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Study of the $f_0(1710)$ and $a_0(1710)$ states

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In the present work, we use the final state interaction formalism to investigate the production of the $f_0(1710)$ and $a_0(1710)$ states in the D_s decay processes. At the quark level, we consider the W-internal and -external emission mechanisms. At the hadron level, we take the final state interaction into account based on the coupled channel approach, where the $f_0(1710)$ and $a_0(1710)$ states are dynamically generated as the molecular states of $K^*\bar{K}^*$. The experimental data of the invariant mass spectra is fitted well, and some corresponded branching fractions are evaluated, which are consistent with the experimental measurements.

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1. Introduction

Due to the development of the experiment, more and more resonances are found, where some of them have been found for a long time and their properties are still under debate, such as $f_0(500)$, $f_0(980)$, $a_0(980)$, $\Lambda(1405)$, and so on. Indeed, the states $f_0(980)$ and $a_0(980)$ were observed in the experiments about 50 years ago [1–3], which have been long debated regarding their nature and structure, particularly in relation to their masses close to the $K\bar{K}$ threshold. The prevalent assignment of these states as the $K\bar{K}$ molecules [4–11] arises from their decay properties and mass distribution, which suggest a strong coupling to the $K\bar{K}$ channel. On the other hand, the $f_0(1710)$ resonance, discovered about 40 years ago [12, 13], has been associated with different interpretations as well, including that of a gluonium state or a conventional $q\bar{q}$ bound state. Using the coupled channel interaction approach, the $f_0(1710)$ was found to be a molecular state of $K^*\bar{K}^*$ in Ref. [14], with an isovector partner a_0 state around 1.78 GeV predicted, which is expected to be analogous to the $f_0(980)$ and $a_0(980)$ produced by the interactions of $K\bar{K}$ and its coupled channels. Also with the coupled channel interaction approach, Refs. [15, 16] also obtained similar prediction for a new a_0 state in the interactions involving the $K^*\bar{K}^*$ channel. Thus, searching for the isovector partner of the $f_0(1710)$ resonance, the a_0 state, is indeed a critical effort to shed light on the nature of this state, as it could provide vital clues to determine whether the $f_0(1710)$ is a glueball, a molecular state, or has a different underlying structure. Until 2021, a new state $a_0(1700)$ was observed in the $\pi\eta$ invariant mass spectrum in the $\eta_c \to \eta\pi^+\pi^-$ decay by the BABAR Collaboration [17], where the mass and width were reported as (1.704 ± 0.005) GeV and (0.110 ± 0.019) GeV, respectively. Additionally, the BESIII Collaboration investigated the decay $D_s^+ \to K_S^0 K_S^0 \pi^+$ [18], and claimed the observation of a resonance structure near 1.710 GeV in the $K_S^0 K_S^0$ invariant mass distribution, with the mass and width given by $M_{S(1710)} = (1.723 \pm 0.011)$ GeV, $\Gamma_{S(1710)} = (0.140 \pm 0.015)$ GeV, respectively. In their findings, it was called as $S(1710)$ due to the overlap of $a_0(1710)^0$ and $f_0(1710)$. Thus, to identify the a_0 state, the BESIII Collaboration conducted further measurements of the analougous decay $D_s^+ \to K_S^0 K^+ \pi^0$ [19]. In their analysis, they observed a resonance $a_0(1710)^+$ in the $K_S^0 K^+$ invariant mass spectrum, with a mass $M_{a_0(1710)} = (1.817 \pm 0.022)$ GeV and a width $\Gamma_{a_0(1710)} = (0.097 \pm 0.027)$ GeV. Due to its mass a bit higher, this resonance was referred to the $a_0(1817)$ state in the existing literature at that time. Not that, in the updated online version of Particle Data Group (PDG) [20], it has since been designated as $a_0(1710)$.

Indeed, as suggested in Ref. [21], the new a_0 state observed in Ref. [19] should be properly named as $a_0(1817)$, which was arranged in the same Regge trajectory with the $a_0(980)$. In contrast, it was labeled as $a_0(1710)$ in Ref. [22], which was based on the analysis of the coupled channel interaction involving the $K^*\bar{K}^*$ channel. Using the coupled channel formalism to account for final state interactions, the decay $D_s^+ \to K_S^0 K^+ \pi^0$ was investigated in Refs. [23], where the new state found in Ref. [19] was identified as the $a_0(1710)$. Similarly, the decay $D_s^+ \to K_S^0 K_S^0 \pi^+$ was analyzed in Refs. [24, 25] with the final-state interaction approach. Applying the MIT bag model, it was found in Ref. [26] that the strong a_0 coupling to the vector channels, such as $K^*\bar{K}^*$, $\rho\phi$, and so on, depends on the mass of the a_0. Consequently, the detection of these vector channels is essential for understanding the nature of the new a_0. Furthermore, Ref. [27] provides additional insights and proposals for future experiments regarding the $a_0(1710)$. Thus, in the present work, to understand more about the nature of $f_0(1710)$ and $a_0(1710)$, we investigate the decays $D_s^+ \to K_S^0 K_S^0 \pi^+$ and

$D_s^+ \to K_S^0 K^+ \pi^0$ with the final state interaction formalism.

Our manuscript is organized as follows. In the next section, our formalism is given briefly, and then, our results are shown in the following section. Finally, a short summary is made.

2. Formalism

In the present work, we study both the decays $D_s^+ \to K_S^0 K_S^0 \pi^+$ and $D_s^+ \to K_S^0 K^+ \pi^0$, which are similar in our formalism and thus only the formalism for the $D_s^+ \to K_S^0 K_S^0 \pi^+$ is discussed in detail. For the case of the $D_s^+ \to K_S^0 K_S^0 \pi^+$ decay, only the dominant contributions of the weak decay topology from the W-external and -internal emission mechanisms are taken into account. The Feynman diagrams of two mechanisms are shown in Figs. 1 and 2, which can be formulated as

$$\begin{aligned}\left|H^{(1a)}\right\rangle =& V_P^{(1a)} V_{cs} V_{ud} (u\bar{d} \to \pi^+) \left|s(\bar{u}u + \bar{d}d + \bar{s}s)\bar{s}\right\rangle \\ &+ V_P^{*(1a)} V_{cs} V_{ud} (u\bar{d} \to \rho^+) \left|s(\bar{u}u + \bar{d}d + \bar{s}s)\bar{s}\right\rangle \\ =& V_P^{(1a)} V_{cs} V_{ud} (\pi^+)(M \cdot M)_{33} + V_P^{*(1a)} V_{cs} V_{ud} (\rho^+)(M \cdot M)_{33},\end{aligned} \tag{1}$$

$$\begin{aligned}\left|H^{(1b)}\right\rangle =& V_P^{(1b)} V_{cs} V_{ud} (s\bar{s} \to -\frac{2}{\sqrt{6}}\eta) \left|u(\bar{u}u + \bar{d}d + \bar{s}s)\bar{d}\right\rangle \\ &+ V_P^{*(1b)} V_{cs} V_{ud} (s\bar{s} \to \phi) \left|u(\bar{u}u + \bar{d}d + \bar{s}s)\bar{d}\right\rangle \\ =& V_P^{(1b)} V_{cs} V_{ud} (-\frac{2}{\sqrt{6}}\eta)(M \cdot M)_{12} + V_P^{*(1b)} V_{cs} V_{ud} (\phi)(M \cdot M)_{12},\end{aligned} \tag{2}$$

$$\begin{aligned}\left|H^{(2a)}\right\rangle =& V_P^{(2a)} V_{cs} V_{ud} (s\bar{d} \to \bar{K}^0) \left|u(\bar{u}u + \bar{d}d + \bar{s}s)\bar{s}\right\rangle \\ &+ V_P^{*(2a)} V_{cs} V_{ud} (s\bar{d} \to \bar{K}^{*0}) \left|u(\bar{u}u + \bar{d}d + \bar{s}s)\bar{s}\right\rangle \\ =& V_P^{(2a)} V_{cs} V_{ud} (\bar{K}^0)(M \cdot M)_{13} + V_P^{*(2a)} V_{cs} V_{ud} (\bar{K}^{*0})(M \cdot M)_{13},\end{aligned} \tag{3}$$

$$\begin{aligned}\left|H^{(2b)}\right\rangle =& V_P^{(2b)} V_{cs} V_{ud} (u\bar{s} \to K^+) \left|s(\bar{u}u + \bar{d}d + \bar{s}s)\bar{d}\right\rangle \\ &+ V_P^{*(2b)} V_{cs} V_{ud} (u\bar{s} \to K^{*+}) \left|s(\bar{u}u + \bar{d}d + \bar{s}s)\bar{d}\right\rangle \\ =& V_P^{(2b)} V_{cs} V_{ud} (K^+)(M \cdot M)_{32} + V_P^{*(2b)} V_{cs} V_{ud} (K^{*+})(M \cdot M)_{32},\end{aligned} \tag{4}$$

where $\left|H^{(1a)}\right\rangle$, $\left|H^{(1b)}\right\rangle$, $\left|H^{(2a)}\right\rangle$, $\left|H^{(2b)}\right\rangle$ correspond to Figs. 1 (a), 1 (b), 2 (a), 2 (b), respectively. The factors $V_P^{(1a)}$, $V_P^{*(1a)}$, $V_P^{(1b)}$ and $V_P^{*(1b)}$ are the weak interaction strengths of the production vertices to produce π^+, ρ^+, η and ϕ mesons, respectively. The ones V_{cs} and V_{ud} are the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The matrix M with elements of $q\bar{q}$ quark pairs in SU(3) is defined as,

$$M = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix}. \tag{5}$$

Under the dominant W-external and -internal emission mechanisms, after the quark pair hadronization and the final state interactions, the amplitude of $D_s^+ \to K_S^0 K_S^0 \pi^+$ decay process in the S-wave

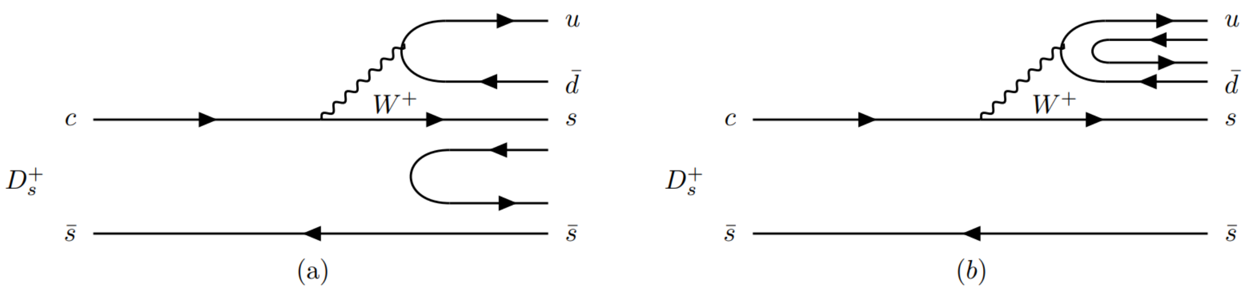


Figure 1: W-external emission mechanism for the $D_s^+ \to K_S^0 K_S^0 \pi^+$ decay. (a) The $s\bar{s}$ quark pair hadronizes into final mesons, (b) the $u\bar{d}$ quark pair hadronizes into final mesons.

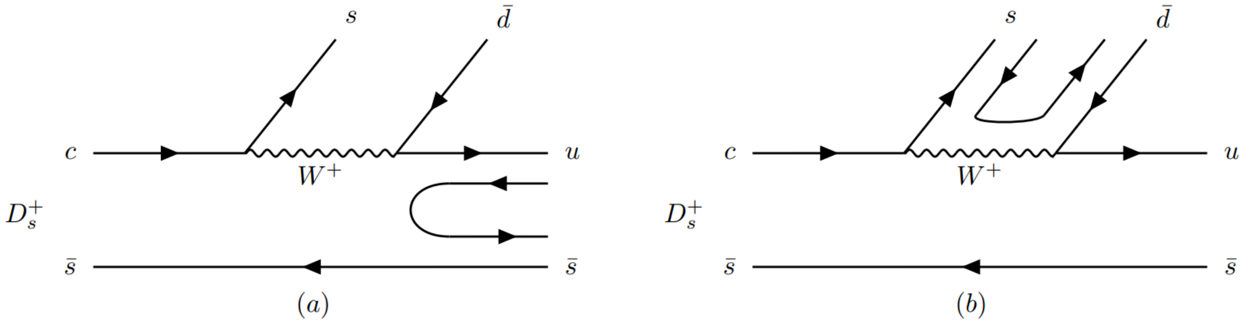


Figure 2: W-internal emission mechanism for the $D_s^+ \to K_S^0 K_S^0 \pi^+$ decay. (a) The $u\bar{s}$ quark pair hadronizes into final mesons, (b) the $s\bar{d}$ quark pair hadronizes into final mesons.

are obtained as,

$$
\begin{aligned}
& t(M_{12})|_{K_S^0 K_S^0 \pi^+} \\
&= -\frac{1}{2} C_1 G_{K^+K^-}(M_{12}) T_{K^+K^- \to K^0\bar{K}^0}(M_{12}) - \frac{1}{2} C_2 - \frac{1}{2} C_2 G_{K^0\bar{K}^0}(M_{12}) T_{K^0\bar{K}^0 \to K^0\bar{K}^0}(M_{12}) \\
&\quad - \frac{1}{3} C_3 G_{\eta\eta}(M_{12}) T_{\eta\eta \to K^0\bar{K}^0}(M_{12}) - \frac{1}{2} C_4 G_{K^{*+}K^{*-}}(M_{12}) T_{K^{*+}K^{*-} \to K^0\bar{K}^0}(M_{12}) \\
&\quad - \frac{1}{2} C_5 G_{K^{*0}\bar{K}^{*0}}(M_{12}) T_{K^{*0}\bar{K}^{*0} \to K^0\bar{K}^0}(M_{12}) - \frac{1}{2} C_6 G_{\phi\phi}(M_{12}) T_{\phi\phi \to K^0\bar{K}^0}(M_{12}) \\
&\quad - \frac{1}{2\sqrt{2}} C_7 G_{\omega\phi}(M_{12}) T_{\omega\phi \to K^0\bar{K}^0}(M_{12}) - \frac{1}{2\sqrt{2}} C_8 G_{\rho^0\phi}(M_{12}) T_{\rho^0\phi \to K^0\bar{K}^0}(M_{12}),
\end{aligned}
\tag{6}
$$

where $G_{PP'(VV')}$ and $T_{PP'(VV')\to PP'}$ are the loop functions and the two-body scattering amplitudes, respectively. Note that, the elements of the CKM matrix and weak decay vertex factors are incorporated into the constants C_i, which are treated as free parameters. These parameters, including a global normalization factor to match the events of the experimental data, are determined by fitting the experimental invariant mass distributions. The two-body scattering amplitudes $T_{PP'(VV')\to PP'}$ are evaluated by the coupled channel Bethe-Salpeter equation [8, 28] with the on-shell approximation,

$$
T = [1 - VG]^{-1} V, \tag{7}
$$

where the matrix G is constructed by the loop functions, and the matrix V is made of the interaction potential for each coupled channel, see more details in Ref [29].

Furthermore, the vector meson can be produced directly in both the W-external and -internal emission mechanisms, which is not generated in the meson-meson rescattering process and also decays to the same final states. Thus, we also consider the contribution of the vector resonance generated from the hadronization in P wave, such as $K^*(892)^+$, of which the relativistic amplitude for the decay $D_s^+ \to K_s^0 K^*(892)^+ \to K_S^0 K_S^0 \pi^+$ is given by

$$\begin{aligned} t_{K^*(892)^+}(M_{12}, M_{23}) =& \frac{\mathcal{D} e^{i\alpha_{K^*(892)^+}}}{M_{23}^2 - M_{K^*(892)^+}^2 + iM_{K^*(892)^+}\Gamma_{K^*(892)^+}} \\ & \times \left[\frac{(m_{D_s^+}^2 - m_{K_S^0}^2)(m_{K_S^0}^2 - m_{\pi^+}^2)}{M_{K^*(892)^+}^2} - M_{12}^2 + M_{13}^2 \right], \end{aligned} \tag{8}$$

where $\mathcal{D}$ and $\alpha_{K^*(892)^+}$ are the normalization constant and the phase, respectively, which can be determined by fitting the experimental data. The mass and width of intermediate vector meson $K^*(892)^+$ are given by the PDG [20], taking as $M_{K^*(892)^+} = 0.89167$ GeV and $\Gamma_{K^*(892)^+} = 0.0514$ GeV, respectively.

Finally, the double differential width distribution for the three-body decay $D_s^+ \to K_S^0 K_S^0 \pi^+$ can be calculated by [20],

$$\frac{d^2\Gamma}{dM_{12}dM_{23}} = \frac{1}{(2\pi)^3} \frac{M_{12}M_{23}}{8m_{D_s^+}^3} \frac{1}{2} \mid \mathcal{M} \mid^2, \tag{9}$$

where the factor $1/2$ accounts for two identical particle K_S^0 in the final states, and $\mathcal{M}$ is the total amplitude of the decay $D_s^+ \to K_S^0 K_S^0 \pi^+$. Taken the contributions from both S- and P-waves into account, the total amplitude of $\mathcal{M}$ is obtained as

$$\mathcal{M} = t(M_{12})|_{K_S^0 K_S^0 \pi^+} + t_{K^*(892)^+}(M_{12}, M_{23}) + (1 \leftrightarrow 2), \tag{10}$$

where the amplitude $t(M_{12})|_{K_S^0 K_S^0 \pi^+}$ is given by Eq. (6), the one $t_{K^*(892)^+}(M_{12}, M_{23})$ by Eq. (8), and $(1 \leftrightarrow 2)$ resembles the symmetry between the two identical K_S^0 in the final states. Using Eq (9), the invariant mass spectra $d\Gamma/dM_{12}$, $d\Gamma/dM_{13}$ and $d\Gamma/dM_{23}$ can be calculated by integrating over each of the invariant mass variables with the limits of the Dalitz Plot, see more details in the PDG [20].

Analogously, for the case of the decay $D_s^+ \to K_S^0 K^+ \pi^0$, the double differential width of this decay is written as

$$\frac{d^2\Gamma}{dM_{12}dM_{13}} = \frac{1}{(2\pi)^3} \frac{M_{12}M_{13}}{8m_{D_s^+}^3} \left(\left| t_{S\text{-wave}} + t_{\bar{K}^*(892)^0} + t_{K^*(892)^+} \right|^2 \right), \tag{11}$$

where the amplitudes of S- and P-waves are given by

$$\begin{aligned} t_{S\text{-wave}}(M_{12})|_{K_S^0 K^+ \pi^0} =& -\frac{1}{2} C_1 G_{\rho^+\phi}(M_{12}) T_{\rho^+\phi \to K^+\bar{K}^0}(M_{12}) \\ & - \frac{1}{2} C_2 - \frac{1}{2} C_2 G_{K^+\bar{K}^0}(M_{12}) T_{K^+\bar{K}^0 \to K^+\bar{K}^0}(M_{12}) \\ & - \frac{1}{2} C_3 G_{K^{*+}\bar{K}^{*0}}(M_{12}) T_{K^{*+}\bar{K}^{*0} \to K^+\bar{K}^0}(M_{12}). \end{aligned} \tag{12}$$

$$t_{\bar{K}^*(892)^0}(M_{12}, M_{13}) = \frac{\mathcal{D}_1 e^{i\phi_{\bar{K}^*(892)^0}}}{M_{13}^2 - m_{\bar{K}^*(892)^0}^2 + i m_{\bar{K}^*(892)^0}\Gamma_{\bar{K}^*(892)^0}} \times \left[(m_{K_S^0}^2 - m_{\pi^0}^2)\frac{m_{D_s^+}^2 - m_{K^+}^2}{m_{\bar{K}^*(892)^0}^2} - M_{12}^2 + M_{23}^2 \right], \tag{13}$$

$$t_{K^*(892)^+}(M_{12}, M_{13}) = \frac{\mathcal{D}_2 e^{i\phi_{K^*(892)^+}}}{M_{23}^2 - m_{K^*(892)^+}^2 + i m_{K^*(892)^+}\Gamma_{K^*(892)^+}} \times \left[(m_{K^+}^2 - m_{\pi^0}^2)\frac{m_{D_s^+}^2 - m_{K_S^0}^2}{m_{K^*(892)^+}^2} - M_{12}^2 + M_{13}^2 \right], \tag{14}$$

with C_1, C_2, and C_3 the strengths of three production factors in the S-wave final state interactions, and $\mathcal{D}_1$, $\mathcal{D}_2$, $\phi_{\bar{K}^*(892)^0}$, and $\phi_{K^*(892)^+}$ the production factors and phases appeared in the P-wave productions, as free parameters determined by the fits of the experimental data. More details can be referred to Ref. [30].

3. Results

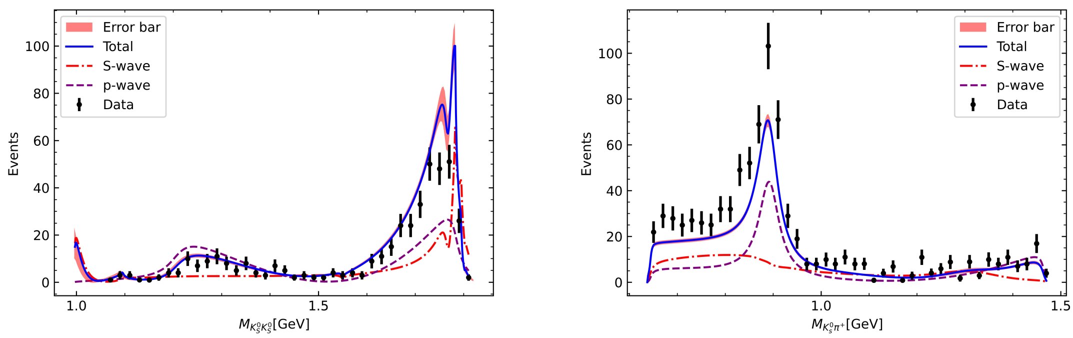
3.1 Results of the decay $D_s^+ \to K_S^0 K_S^0 \pi^+$

As one can see in the former section that there are some free parameters in our framework, which should be fitted by the experimental data. We can determine these parameters by performing a combined fit to the invariant mass distributions of the $D_s^+ \to K_S^0 K_S^0 \pi^+$ decay, as reported by the BESIII Collaboration [18]. The fitted parameters are summarized in Table 1, where the calculated $\chi^2/dof. = 175.97/(80-11) = 2.55$. Using these fitted parameters, the resulting invariant mass distributions for the $K_S^0 K_S^0$ and $K_S^0 \pi^+$ channels are presented in Fig. 3, although the error bands are not clearly visible, which suggest that the uncertainties in the fit are relatively minor due to the constraint from the small errorbars of precise measurement of the experimental data. It looks like that the description of the data in Fig. 3a is better than the one of Fig. 3b, even though we do a combined fit of both the invariant mass distributions of the $K_S^0 K_S^0$ and $K_S^0 \pi^+$ channels. No matter we only fit the data of the $K_S^0 K_S^0$ invariant mass distributions, the fitted results for the $K_S^0 \pi^+$ invariant mass distributions are not much improvement, as shown in Fig. 4. When we check in detail, the data around the bump near 1.3 GeV of the $K_S^0 K_S^0$ invariant mass distributions constraints the fit of the $K_S^0 \pi^+$ invariant mass distributions, which lead to a destroyed interference effect between the contributions of S- and P-waves. From the results of Figs. 3 and 4, two overlapped peaks are found for the mixing effect of the $a_0(1710)$ and $f_0(1710)$, which are all produced from the S-wave amplitude.

Then, using the results of the fitting parameters, one can obtain the corresponding poles for the $a_0(980)$, $f_0(980)$, $a_0(1710)$ and $f_0(1710)$ states in the complex Riemann sheets, as shown in Table. 2. In Table 2, the pole for the $a_0(980)$, $f_0(980)$, and $a_0(1710)$ states are consistent with the one obtained in our former results of Ref. [30, 31]. But, the obtained width from the $a_0(1710)$ pole is several times smaller than the values reported in Refs. [14, 15, 22]. Additionally, there is an approximate difference of 30 MeV between the pole positions of $a_0(1710)$ and $f_0(1710)$. This slight

Table 1: Values of the parameters from the fit.

Parameters	μ	C_1	C_2	C_3
Fit	0.648 ± 0.01 GeV	8640.90 ± 1115.80	2980.71 ± 638.37	-1902.86 ± 293.27
Parameters	C_4	C_5	C_6	C_7
Fit	56906.35 ± 10869.67	-13433.15 ± 5017.76	-58284.22 ± 7319.04	102835.76 ± 23333.56
Parameters	C_8	D	$\alpha_{K^*(892)^+}$	$\chi^2/dof.$
Fit	202807.71 ± 30750.45	54.8 ± 2.0	0.0024 ± 4.30	2.55



(a) Invariant mass distribution of $K_S^0K_S^0$. **(b)** Invariant mass distribution of $K_S^0\pi^+$.

Figure 3: Combined fit for the invariant mass distributions of the decay $D_s^+ \to \pi^+ K_S^0 K_S^0$. The solid (red) line is the total contributions of the S and P waves, the dashed (blue) line represents the contribution of the $K^*(892)^+$, the dotted (magenta) lines is the contributions from the S-wave interactions with the resonances $S(980)$ and $S(1710)$. The dot (black) points are the experimental data measured by the BESIII Collaboration [18].

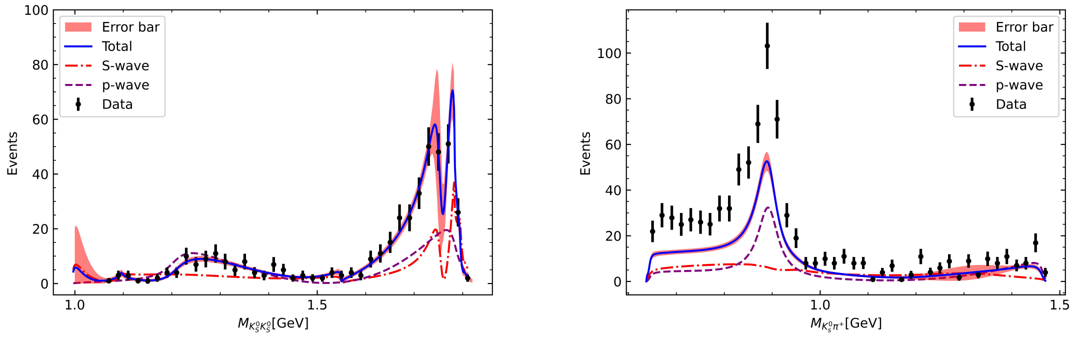
discrepancy in their widths contributes to the observed split structure in the $K_S^0K_S^0$ mass distributions around the high-energy region of 1.7 GeV, as discussed previously.

Table 2: Poles compared with the other works (unit: GeV).

	This work	Ref. [30]	Ref. [31]	Ref. [14]	Ref. [22]	Ref. [15]
Parameters	$\mu = 0.648$	$\mu = 0.716$	$q_{max} = 0.931$	$\mu = 1.0$	$q_{max} = 1.0$	$q_{max} = 1.0$
$a_0(980)$	$1.0598 + 0.024i$	$1.0419 + 0.0345i$	$1.0029 + 0.0567i$	$\cdots$	$\cdots$	$\cdots$
$f_0(980)$	$0.9912 + 0.003i$	$\cdots$	$0.9912 + 0.0135i$	$\cdots$	$\cdots$	$\cdots$
$a_0(1710)$	$1.7981 + 0.0018i$	$1.7936 + 0.0094i$	$\cdots$	$1.780 - 0.066i$	$1.72 - 0.010i$	$1.76 \pm 0.03i$
$f_0(1710)$	$1.7676 + 0.0093i$	$\cdots$	$\cdots$	$1.726 - 0.014i$	$\cdots$	$\cdots$

Additionally, the ratios of branching fractions for the corresponding decay channels are evaluated based on our fitting results. By integrating the invariant mass distributions for the $K_S^0K_S^0$ and $K_S^0\pi^+$ channels, we obtain the following ratios, normalized to the vector channels:

$$\begin{aligned}
\frac{\mathcal{B}(D_s^+ \to S(980)\pi^+, S(980) \to K_S^0K_S^0)}{\mathcal{B}(D_s^+ \to K_S^0K^*(892)^+, K^*(892)^+ \to K_S^0\pi^+)} &= 0.122^{+0.032}_{-0.023}, \\
\frac{\mathcal{B}(D_s^+ \to S(1710)\pi^+, S(1710) \to K_S^0K_S^0)}{\mathcal{B}(D_s^+ \to K_S^0K^*(892)^+, K^*(892)^+ \to K_S^0\pi^+)} &= 0.552^{+0.460}_{-0.297},
\end{aligned} \tag{15}$$



(a) Invariant mass distribution of $K_S^0K_S^0$. **(b)** Invariant mass distribution of $K_S^0\pi^+$.

Figure 4: Fit only for the data of $K_S^0K_S^0$ spectrum. Others are the same as Fig. 3.

where a cut at 1.5 GeV is used for the lower contributions of the $S(1710)$. Thus, with the input of the branching fraction $\mathcal{B}(D_s^+ \to K^*(892)^+K_s^0 \to \pi^+K_S^0K_S^0) = (3.0 \pm 0.3 \pm 0.1) \times 10^{-3}$ measured by the BESIII Collaboration [18] , the branching fractions for other processes are obtained as,

$$\begin{aligned}
\mathcal{B}(D_s^+ \to S(980)\pi^+, S(980) \to K_S^0K_S^0) &= (0.36 \pm 0.04^{+0.10}_{-0.06}) \times 10^{-3}, \\
\mathcal{B}(D_s^+ \to S(1710)\pi^+, S(1710) \to K_S^0K_S^0) &= (1.66 \pm 0.17^{+1.38}_{-0.89}) \times 10^{-3},
\end{aligned} \tag{16}$$

where the first uncertainties are estimated from the errors of experimental results, and the second ones are from Eq. (15). The results for the decay $(D_s^+ \to S(1710)\pi^+, S(1710) \to K_S^0K_S^0)$ is about one half of the value measured in Ref. [18], $(3.1 \pm 0.3 \pm 0.1) \times 10^{-3}$.

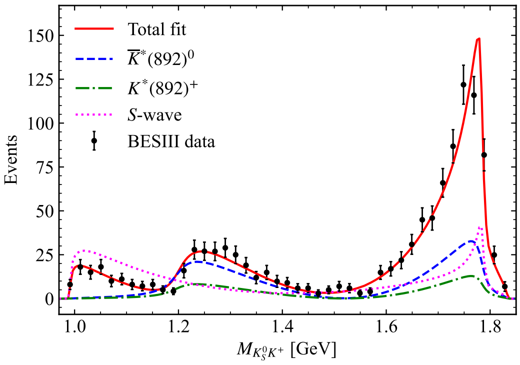
3.2 Results of the decay $D_s^+ \to K_S^0K^+\pi^0$

In this case of the decay $D_s^+ \to K_S^0K^+\pi^0$, we also do a combine fit of the data of the invariant mass distributions measured by the BESIII Collaboration [19] to determine the free parameters of the formalism, as given in Table 3. Our fitting results for the invariant mass distributions are shown in Fig. 5, which describes well the data of the three invariant mass distributions reported in Ref. [19], also well in the resonance structure around 1.7 GeV of the $K_S^0K^+$ invariant mass distributions. In addition, using the fitting results, we also evaluate the partial decay widths of each coupled channel. Taken the formulae from Refs. [8, 10], we obtain the results as shown in Table 4.

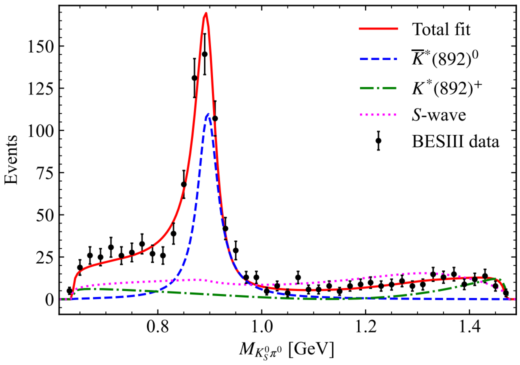
Table 3: Values of the parameters from the fit.

Par.	μ	C_1	C_2	C_3
Fit I	0.716 ± 0.013 GeV	47518.79 ± 7523.18	1595.34 ± 138.51	46454.25 ± 3868.04
	$\mathcal{D}_1$	$\mathcal{D}_2$	$\phi_{\bar{K}^*(892)^0}$	$\phi_{K^*(892)^+}$
	61.65 ± 2.33	40.43 ± 2.95	1.46 ± 0.12	1.67 ± 0.15

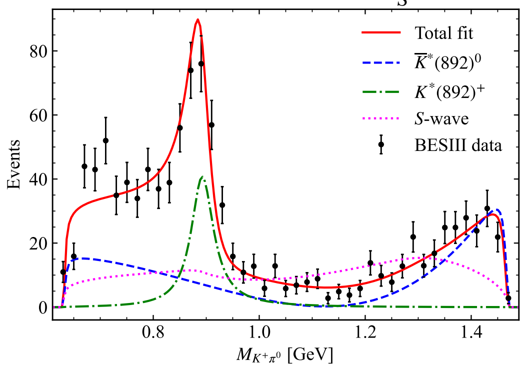
Moreover, the branching ratios of the corresponding decay channels are calculated. For the decays $D_s^+ \to \bar{K}^*(892)^0K^+$, $K^*(892)^+K_S^0$, and $a_0(980)^+\pi^0$, we integrate the corresponding invariant mass spectra from the threshold to 1.2 GeV. The uncertainties come from the changes of upper limits 1.20 ± 0.05 GeV. For the $D_s^+ \to a_0(1710)^+\pi^0$ decay, the integration limits are defined from



(a) $K_S^0 K^+$ invariant mass distribution.



(b) $K_S^0 \pi^0$ invariant mass distribution.



(c) $K^+\pi^0$ invariant mass distribution.

Figure 5: Invariant mass distributions for the $D_s^+ \to K_S^0 K^+ \pi^0$ decay. The solid (red) line corresponds to the total contributions of the S- and P-waves, the dash (blue) line represents the contributions from the $\bar{K}^*(892)^0$, the dash-dot (green) line is the contributions from the $K^*(892)^+$, the dot (magenta) line is the contributions from the S-wave interactions ($a_0(980)^+$ and $a_0(1710)^+$), and the dot (black) points are the data taken from Ref. [19].

Table 4: The partial decay widths.

$\Gamma_{a_0(980)^+ \to K^+ \bar{K}^0}$	$\Gamma_{a_0(980)^+ \to \pi^+ \eta}$	$\Gamma_{a_0(1710)^+ \to \rho^+ \omega}$	$\Gamma_{a_0(1710)^+ \to K^+ \bar{K}^0}$	$\Gamma_{a_0(1710)^+ \to \pi^+ \eta}$
28.38 MeV	43.60 MeV	19.65 MeV	0.54 MeV	0.05 MeV

1.6 GeV to $(m_{D_s^+} - m_{\pi^0})$, with uncertainties stemming from changes in the lower limit, taken as 1.60 ± 0.05 GeV. The obtained results are presented as follows

$$\frac{\mathcal{B}(D_s^+ \to K^*(892)^+ K_S^0, K^*(892)^+ \to K^+ \pi^0)}{\mathcal{B}(D_s^+ \to \bar{K}^*(892)^0 K^+, \bar{K}^*(892)^0 \to K_S^0 \pi^0)} = 0.40^{+0.002}_{-0.003}, \tag{17}$$

$$\frac{\mathcal{B}(D_s^+ \to a_0(980)^+ \pi^0, a_0(980)^+ \to K_S^0 K^+)}{\mathcal{B}(D_s^+ \to \bar{K}^*(892)^0 K^+, \bar{K}^*(892)^0 \to K_S^0 \pi^0)} = 0.53^{+0.06}_{-0.08}, \tag{18}$$

$$\frac{\mathcal{B}(D_s^+ \to a_0(1710)^+ \pi^0, a_0(1710)^+ \to K_S^0 K^+)}{\mathcal{B}(D_s^+ \to \bar{K}^*(892)^0 K^+, \bar{K}^*(892)^0 \to K_S^0 \pi^0)} = 0.41^{+0.04}_{-0.05}. \tag{19}$$

Next, taking the input of the branching fraction $\mathcal{B}(D_s^+ \to \bar{K}^*(892)^0 K^+, \bar{K}^*(892)^0 \to K_S^0\pi^0) = (4.77 \pm 0.38 \pm 0.32) \times 10^{-3}$ reported by the BESIII Collaboration [19], the branching ratios for the other channels are easily obtained as

$$\begin{aligned} \mathcal{B}(D_s^+ \to K^*(892)^+K_S^0, K^*(892)^+ \to K^+\pi^0) &= (1.91 \pm 0.20^{+0.01}_{-0.01}) \times 10^{-3}, \\ \mathcal{B}(D_s^+ \to a_0(980)^+\pi^0, a_0(980)^+ \to K_S^0K^+) &= (2.53 \pm 0.26^{+0.27}_{-0.38}) \times 10^{-3}, \\ \mathcal{B}(D_s^+ \to a_0(1710)^+\pi^0, a_0(1710)^+ \to K_S^0K^+) &= (1.94 \pm 0.20^{+0.18}_{-0.24}) \times 10^{-3}, \end{aligned} \tag{20}$$

with the first uncertainties estimated from the experimental errors, and the second ones from the Eqs. (17-19), some of which are consistent with the experimental measurements [19] within the uncertainties, given by

$$\begin{aligned} \mathcal{B}(D_s^+ \to K^*(892)^+K_S^0, K^*(892)^+ \to K^+\pi^0) &= (2.03 \pm 0.26 \pm 0.20) \times 10^{-3}, \\ \mathcal{B}(D_s^+ \to a_0(980)^+\pi^0, a_0(980)^+ \to K_S^0K^+) &= (1.12 \pm 0.25 \pm 0.27) \times 10^{-3}, \\ \mathcal{B}(D_s^+ \to a_0(1710)^+\pi^0, a_0(1710)^+ \to K_S^0K^+) &= (3.44 \pm 0.52 \pm 0.32) \times 10^{-3}. \end{aligned} \tag{21}$$

For the result of the decay $D_s^+ \to a_0(1710)^+\pi^0$, our result is approximately one-third smaller than the experimental measurement. Note that, the predicted branching ratio of $D_s^+ \to a_0(1710)^+\pi^0$ is obtained as $(1.3 \pm 0.4) \times 10^{-3}$ in Ref. [24].

4. Summary

Base on the recent measurements and new findings reported by the BESIII Collaboration, we investigate the weak decay processes of $D_s^+ \to K_S^0K_S^0\pi^+$ and $D_s^+ \to K_S^0K^+\pi^0$ by incorporating the mechanisms of external and internal W-emission at the quark level. At the hadron level, our analysis employs the final state interaction formalism, including the contributions from both tree-level processes and rescattering effects. To determine the free parameters of the framework, we make the combined fits to the invariant mass spectra measured by the experiments, where our fitting results describe well the experimental data. In our formalism, we dynamically reproduce the resonances $a_0(1710)$, new one, and $f_0(1710)$ from the coupled channel interactions involving the channel $K^*\bar{K}^*$, and thus these resonances can be assigned as the molecules of $K^*\bar{K}^*$. Additionally, the branching ratios for the corresponding channels are estimated, where some of our results agree with the experimental measurements.

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