LCDA in (3) simply reduces to $\Phi^{(0)}(x|u) = \delta(x-u)$ (up to a normalization factor which also appears in $F(u,v)$). By equating both sides of (2), one reproduces the well-known tree-level expression $T^{(0)}(x,y)$ [3–9]

$$T^{(0)}(x,y) = \frac{e_u}{\bar{x}\bar{y}}(1-\epsilon) - \begin{bmatrix} e_u \to e_d \\ \bar{x} \to x, \bar{y} \to y \end{bmatrix}, \tag{8}$$

which holds true in $d = 4 - 2\epsilon$ spacetime dimension.

Once beyond the tree level, the UV and IR divergences inevitably arise and we use the dimensional regularization (DR) to regularize both types of divergences. Nevertheless, the bare “pion" LCDA remains intact since the scaleless integrals vanish in DR. The renormalized “pion" LCDA is related to the bare one via

$$\Phi(x|u) = \int dy Z(x,y)\Phi_{\text{bare}}(y|u) = Z(x,u), \tag{9}$$

which is solely comprised of various IR poles.

$Z(x,y)$ in (9) signifies the renormalization function in the $\overline{\text{MS}}$ scheme, which can be cast into the following Laurent-expanded form in ϵ:

$$Z(x,y) = \delta(x-y) + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} Z_k(x,y), \tag{10}$$

Note that the prefactor of single pole in (10) is related to the ERBL kernel $V(x,y)$ in (4) via $V(x,y) = -\alpha_s \partial Z_1/\partial\alpha_s$ [10]. Note that the two-loop [11–15] and three-loop corrections [16] to $V(x,y)$ have been available.

The two-loop renormalized “pion" LCDA $\Phi^{(2)}$ also contains double IR pole. The Z_2 can be obtained through the recursive relation [17]

$$\alpha_s \frac{\partial Z_2}{\partial \alpha_s} = \alpha_s \frac{\partial Z_1}{\partial \alpha_s} \otimes Z_1 + \beta(\alpha_s)\frac{\partial Z_1}{\partial \alpha_s}, \tag{11}$$

where $d\alpha_s/d\ln\mu^2 = -\epsilon\alpha_s + \beta(\alpha_s)$.

With the aid of (9) and (10), we then determine the $O(\alpha_s)$ and $O(\alpha_s^2)$ corrections to the renormalized "pion" LCDA in (7). At one-loop order, the matching equation for a fictitious pion state becomes

$$Q^2F^{(1)}(u,v) \quad = \quad T^{(1)}(u,v)+\Phi^{(1)}(x|u)\underset{x}{\otimes}T^{(0)}(x,v)+\Phi^{(1)}(y|v)\underset{y}{\otimes}T^{(0)}(u,y), \tag{12}$$

where $\underset{x}{\otimes}$ signifies the convolution over x. Note that the renormalized scalar form factor $F^{(1)}(u,v)$ still contains single collinear pole. However, the renormalized $\Phi^{(1)}(x|u)$ and $\Phi^{(1)}(y|v)$ also contains the same IR poles. Upon solving this matching equation, one ends with both UV and IR-finite $T^{(1)}(x,y)$. Our expressions agree with the known NLO result [18].

To the desired two-loop order, the following matching equation descends from (2):

$$Q^2F^{(2)}(u,v) = T^{(2)}(u,v)+\Phi^{(2)}(x|u)\underset{x}{\otimes}T^{(0)}(x,v)+\Phi^{(2)}(y|v)\underset{y}{\otimes}T^{(0)}(u,y)$$
$$+\Phi^{(1)}(x|u)\underset{x}{\otimes}T^{(1)}(x,v)+\Phi^{(1)}(y|v)\underset{y}{\otimes}T^{(1)}(u,y)+\Phi^{(1)}(x|u)\underset{x}{\otimes}T^{(0)}(x,y)\underset{y}{\otimes}\Phi^{(1)}(y|v) \tag{13}$$

More Severe IR divergences are expected to arise in both $F^{(2)}(u,v)$ and $\Phi^{(2)}(x|u)$. Clearly one also needs compute $T^{(1)}(x,y)$ to $O(\epsilon)$.

We use `HepLib` [19] and `FeynArts` [20] to generate Feynman diagrams and the corresponding amplitudes for the reaction $\gamma^*+u(uP)\bar{d}(\bar{u}P)\to u(vP')\bar{d}(\bar{v}P')$. We employ the covariant projector technique to enforce each $u\bar{d}$ pair to bear zero helicity. For our purpose it suffices to adopting the naive anticommutation relation to handle γ_5 in DR. There are about 1600 two-loop diagrams, one of which is depicted in Fig. 1c). We employ the package `Apart` [21] to conduct partial fraction, and `FIRE` [22] for integration-by-part reduction. We end up with 116 independent master integrals (MIs). The MIs are calculated by utilizing the differential equations method [23–25].

Upon renormalizing the QCD coupling in $\overline{\text{MS}}$ scheme, we end up with a lengthy expression for $F^{(2)}(u,v)$. Being UV finite, it still contains severe IR divergences which start at order-$1/\epsilon_{\text{IR}}^2$. Inspecting the matching equation (13), piecing all the known ingredients together, we are able to solve for the intended two-loop hard-scattering kernel. Hearteningly, $T^{(2)}(x,y)$ is indeed IR finite, verifies that the collinear factorization does hold at two-loop level for the pion EM form factor.

The leading-twist pion LCDA is conveniently expanded in the Gegenbauer polynomial basis: $\Phi_\pi(x,\mu_F)=\frac{f_\pi}{2\sqrt{2N_c}}\sum_{n=0}{}' a_n(\mu_F)\psi_n(x)$, $\psi_n(x)=6x\bar{x}C_n^{3/2}(2x-1)$, where the pion decay constant $f_\pi=0.131$ GeV, and $\sum'$ signifies the sum over even integers. Note all the nonperturbative dynamics is encoded in the Gegenbauer moments $a_n(\mu_F)$. We reexpress the pion EM form factor as

$$Q^2F_\pi(Q^2)=\frac{2C_F\pi^2(e_u-e_d)f_\pi^2}{3}\sum_{k=0}\left(\frac{\alpha_s}{\pi}\right)^{k+1}\sum_{m,n}{}' a_n(\mu_F)a_m(\mu_F)\mathcal{T}_{mn}^{(k)}, \tag{14}$$

with $\mathcal{T}_{mn}^{(k)}$ defined by

$$\mathcal{T}_{mn}^{(k)}=\frac{1}{e_u-e_d}\psi_m(x)\underset{x}{\otimes}T^{(k)}\left(x,y,\frac{\mu_R^2}{Q^2},\frac{\mu_F^2}{Q^2}\right)\underset{y}{\otimes}\psi_n(y). \tag{15}$$

For simplicity, we will set $\mu_R=\mu_F=\mu$ and $n_L=3$ from now on. The two-fold integrations in (15) can be readily worked out at tree and one-loop levels. For instance, we have $\mathcal{T}_{mn}^{(0)}=9, \mathcal{T}_{00}^{(1)}=\frac{1}{4}(81L_\mu+237)$, with $L_\mu\equiv\ln(\mu^2/Q^2)$.

Remarkably, the two-loop coefficients $\mathcal{T}_{mn}^{(2)}$ can also be computed analytically, thanks to the fact that $T^{(2)}$ can be expressed in terms of the GPLs. Although the integrand in (15) contains about $O(10^5)$ individual terms, the final result after two-fold integration becomes exceedingly compact, which can be expressed in terms of the rational numbers and Riemann zeta function. For instance, the expression of $\mathcal{T}_{00}^{(2)}$ reads

$$\mathcal{T}_{00}^{(2)} = \frac{729L_\mu^2}{16} - (8\zeta_3 + \frac{35\pi^2}{6} - \frac{2961}{8})L_\mu + 205\zeta_5 - \frac{3\pi^4}{20} - \frac{651\zeta_3}{2} - \frac{275\pi^2}{24} + 821. \tag{16}$$

In Table 1 we tabulate the numerical values of $\mathcal{T}_{mn}^{(1,2)}$ for $0 \le m, n \le 6$, which is sufficient for most phenomenological analysis.

(m,n)	c_1	c_2	d_1	d_2	d_3
(0,0)	20.25	59.25	45.5625	302.936	514.600
(0,2)	32.75	112.473	96.4306	735.637	1498.75
(0,4)	38.45	147.638	125.390	1049.88	2346.49
(0,6)	42.2571	174.359	146.743	1306.76	3103.33
(2,2)	45.25	192.871	164.660	1513.95	3690.60
(2,4)	50.95	240.181	201.536	2024.03	5371.58
(2,6)	54.7571	274.974	228.176	2424.45	6796.36
(4,4)	56.65	292.970	242.021	2640.85	7589.17
(4,6)	60.4571	331.411	271.074	3118.35	9432.36
(6,6)	64.2643	372.282	301.736	3651.19	11598.1

Table 1: The numerical values for $\mathcal{T}_{mn}^{(1)} = c_1 L_\mu + c_2$ and $\mathcal{T}_{mn}^{(2)} = d_1 L_\mu^2 + d_2 L_\mu + d_3$, with $0 \le m, n \le 6$.

3. Phenomenological exploration

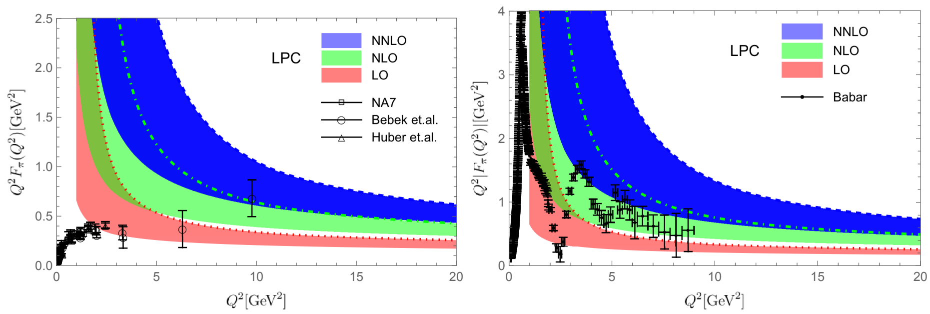

Figure 2: Theoretical predictions vs. data for $Q^2F_\pi(Q^2)$ in the space-like (left panel) and time-like (right panel) regions. We take the central values of a_2, a_4 and a_6 determined by LPC. The red, green and blue curves correspond to the LO, NLO and NNLO results, and the respective bands are obtained by sliding μ from $Q/2$ to Q. Experimental data points are taken from `NA7` [26], `Bebek et al.` [27], `Huber et al.` [28] and `BaBar` [29].

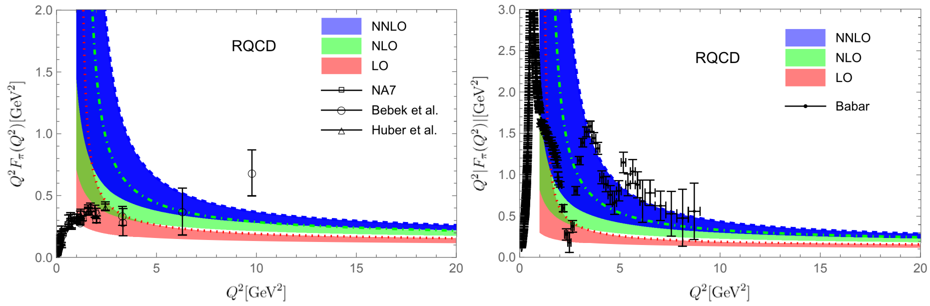

Figure 3: Same as Fig. 2, except the predictions are made by taking the central value of a_2 determined by RQCD, with a_4 and a_6 set to zero.

In Fig. 2 and Fig. 3, we confront our predictions at various perturbative accuracy with the available data, including both space-like and time-like regime. One clearly visualizes that NNLO correction is positive and substantial. In Fig. 2, we set the various Gegenbauer moments of pion LCDA to the central values given by LPC Collaboration [30]. It appears the NNLO predictions are significantly overshooting the experimental data at large Q^2 ($>$ 5 GeV^2), especially for the time-like regime with high statistics data. This symptom is mainly due to the large value of a_2.

In Fig. 3 we present our predictions with a_2 taken from RQCD while setting the values of a_4 and a_6 to zero. We find satisfactory agreement between our NNLO predictions and the data, both in space-like and time-like regimes. This may indicates that the value of a_2 given by RQCD might be more trustworthy. It is of utmost importance for RQCD and LPC collaborations to settle the discrepancy in the value of a_2 in the future. The prospective Electron-Ion Collider (EIC) program plans to measure the space-like pion EM form factor with Q^2 as large as 30 GeV^2 [31], where perturbative QCD should be definitely reliable. We are eagerly awaiting to confronting our NNLO predictions with the future EIC data.

4. Conclusion

We report the first calculation of the two-loop QCD corrections to the pion electromagnetic form factor. We have explicitly verified the validity of the collinear factorization to two-loop order for this observable, and obtain the UV, IR-finite two-loop hard-scattering kernel in closed form. The NNLO QCD correction turns to be positive and important. We then confront our finest theoretical predictions with various space-like and time-like pion form factor data. Our phenomenological study reveals that adopting the second Gegenbauer moment computed by RQCD can yield a decent agreement with large-Q^2 data (above the resonance region in the time-like case). Nevertheless, to make a definite conclusion, it seems imperative to resolve the discrepancy between LPC and RQCD Collaboration on the value of a_2 in the future study. Furthermore, we look forward to the future high-statistics larger-Q^2 pion EM form factor data for critically testing our NNLO predictions. It is also very interesting to confront our NNLO predictions with the available high-quality kaon EM form factor data.

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