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Dipion transitions in heavy quarkonium high excitations and heavy quarkonium hybrids

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We report on preliminary results for dipion transitions involving heavy quarkonium and heavy hybrid states of size much larger than the typical hadronic scale. For these states the usual QCD multipole expansion does not hold. As an alternative, we propose an interaction Lagrangian for pions and the QCD string. It allows us to calculate the light quark mass dependence of the string tension, elastic pion scattering off the string, and the decay of string excitations by pion emission. We then introduce an interaction Lagrangian of pions with heavy quarkonium and heavy hybrid states such that it reproduces the previous results in the static limit. For illustration purposes, we calculate a couple of selected transitions with this Lagrangian. We also discuss the missing ingredients in order to achieve a full leading order calculation.

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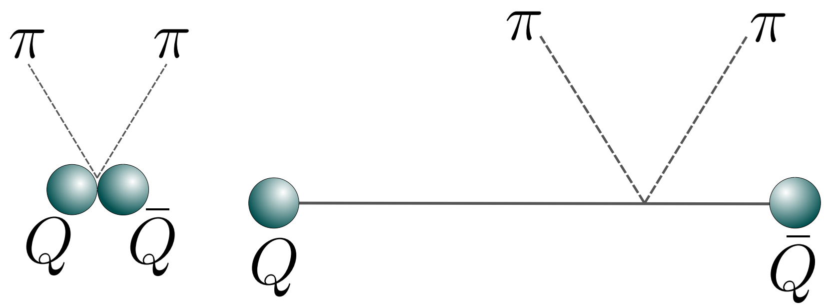

Figure 1: Multipole expansion emission (left) versus string emission (right)

1. Introduction

Dipion transitions in heavy quarkonium are usually calculated through a two-step procedure. First, the multipole expansion is used [1]. This consist of expanding soft gluon operators around the heavy quarkonium center of mass. Next, the soft gluon operators are hadronized in terms of pions. For this approach to be valid, the typical size of the bound states r must be smaller than the typical hadronic scale $1/\Lambda_{\rm QCD}$ and smaller than the time scale of the transition $1/\Delta E$, where ΔE is the energy difference between the initial and final quarkonium states. Often an additional expansion is used, the so-called twist expansion [2], which localizes soft gluon operators at different times, at the same time. This approximation assumes that the typical binding energies are larger than $\Lambda_{\rm QCD}$ and ΔE, which is difficult to justify in practice [3]. For quarkonium states with large principal quantum number, even the multipole expansion may fail. For instance, typical values of $1/r$ for $\psi(nS)$ ($\Upsilon(nS)$) are 520 (1030) MeV, 260 (430) MeV, 190 (300) MeV and 150 (230) MeV for $n = 1, 2, 3, 4$ [4]. Except for the $1S$ states, the remaining figures for $1/r$ are of the order or smaller than $\Lambda_{\rm QCD}$. It is then worth exploring the consequences of giving up the multipole expansion. In order to do so, we focus on the opposite limit $r\Lambda_{\rm QCD} \gg 1$. We illustrate this limit and the one corresponding to the multipole expansion in Fig. 1. In this limit, the QCD effective string theory (EST) can be used [5]. We propose an interaction Lagrangian of the QCD string with pions, which respects both the symmetries of the EST and chiral symmetry. With this Lagrangian, we then calculate the quark mass dependence of the string tension, elastic pion scattering off the string, and the decay of string excitations into pion pairs. We put forward interaction Lagrangians for quarkonium-to-quarkonium and hybrid-to-quarkonium dipion transitions, which reproduce the string result in the static limit. We present preliminary results for quarkonium/hybrid-to-quarkonium dipion transitions.

2. The interaction of pions with the QCD string

The effective QCD string provides an accurate description of the static potential at long distances ($r\Lambda_{\rm QCD} \gg 1$), both for quarkonium and hybrids. This is also the case for a number of $1/m$ suppressed quarkonium potentials. The Nambu-Goto action provides the leading terms of the EST [6, 7].

$$S_{NG} = -\sigma \int d^2\xi \sqrt{-\det(\partial_a x^\mu \partial_b x_\mu)}\,, \tag{1}$$

where σ is the string tension. This action enjoys reparameterization invariance and Poincaré invariance. We choose the frame $\xi^1 = x^0 = t$, $t \in \mathbb{R}$, $\xi^2 = x^3 = z$, $z \in [-r/2, r/2]$, $x^i = x^i(t,z)$, $i = 1, 2$, and assume $x^i \sim 1/\Lambda_{\rm QCD}$ and $\partial_t \sim \partial_r \sim 1/r$. For $r\Lambda_{\rm QCD} \gg 1$, we then have,

$$S_{NG} \simeq -\sigma \int dtdz \left[1 - \frac{1}{2}\partial_0 x^i \partial_0 x^i + \frac{1}{2}\partial_z x^i \partial_z x^i\right] = -\sigma \int dtdz \left[1 - \partial_0\varphi^* \partial_0\varphi + \partial_z\varphi^* \partial_z\varphi\right]\,, \tag{2}$$

$\varphi = \varphi(z,t) = \left(x^1(z,t) + ix^2(z,t)\right)/\sqrt{2}$.

At low energies ($p \sim m_\pi \ll \Lambda_{\rm QCD}$), pion physics is described by the Chiral Lagrangian [8],

$$\mathcal{L}^{LO}_{\rm Ch} = \frac{f_\pi^2}{4}{\rm Tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{f_\pi^2 m_\pi^2}{4m_q}{\rm Tr}(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U) \quad , \quad U = e^{\frac{i\vec{\pi}\vec{\tau}}{f_\pi}} \quad , \quad \vec{\pi}\vec{\tau} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \tag{3}$$

which is also Poincaré invariant, and approximately invariant under $SU_L(2) \otimes SU_R(2)$ chiral symmetry. The latter symmetry is spontaneously broken by the ground state to the diagonal $SU(2)$ (isospin) and explicitly broken by the light quark masses, $\mathcal{M} = m_q\mathbb{I}$, $m_q = (m_u + m_d)/2$.

The guidelines for building the interaction of pions with the string in an EFT framework are symmetries and power counting. We shall write down a local effective Lagrangian that respects the symmetries of both Eq. (2) and Eq. (3). For the power counting, we are going to assume $1/r$, p, $m_\pi \ll \Lambda_{\rm QCD}$, which holds for both the EST and the Chiral Lagrangian. This means an expansion in $\partial x^i(\xi)/\partial\xi^a \sim 1/r\Lambda_{\rm QCD}$, $\partial_\mu/4\pi f_\pi \sim p/\Lambda_{\rm QCD}$, $\mathcal{M}/4\pi f_\pi \sim m_\pi/\Lambda_{\rm QCD}$. Since the QCD string does not transform under chiral symmetry, the simplest lower-dimensional local operator we may build corresponds to the embedding of the string into the Chiral Lagrangian.

$$S_{\rm int} = \int d^2\xi \sqrt{-\det(\partial_a x^\mu \partial_b x_\mu)}\,\mathcal{L}_{\rm ChS}(x(\xi)) \tag{4}$$
$$\mathcal{L}_{\rm ChS} = \lambda {\rm Tr}(\partial_\mu U^\dagger \partial^\mu U) + \lambda' {\rm Tr}(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U) + \lambda''' {\rm Tr}(\mathcal{M}^\dagger \mathcal{M}) + \lambda'''' {\rm Tr}(U^\dagger \mathcal{M} U^\dagger \mathcal{M} + {\rm h.c.})\,.$$

We have displayed in $\mathcal{L}_{\rm ChS}$ the leading order (LO) terms, and the next-to-leading (NLO) terms needed for the renormalization of the string tension.

2.1 The quark mass dependence of the string tension

Note that in the subspace of zero pions, the Lagrangian in Eq. (4) reduces to the usual Nambu-Goto action, and hence redefines the string tension. This redefinition introduces a light quark mass dependence through $\mathcal{M}$ in Eq. (4) and the pion mass. Indeed, space-time translational invariance implies $\langle 0|\, \mathcal{L}_{\rm ChS}(x(\xi))\, |0\rangle = \langle 0|\, \mathcal{L}_{\rm ChS}(0)\, |0\rangle$. At leading order in the light quark mass, there is a tree-level linear contribution from the second term in the LO Lagrangian in Eq. (4). At NLO, there is a one-loop contribution from the first and second terms of the LO Lagrangian and a tree-level contribution from the NLO Lagrangian. The final result reads,

$$\sigma \to \sigma - \left(4\lambda' m_q + \left[\frac{6B_0}{8\pi^2 f_\pi^2}(2B_0\lambda - \lambda')\left(\ln\frac{2B_0 m_q}{\mu^2} - 1\right) + 2\lambda''\right] m_q^2\right), \tag{5}$$

$\lambda'' = \lambda''' + 2\lambda''''$ is necessary to absorb the UV divergences of the one-loop calculation. Dimensional regularization in the $\overline{\rm MS}$ scheme has been used. We have also used $m_\pi^2 = 2m_q B_0$ ($B_0 \sim \Lambda_{\rm QCD}$) in higher order terms. This result is expected to be useful to understand the differences between heavy quarkonium spectrum observed in lattice calculations at different pion masses (see for instance [9] and [10] for charmonium), which are more pronounced for hybrids and highly excited quarkonium states. The light quark mass dependence of the string tension has been recently observed in [11].

2.2 Pion scattering off the string

The simplest process involving pions is the elastic scattering of a pion off any string state. The LO contribution arises at tree level, with $\sqrt{-\det(\partial_a x^\mu \partial_b x_\mu)} \simeq 1$. We have

$$\langle \pi(\vec{q})|\mathcal{L}_{\rm ChS}^{LO}(x)|\pi(\vec{q}\,')\rangle = \frac{4}{f_\pi^2}(\lambda q_\mu q'^\mu - \lambda'\frac{m_\pi^2}{2B_0})e^{i(q-q')x}, \tag{6}$$

$x = x(\xi)$. This expression must now be sandwiched between the given string state. At LO, the exponential can be approximated by $e^{i((E_q - E_{q'})t - (q_z - q_z')z)}$, which leads to,

$$\langle \pi(\vec{q})|S_{\rm int}|\pi(\vec{q}\,')\rangle = \frac{16\pi}{f_\pi^2}(\lambda q_\mu q'^\mu - \lambda'\frac{m_\pi^2}{2B_0})\frac{\sin\left[(q_z - q_z')\frac{r}{2}\right]}{(q_z - q_z')}\delta(E_q - E_q'), \tag{7}$$

$q^\mu = (E_q, \mathbf{q})$, $\mathbf{q} = (q^1, q^2, q_z)$, and analogously for q′. The LO expression above does not depend on the particular string state the pion scatters off. Notice the non-trivial interplay between the third component of the pion momentum transfer $q_z - q_z'$ and the string length r.

2.3 Decay of string excitations through pion emission

The string excitations may decay into the string ground state by a two-pion (dipion) emission. Let us discuss the amplitude corresponding to the lowest lying excitations $N = 1$, $E_N = \pi N/r$ at LO. The calculation for the $N = 2$ excitations can be found in [12]. The $N = 1$ excitations consist of two degenerate Π_u states, a clockwise (R) and an anticlockwise (L) rotation of $|L_z| = 1$. The part of the calculation involving pions, can be obtained from Eq. (6) by crossing, $q' \to -q'$. Now we need the second term in the expansion of the exponential, $e^{i((E_q+E_{q'})t-(q_z+q_z')z)}(1 - i((q+q')^i x^i(z,t)))$. $\sqrt{-\det(\partial_a x^\mu \partial_b x_\mu)} \simeq 1$ still holds. We obtain,

$$\langle \pi(\vec{q}), \pi(\vec{q}\,'); 0_{\rm EST}|S_{\rm int}|0_\pi; 1\Pi_u^R\rangle = \langle \pi(\vec{q}), \pi(\vec{q}\,'); 0_{\rm EST}|S_{\rm int}|0_\pi; 1\Pi_u^L\rangle^* \tag{8}$$
$$= \frac{16\pi\sqrt{\pi}}{f_\pi^2\sqrt{\sigma}r}\left(\lambda q_\mu q'^\mu + \lambda'\frac{m_\pi^2}{2B_0}\right)\frac{\cos\left[(q_z+q'_z)\frac{r}{2}\right]}{\frac{\pi^2}{r^2} - (q_z+q'_z)^2}\frac{i}{\sqrt{2}}\left(q^1 + q'^1 + i(q^2 + q'^2)\right)\delta\left(E_q + E_q' - \frac{\pi}{r}\right),$$

The amplitudes corresponding to the $N = 2$ excitations, and selected $N = 3$ ones, are given in the Appendix A.

3. The interaction of pions with heavy quarkonium and heavy quarkonium exotics

In the framework of Born-Oppenheimer effective field theory (BOEFT) [4, 14–16], the interaction of pions with heavy quarkonium at LO in $1/m_Q$ and the chiral expansion is given by,

$$\begin{aligned} \mathrm{L}_{\mathrm{int}} &= \int d^3\mathbf{R} \int d^3\mathbf{r}\, \mathrm{tr} \Big[S^\dagger(\mathbf{R},\mathbf{r},t) \Big(g_0(r)\partial_0 U^\dagger \partial^0 U + g_1(r)\partial_i U^\dagger \partial^i U + \\ & \quad + g_2(r) r^i r^j \partial_i U^\dagger \partial_j U + g_3(r) \left(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U \right) \Big) S(\mathbf{R},\mathbf{r},t) \Big] \end{aligned} \tag{9}$$

where $S(\mathbf{R},\mathbf{r},t) = (S_0(\mathbf{R},\mathbf{r},t)\mathbb{I}_2 + S_1^j(\mathbf{R},\mathbf{r},t)\sigma^j)/\sqrt{2}$, S_0 (S_0^j) is the quarkonium wave function field for the spin 0 (spin 1) heavy quarkonium. $U = U(\mathbf{R},t)$, and $g_i(r)$, $i = 0,1,2,3$ are unknown form factors (functions). The trace must be understood to act independently on both spin-symmetry multiplets of quarkonium and on the chiral operators. This Lagrangian implicitly assumes that the energies involved in the processes, ΔE, fulfill $\Delta E \ll 1/r\,, \Lambda_{\mathrm{QCD}}$. The short distance behavior of $g_i(r)$ can be obtained using the QCD multipole expansion [1]. It provides the following r-dependence: $g_0(r) \sim g_1(r) \sim g_3(r) \sim r^2$ and $g_2(r) \sim 1$. The long distance behavior of $g_i(r)$ is unknown. We are going to estimate it in the following section using the EST.

Analogous Lagrangians can be written down when one of the fields represents a heavy quarkonium exotic, namely a quarkonium with non-trivial light degrees of freedom (LDF) quantum numbers, rather than a heavy quarkonium (0^{++} LDF). Let us present here the case with LDF 1^{+-} at LO in the $1/m_Q$ and the chiral expansion,

$$\mathrm{L}_{\mathrm{int}} = \int d^3\mathbf{R} \int d^3\mathbf{r}\, \mathrm{tr} \Big[S^\dagger(\mathbf{R},\mathbf{r},t) \left(\epsilon_{ijk} r^j g_4(\mathbf{r}) \partial_0 U^\dagger \partial^i U \right) H^i(\mathbf{R},\mathbf{r},t) \Big] + \mathrm{H.c.} \tag{10}$$

where $\mathbf{H}(\mathbf{R},\mathbf{r},t) = (\mathbf{H}_0(\mathbf{R},\mathbf{r},t)\mathbb{I}_2 + \mathbf{H}_1^j(\mathbf{R},\mathbf{r},t)\sigma^j)/\sqrt{2}$, $\mathbf{H}_0$ ($\mathbf{H}_0^j$) is the wave function field for the spin 0 (spin 1) heavy quarkonium hybrid. The trace must be understood to act independently on both spin-symmetry multiplets of quarkonium and on the chiral operators. At short distance, $g_4(r) \sim 1$ [13]. The long distance behavior can be obtained from Eq. (25) in an analogous way to the one described in the next section.

4. The long distance interaction of pions with heavy quarkonium and heavy quarkonium hybrids

In order to establish the long distance interactions of pions with heavy quarkonium and heavy quarkonium hybrids, we just write down interaction Lagrangians that reproduce the amplitudes of the pion scattering off the string and of string excitations decays, calculated in Sec. 2.2 and in Sec. 2.3 respectively, in the static limit.

In the case where only quarkonia are involved, it reads

$$\mathrm{L}_{\mathrm{int}} = \int d^3\mathbf{R} \int d^3\mathbf{r}\, \mathrm{tr} \left[S^\dagger(\mathbf{R},\mathbf{r},t) S(\mathbf{R},\mathbf{r},t) \right] \int_{-r/2}^{r/2} dz\, g(r,z) \mathcal{L}_{\mathrm{ChS}}(t, \mathbf{R} + z\hat{\mathbf{r}})\,, \tag{11}$$

This expression matches Eq. (7) if $g(r,z) = 1$. Note that this is not the expected form of the BOEFT Eq. (9). In fact, it is a more general expression that also holds when $\Delta E \sim 1/r$. For $r \ll 1/\Delta E \sim 1/m_\pi$, $\mathcal{L}_{\rm ChS}(t, \mathbf{R} + z\hat{\mathbf{r}})$ can be expanded around $\mathbf{R}$ and the expected form in the BOEFT in Eq. (9) is recovered with $g_0(r) = g_1(r) = \int_{-r/2}^{r/2} dz\, g(r,z)\lambda = r\lambda$, $g_2(r) = 0$ and $g_3(r) = \int_{-r/2}^{r/2} dz\, g(r,z)\lambda'/2B_0 = r\lambda'/2B_0$. Hence, the r^2 behavior at short distances becomes an r behavior at long distances. The absence of a tensor term at long distances ($g_2(r) = 0$) is due to the particular form of the Lagrangian Eq. (4). A $g_2(r) \neq 0$ is obtained when the complete set of string operators contributing to LO is considered. We discuss this issue in Appendix B.

In the case of a hybrid with 1^{+-} LDF in the initial state, it reads,

$$\mathrm{L}_{\rm int} = \int d^3\mathbf{R} \int d^3\mathbf{r}\, \mathrm{tr}\left[S^\dagger(\mathbf{R},\mathbf{r},t)\mathbf{H}(\mathbf{R},\mathbf{r},t)\right] \int_{-r/2}^{r/2} dz\, g(r,z)\hat{\mathbf{r}} \times \nabla_{\mathbf{R}}\, \mathcal{L}_{\rm ChS}(t, \mathbf{R} + z\hat{\mathbf{r}}) \tag{12}$$

$g(r,z) = i\cos(\pi z/r)/\sqrt{\pi\sigma}$ matches the string theory result Eq. (8). Note again that for $r \ll 1/\Delta E \sim 1/m_\pi$ the expected form in the BOEFT is recovered,

$$\mathrm{L}_{\rm int} \simeq \int d^3\mathbf{R} \int d^3\mathbf{r}\, \mathrm{tr}\left[S^\dagger(\mathbf{R},\mathbf{r},t)\mathbf{H}(\mathbf{R},\mathbf{r},t)\right] \frac{i2r}{\sqrt{\pi^3\sigma}}\hat{\mathbf{r}} \times \nabla_{\mathbf{R}}\, \mathcal{L}_{\rm ChS}(t, \mathbf{R} + z\hat{\mathbf{r}})\,. \tag{13}$$

However, this result does not correspond to the LO Lagrangian Eq. (9) but to a higher order term in the chiral expansion. This makes us suspect that additional operators in the Lagrangian in Eq. (4) may be relevant. This is discussed in the Appendix B.

4.1 Dipion transitions between highly excited quarkonium states

We then obtain from Eq. (11) for the spin zero case,

$$\begin{aligned}
\mathcal{M}(i = \{nLM\} \to f = \{n'L'M'\}\,\pi(q)\pi(q')) &= -\frac{8\pi}{f_\pi^2}\left(\lambda q_\mu q'^\mu + \lambda' \frac{m_\pi^2}{2B_0}\right) I_{i\to f}(s) \\
I_{i\to f}(s) &= \int dr \int d\Omega\, S_{n'L'}(r) S_{nL}(r) Y^*_{L'M'}(\hat{\mathbf{r}}) Y_{LM}(\hat{\mathbf{r}}) \frac{\sin\left(sr\frac{\cos\theta}{2}\right)}{s\cos\theta}\,,
\end{aligned} \tag{14}$$

nLM ($n'L'M'$) are the quarkonium principal quantum number, orbital angular momentum and its third component in the initial (final) state, $S_{nL}(r)$ the quarkonium reduced radial wave function, $Y_{LM}(\hat{\mathbf{r}})$ the spherical harmonics ($\hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$), and $s = |\mathbf{q} + \mathbf{q}'|$.

4.2 Dipion transitions between hybrids and highly excited quarkonium states

If the initial state is a hybrid with 1^{+-} LDF, we obtain from Eq. (12) for the spin zero case,

$$\begin{aligned}
\mathcal{M}(i = \{nJLM\} \to f = \{n'L'M'\}\,\pi(q)\pi(q')) &= \frac{8\sqrt{\pi}}{f_\pi^2\sqrt{\sigma}}\left(\lambda q_\mu q'^\mu + \lambda' \frac{m_\pi^2}{2B_0}\right) I_{i\to f}(s) \\
I_{i\to f}(s) &= \int dr S_{n'L'}(r) P^L_{nJ}(r) \int d\Omega Y^*_{L'M'}(\hat{\mathbf{r}}) \left(C(L1J; M-1\,1) e^{i\phi} Y_{LM-1}(\hat{\mathbf{r}})\right. \\
&\quad \left. + C(L1J; M+1\ -1) e^{-i\phi} Y_{LM+1}(\hat{\mathbf{r}})\right) \frac{-is\sin\theta\cos\left(sr\frac{\cos\theta}{2}\right)}{r\sqrt{2}\left(\frac{\pi^2}{r^2} - (s\cos\theta)^2\right)}
\end{aligned} \tag{15}$$

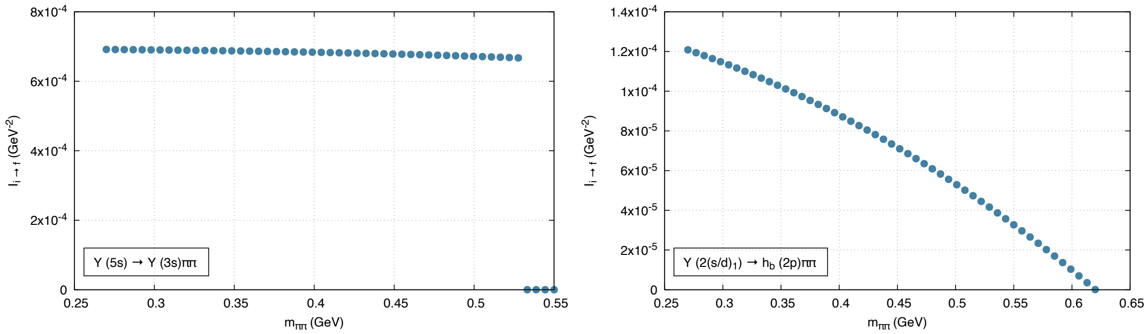

Figure 2: Dipion invariant mass distribution for $\Upsilon(10860)$ transitions assuming it corresponds to a $5s$ quarkonium state (left panel) and to a $2(s/d)_1$ (H_1') hybrid state (right panel).

$P^L_{nJ}(r)$ is the reduced radial wave function of the hybrid. J is the sum of the orbital angular momentum and the angular momentum of the LDF and $C(LL_{\rm LDF}J; MM_{\rm LDF})$ are Clebsch-Gordan coefficients. For a given nJ there are two states, one with $L = J$, and one with $L = J \pm 1$. In the second case, the contributions from both L to $I_{i\to f}(s)$ must be added.

4.3 Dipion invariant mass distribution

The dipion invariant mass distribution reads,

$$\frac{d\Gamma}{dm^2_{\pi\pi}} = \frac{2}{f_\pi^4(2\pi)^3}\left[\frac{\lambda}{2}m^2_{\pi\pi} + \left(\frac{\lambda'}{2B_0} - \lambda\right)m^2_\pi\right]^2 \sqrt{1 - \frac{4m^2_\pi}{m^2_{\pi\pi}}}\, s|\bar{I}_{i\to f}(s)|^2\,, \tag{16}$$

$m^2_{\pi\pi} = (q+q')^2 \simeq \Delta E - s^2$. ΔE is the mass difference between the quarkonium/hybrid in the initial state and the quarkonium in the final state once recoil corrections are neglected. $|\bar{I}_{i\to f}(s)|^2$ stands for the spin average of $|I_{i\to f}(s)|^2$. Note that the dependence on the unknown low-energy constants λ and λ' is universal, namely independent of the initial and final states. As an example, in Fig. 2 we show a preliminary estimate of $|\bar{I}_{i\to f}(s)|^2$ assuming that $\Upsilon(10860)$ is a $5s$ quarkonium and a $2(s/d)_1$ hybrid (H_1'). We observe that the shapes are qualitatively different.

5. Discussion

We have shown how to estimate the long-distance behavior of the form factors $g_i(r)$, $i = 0, 1, 2, 3, 4$ in Eq. (9) and Eq. (10) controlling the dipion emission in heavy quarkonium and heavy quarkonium hybrid transitions using the QCD effective string theory. This is necessary to have realistic estimates of these transitions for states with a size larger than the typical QCD scale, which applies to a number of charmonium and bottomonium mesons listed in the PDG. We have illustrated the procedure with a simplified interaction Lagrangian. The complete analysis will be presented elsewhere.

The interaction Lagrangian in Eq. (4) is built in such a way that both the symmetries of the Effective String Theory and of the Chiral Lagrangian are implemented. We have focused on

$SU_L(2) \otimes SU_R(2)$ chiral symmetry. The generalization to the $SU_L(3) \otimes SU_R(3)$ case, which would allow, for instance, to study the η emission, is straightforward [17].

An interesting application of our results is to states that are suspected of being a mixture of hybrids and quarkonium, since the analysis of the dipion spectrum may help in elucidating the mixing pattern. In Fig. 2 we have displayed an example related to this issue. Indeed, in [4] it was found that $\Upsilon(10860)$ could be a mixture of a $5s$ quarkonium and a $2(s/d)_1$ hybrid, which may explain the spin-symmetry violating decays of this meson.

From Eq. (16) it follows that the ratio of the spectra is independent of the low-energy constants λ and λ', and depends only on the initial and final states. Unfortunately, this interesting feature may not persist when additional operators are included in Eq. (4), see Appendix B.

If the dipion three-momentum is small ($rs \ll 1$), as it is the case if the masses of the initial and final states differ by about $2m_\pi$, Eq. (11) and Eq. (12) simplify as the pion fields can be expanded in z and the integrals over z reduce to functions of r, namely to the expected form in the BOEFT Eq. (9) and Eq. (10). In the EFT spirit, a systematic analysis of the operators contributing to each order of the expansion $rs \ll 1 \ll r\Lambda_{\rm QCD}\,, \Lambda_{\rm QCD}/m_\pi$ is necessary and is underway, see Appendix B. When $rs \ll 1$ does not hold, the full non-local form of Eq. (9) and Eq. (10) must be kept. This occurs when the energy difference between the initial and final states ΔE is the same size as the typical $1/r$ of (at least one of) the states. Note also that formulas Eq. (10) and Eq. (12) have been displayed with hybrid states in mind. However, they also hold for any exotic quarkonium with LDF quantum numbers 1^{+-}. Different 1^{+-} exotics would correspond to different functions $g_i(r)$ in Eq. (10) and $g(r,z)$ in Eq. (12).

The numerical calculations leading to Fig. 2 have been carried out with the LO wave functions of ref. [4], which uses the lattice data of ref. [18]. More recent lattice data for the hybrid static potentials exist, see [19–21]. Wave functions incorporating spin-dependent effects (partial NLO) also exist [22–24], and, recently, lattice evaluations of the spin-dependent potential have been carried out [25]. All together, it indicates that increasing precision can be achieved for dipion transitions once the interaction Lagrangian of pions with heavy quarkonium and with heavy quarkonium hybrids is pinned down. We have presented steps forward towards this goal.

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A. Dipion emission from higher string excitations

A.1 String N=2 excitations

The $N = 2$ excitations consist of five states. A Σ_g^+ state, two degenerate Π_g states, and two degenerate Δ_g states corresponding to the clockwise (R) and anticlockwise (L) rotations of $|L_z| = 1$

and $|L_z| = 2$ respectively.

Remaining at the expansion order of the exponential already seen in Sec. 2.3, ($O(\vec{x})$), we obtain the amplitudes for the Π_g states. $\sqrt{-\det(\partial_a x^\mu \partial_b x_\mu)} \simeq 1$ still holds.

$$\langle \pi(\vec{q}), \pi(\vec{q}\,'); 0_{\rm EST}|S_{\rm int}|0_\pi; 1\Pi_g^R\rangle = \langle \pi(\vec{q}), \pi(\vec{q}\,'); 0_{\rm EST}|S_{\rm int}|0_\pi; 1\Pi_g^L\rangle^* \tag{17}$$
$$= \frac{16\pi\sqrt{2\pi}}{f_\pi^2\sqrt{\sigma}r}\left(\lambda q_\mu q'^\mu + \lambda' \frac{m_\pi^2}{2B_0}\right)\frac{\sin\left[(q_z+q'_z)\frac{r}{2}\right]}{\frac{4\pi^2}{r^2}-(q_z+q'_z)^2}\frac{i}{\sqrt{2}}\left(q^1+q'^1+i(q^2+q'^2)\right)\delta\left(E_q+E'_q-\frac{2\pi}{r}\right)$$

For the other states, we need to expand the exponential to higher orders ($O(\vec{x}^2)$) being the only terms that contribute $e^{-i(q+q')^i x^i} \sim -\frac{1}{2}(q+q')^i(q+q')^j x^i(z,t)x^j(z,t)$. At this level of the exponential and with $\sqrt{-\det(\partial_a x^\mu \partial_b x_\mu)} \simeq 1$ we find the amplitudes for the Δ_g states.

$$\langle \pi(\vec{q}), \pi(\vec{q}\,'); 0_{\rm EST}|S_{\rm int}|0_\pi; 1\Delta_g^R\rangle = \langle \pi(\vec{q}), \pi(\vec{q}\,'); 0_{\rm EST}|S_{\rm int}|0_\pi; 1\Delta_g^L\rangle^*$$
$$= -\frac{4}{f_\pi^2\sigma}\left(\lambda q_\mu q'^\mu + \lambda' \frac{m_\pi^2}{2B_0}\right)\left[\frac{1}{(q_z+q'_z)^2}+\frac{1}{\frac{4\pi^2}{r^2}-(q_z+q'_z)^2}\right](q_z+q'_z)\sin\left[(q_z+q'_z)\frac{r}{2}\right]\times \tag{18}$$
$$\times\left(q^1+q'^1+i(q^2+q'^2)\right)^2\delta\left(E_q+E'_q-\frac{2\pi}{r}\right)$$

For the Σ_g^+ state there are two contributions, one at first order in the metric and $O(\vec{x}^2)$ in the exponential, as in the previous case, the other comes from the partial terms of the metric $\sqrt{-\det(\partial_a x^\mu \partial_b x_\mu)} \simeq 1 - \frac{1}{2}\partial_0 x^i(z,t)\partial_0 x^i(z,t) + \frac{1}{2}\partial_z x^i(z,t)\partial_z x^i(z,t)$ and $e^{-i(q+q')^i x^i} \sim 1$. The amplitude is given by both contributions

$$\langle \pi(\vec{q}), \pi(\vec{q}\,'); 0_{\rm EST}|S_{\rm int}|0_\pi; 1\Sigma_g^+\rangle =$$
$$= -\frac{4}{f_\pi^2\sigma}\left(\lambda q_\mu q'^\mu + \lambda' \frac{m_\pi^2}{2B_0}\right)\left[\frac{(q^1+q'^1)^2+(q^2+q'^2)^2}{\frac{4\pi^2}{r^2}-(q_z+q'_z)^2}+\frac{(q^1+q'^1)^2+(q^2+q'^2)^2-\frac{4\pi^2}{r^2}}{(q_z+q'_z)^2}\right]\times \tag{19}$$
$$\times(q_z+q'_z)\sin\left[(q_z+q'_z)\frac{r}{2}\right]\delta\left(E_q+E'_q-\frac{2\pi}{r}\right).$$

A.2 String N=3 excitations

For the $N = 3$ excitations decaying to the ground state, we only consider the case of the Σ_u^- state, because it enters in the Schrödinger equation of the lower lying hybrid quarkonium mesons, and the Σ_u^+ state as a non-trivial example. In both cases, we may have contributions from the first order in the expansion of the metric and zeroth order in the exponential, and from the zeroth order in expansion of the metric and the second order in the exponential, analogously to the Σ_g^+ case above. We obtain,

$$\langle \pi(\vec{q}), \pi(\vec{q}\,'); 0_{\rm EST}|S_{\rm int}|0_\pi; 1\Sigma_u^-\rangle = 0 \tag{20}$$

$$\langle \pi(\vec{q}), \pi(\vec{q}\,'); 0_{\rm EST}|S_{\rm int}|0_\pi; 1\Sigma_u^+\rangle = \tag{21}$$
$$= -\frac{32\pi^2}{f_\pi^2\sigma r^2}\left(\lambda q_\mu q'^\mu + \lambda' \frac{m_\pi^2}{2B_0}\right)\left[1+\frac{(q^1+q'^1)^2+(q^2+q'^2)^2}{\frac{9\pi^2}{r^2}-(q_z+q'_z)^2}\right]\frac{(q_z+q'_z)\cos\left[(q_z+q'_z)\frac{r}{2}\right]}{\frac{\pi^2}{r^2}-(q_z+q'_z)^2}\delta\left(E_q+E'_q-\frac{3\pi}{r}\right)$$

The vanishing result in Eq. (20) can be traced back to reflection symmetry. Indeed, both the metric and the scalar term arising in the expansion of the exponential at second order are invariant under this symmetry, hence an odd state as Σ_u^- cannot decay into an even state as Σ_g^+. The result in Eq. (21) shows that a non-vanishing transition is possible if we have an even state like Σ_u^+ rather than Σ_u^- in the initial state.

We next display an example of a transition between excited states, which is relevant for an eventual analysis of hybrid-to-hybrid transitions.

$$\langle \pi(\vec{q}), \pi(\vec{q}\,'); 1\Pi_u^R | S_{\rm int} | 0_\pi; 1\Sigma_u^- \rangle = \langle \pi(\vec{q}), \pi(\vec{q}\,'); 1\Pi_u^L | S_{\rm int} | 0_\pi; 1\Sigma_u^- \rangle^* = \tag{22}$$
$$= -\frac{16\pi}{f_\pi^2 r\sqrt{\sigma\pi}} \left(\lambda q_\mu q'^\mu + \lambda' \frac{m_\pi^2}{2B_0}\right) \frac{\sin\left[(q_z+q'_z)\frac{r}{2}\right]}{\frac{4\pi^2}{r^2}-(q_z+q'_z)^2} \frac{i}{\sqrt{2}} \left(q^1+q'^1+i(q^2+q'^2)\right) \delta\left(E_q+E'_q-\frac{2\pi}{r}\right)$$

B. Additional terms in the interaction Lagrangian

The Lagrangian in Eq. (4) appears to be the simplest possibility of a lower-order Lagrangian that respects Poincaré, chiral and reparameterization invariance symmetries. However, the string part contains different orders in the $1/r\Lambda_{\rm QCD}$ expansion, and, as a consequence, the pion field contains different orders in the $p/\Lambda_{\rm QCD}$ expansion. This is due to the fact that in the physical frame $x^0 \sim x^3 \sim r$ but $x^1 \sim x^2 \sim 1/\Lambda_{\rm QCD}$ and the pion fields depend on both x^0, x^3 and x^1, x^2. In order to be systematic, let us first write the possible operators in the physical frame, in which the counting is explicit, and make sure later on that a fully covariant expression exist for them. In that frame, the string symmetry group reduces to $SO(1,1)\otimes O(2)$. With this space-time group the LO $\mathcal{L}_{\rm ChS}$ reads,

$$\mathcal{L}_{\rm ChS} = (\lambda+\eta){\rm Tr}(\partial_0 U^\dagger \partial^0 U) + (\lambda+\eta){\rm Tr}(\partial_3 U^\dagger \partial^3 U) + \lambda {\rm Tr}(\partial_i U^\dagger \partial^i U) + \lambda' {\rm Tr}(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U)$$
$$U = U(t, x^i(t,z), z) \quad , \quad i = 1,2 \quad , \quad z = x^3, t = x^0 \tag{23}$$

This Lagrangian density is to be integrated over $z \in [-r/2, r/2]$ and t in order to get the LO action. Note that an additional parameter (η) appears with respect to $\mathcal{L}_{\rm ChS}$ in Eq. (4). It leads to a $g_2(r) = -\eta r$ and to $g_0(r) = (\lambda+\eta)r$, which modifies the values obtained below Eq. (11). The terms proportional to η can be obtained from the following Poincaré, chiral and reparameterization invariant Lagrangian,

$$\delta S_{\rm int} = \eta \int d^2\xi \sqrt{-{\rm det}(\partial_a x^\mu \partial_b x_\mu)}\, \partial_a x^\mu \partial^a x^\nu {\rm Tr}\left(\partial_\mu U^\dagger \partial_\nu U\right) \tag{24}$$

Indeed, at LO in the $1/r\Lambda_{\rm QCD}$ expansion the determinant reduces to 1, and $\partial_a x^\mu \partial^a x^\nu$ only contributes when $\mu = \nu = 0$ and $\mu = \nu = 3$ giving rise to the terms proportional to η in Eq. (23). At NLO, the determinant remains 1, and $\partial_a x^\mu \partial^a x^\nu$ contributes when $\mu = 0$ or $\mu = 3$ and $\nu = i$, $i = 1,2$ (and exchanging μ by ν). It leads to,

$$\delta S_{\rm int}|_{NLO} = 2\eta \int dt\, dz \left(\partial_0 x^i {\rm Tr}\left(\partial_0 U^\dagger \partial_i U\right) - \partial_z x^i {\rm Tr}\left(\partial_z U^\dagger \partial_i U\right)\right) \tag{25}$$

The contributions arising from these terms, proportional to η, modify the formulas in Section 4, which we only display for illustration purposes, and our final results. In particular, the ones in Eq. (23) give additional contributions to quarkonium-to-quarkonium transitions and the ones in Eq. (25) to hybrid-to-quarkonium transitions. They also modify the shift of the string tension due to the light quark masses Eq. (5), but in a rather trivial way $\lambda \longrightarrow \lambda + \eta/2$ and $\lambda'' \longrightarrow \lambda'' - 3B_0\eta/(4\pi f_\pi)^2$.

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