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# Tetraquark equations

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In recent years, steady progress has been made to formulate ever more accurate practical covariant equations describing the tetraquark where pairwise interactions are dominated by diquark ($D$), antidiquark ($\bar{D}$), or meson ($M$) formation, and where quark-antiquark ($q\bar{q}$) annihilation is taken into account. In all cases, the basic idea has been to exploit the dominance of $D$, $\bar{D}$, and $M$ formation to reduce the inherently four-body equations down to coupled two-body equations. Here we describe the progressive development of such equations, and introduce a new refinement whereby all pole contributions to pairwise interactions between the quarks (corresponding to $D$, $\bar{D}$, and $M$ formation) are included non-perturbatively, while only their non-pole parts are used for a perturbative multiple scattering series.

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## 1. Covariant four-body equations

A natural first step in describing a tetraquark in quantum field theory (QFT), is to treat it as a purely 4-body system, consisting of 2 quarks and 2 antiquarks ($2q2\bar{q}$), where only pairwise interactions between the quarks are taken into account. In this case one can formally write down the bound state equation for a tetraquark as

$$\Phi = K^{(4)} G_0^{(4)} \Phi \tag{1}$$

where $\Phi$ is the tetraquark bound-state form factor, $K^{(4)}$ is the 4-particle-irreducible kernel, $G_0^{(4)}$ is the 4-particle free propagator, and where for simplicity of presentation, we have assumed distinguishable quarks.

In order to express $K^{(4)}$ in terms of two-body kernels, it is useful to assign the labels 1, 2 to the two quarks and 3, 4 to the antiquarks, and note that there are only 3 possible pairwise channels in the 4-body system:

| 1 | 2 | 3 |
|---|---|---|
| $(q\bar{q})(q\bar{q})$ | $(q\bar{q})(q\bar{q})$ | $(qq)(\bar{q}\bar{q})$ |
| (13)(24) | (14)(23) | (12)(34) |

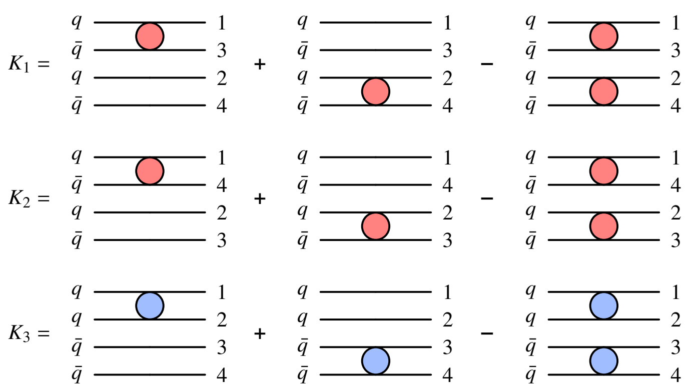
One can thus write

$$K^{(4)} = K_1 + K_2 + K_3$$

where each $K_\alpha$ is expressed in terms of 2-body kernels $K_{ij}$ as:

$$\begin{aligned} K_1 &= K_{13} + K_{24} - K_{13}K_{24} \\ K_2 &= K_{14} + K_{23} - K_{14}K_{23} \\ K_3 &= K_{12} + K_{34} - K_{12}K_{34} \end{aligned} \tag{2}$$

where the minus signs in the product terms are needed to avert overcounting [1]. Graphically:




The disconnected diagrams making up $K^{(4)}$ contain delta functions, making the kernel non-compact. To enable practical 4-body calculations, Khvedelidze and Kvinikhidze (KK) [1] implemented a Faddeev-like rearrangement to obtain covariant 4-body equations with a compact kernel. In KK's description, the bound state form factor $\Phi$ is expressed as a sum of components, $\Phi = \Phi_1 + \Phi_2 + \Phi_3$, which are related to each other through the equations

$$\Phi_\alpha = T_\alpha \sum_{\beta \neq \alpha} G_0^{(4)} \Phi_\beta \tag{3}$$

where $T_\alpha$ are the t matrices generated by the corresponding $K_\alpha$, i.e.

$$T_\alpha = K_\alpha + K_\alpha G_0^{(4)} T_\alpha, \tag{4}$$

and which can be expressed in terms of two-body t matrices $T_{ij}$ analogously to Eqs. (2) as

$$\begin{aligned} T_1 &= T_{13} + T_{24} + T_{13}T_{24} &&= T_1^+ + T_1^\times \\ T_2 &= T_{14} + T_{23} + T_{14}T_{23} &&= T_2^+ + T_2^\times \\ T_3 &= T_{12} + T_{34} + T_{12}T_{34} &&= T_3^+ + T_3^\times \end{aligned}$$

To be noted are the definitions of the sum and product terms $T_\alpha^+$ and $T_\alpha^\times$, and that the product terms come with a plus sign in contrast to the product terms of Eqs. (2).

Application to the $2q2\bar{q}$ system requires antisymmetriation of $2q$ and $2\bar{q}$ states, reducing above to [2]

$$\begin{pmatrix} \Phi_1 \\ \Phi_3 \end{pmatrix} = \left[ \begin{pmatrix} \frac{1}{2}T_1^+ & 0 \\ 0 & T_3^+ \end{pmatrix} + \begin{pmatrix} \frac{1}{2}T_1^\times & 0 \\ 0 & T_3^\times \end{pmatrix} \right] 2 \begin{pmatrix} -\mathcal{P}_{12} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_3 \end{pmatrix}$$

or symbolically

$$\tilde{\Phi} = \left(\mathcal{T}^+ + \mathcal{T}^\times\right) \mathcal{R} \tilde{\Phi} \tag{5}$$

## 2. Giessen tetraquark model

In 2012, Heupel *et al.* [3] (Giessen group) modelled the tetraquark by using two approximations in the above covariant 4-body equations:

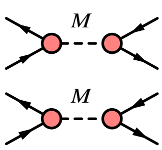
(i) The single scattering terms making up $\mathcal{T}^+$ were neglected:

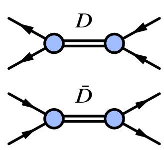
$$\begin{aligned} T_1 &= \cancel{T_{13} + T_{24}} + T_{13}T_{24} \\ T_3 &= \cancel{T_{12} + T_{34}} + T_{12}T_{34} \end{aligned}$$

resulting in a simplified bound state equation

$$\tilde{\Phi} = \left(\cancel{\mathcal{T}^+} + \mathcal{T}^\times\right) \mathcal{R} \tilde{\Phi} \tag{6}$$

(ii) The two-body t matrices were approximated by using the meson and diquark pole approximation: $T_{ij} = \Gamma_{ij} D_{ij} \bar{\Gamma}_{ij}$, so that graphically

$T_1 = T_{13}T_{24} =$ 


$T_3 = T_{12}T_{34} =$ 

where $M$, $D$, and $\bar{D}$ represent a meson, diquark, and antidiquark, respectively. In this approximation one can write

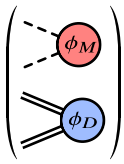
$$\mathcal{T}^{\times} = \begin{pmatrix} \Gamma_{13}\Gamma_{24} & 0 \\ 0 & \Gamma_{12}\Gamma_{34} \end{pmatrix} \begin{pmatrix} \frac{1}{2}D_{13}D_{24} & 0 \\ 0 & D_{12}D_{34} \end{pmatrix} \begin{pmatrix} \bar{\Gamma}_{13}\bar{\Gamma}_{24} & 0 \\ 0 & \bar{\Gamma}_{12}\bar{\Gamma}_{34} \end{pmatrix}$$
$$\equiv -\Gamma D \bar{\Gamma}. \tag{7}$$

In this way the bound state equation Eq. (6) becomes

$$\tilde{\Phi} = -\Gamma D \bar{\Gamma} \mathcal{R} \tilde{\Phi}. \tag{8}$$

Defining the matrix $\boldsymbol{\phi}$ of tetraquark $\to MM$, $D\bar{D}$ amplitudes

$$\boldsymbol{\phi} \equiv \bar{\Gamma}\mathcal{R}\tilde{\Phi} = \begin{pmatrix} \phi_M \\ \phi_D \end{pmatrix} = \tag{9}$$




Eq. (8) can be recast as coupled two-body equations for the tetraquark transition amplitudes:

$$\boldsymbol{\phi} = VD\boldsymbol{\phi} \tag{10}$$

where the kernel $V$ is defined as the $2 \times 2$ matrix

$$V = -\bar{\Gamma}\mathcal{R}\Gamma \tag{11}$$

which consists of crossed-quark terms. Eq. (10) are the final tetraquark equations of the Giessen group, which can be represented graphically as in Fig. 1.

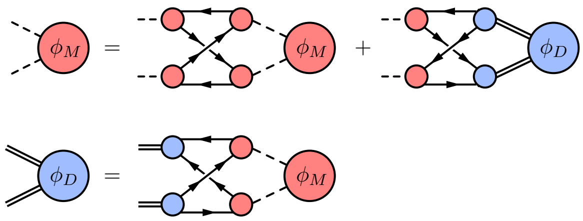



**Figure 1:** Tetraquark equations of the Giessen group [3]: $\phi_M$ and $\phi_D$ are transition amplitudes for tetraquark $\to MM$ and tetraquark $\to D\bar{D}$, respectively .

## 3. Unified tetraquark equations I

In the Giessen tetraquark model, the term $\mathcal{T}^{+}$ was neglected. Recently we explored the consequences of retaining this term [2]. With $\mathcal{T}^{+}$ retained, Eq. (5) can be rearranged to obtain

$$\tilde{\Phi} = (1 - \mathcal{T}^{+}\mathcal{R})^{-1}\mathcal{T}^{\times}\mathcal{R}\tilde{\Phi}. \tag{12}$$

Using the meson and diquark pole approximation in $\mathcal{T}^\times$ [see Eq. (7)], but not in $\mathcal{T}^+$, the above equation becomes

$$\tilde{\Phi} = -(1 - \mathcal{T}^+\mathcal{R})^{-1}\Gamma D\bar{\Gamma}\mathcal{R}\tilde{\Phi}. \tag{13}$$

Again defining $\boldsymbol{\phi}$, the matrix of tetraquark $\longrightarrow MM, D\bar{D}$ amplitudes, as in Eq. (9), the above equation can be recast into a coupled two-body equation given by Eq. (10), but this time with the kernel $V$ defined as

$$V = -\bar{\Gamma}\mathcal{R}(1 - \mathcal{T}^+\mathcal{R})^{-1}\Gamma. \tag{14}$$

Expanding the inverse term into a perturbation series, one obtains the "unified tetraquark model" expressed by the equations

$$\boldsymbol{\phi} = VD\boldsymbol{\phi}$$

$$\begin{aligned} V &= -\bar{\Gamma}\mathcal{R}\left[1 + \mathcal{T}^+\mathcal{R} + (\mathcal{T}^+\mathcal{R})^2 + \ldots\right]\Gamma \\ &= V^{(0)} + V^{(1)} + V^{(2)} + \ldots \end{aligned} \tag{14}$$

so called because the tetraquark model of the Giessen group, as illustrated in Fig. 1, corresponds to keeping just the zero order term in the series, i.e., $V \longrightarrow V^{(0)} = -\bar{\Gamma}\mathcal{R}\Gamma$, and the tetraquark model of Faustov *et al.* [4] (Moscow group), as illustrated in Fig. 2, corresponds to keeping just the first order term $V \longrightarrow V^{(1)} = -\bar{\Gamma}\mathcal{R}\mathcal{T}^+\mathcal{R}\Gamma$.

This suggests that the Giessen and Moscow groups have been calculating non-overlapping parts of the same tetraquark equations, namely

$$\boldsymbol{\phi} = [V^{(0)} + V^{(1)}]D\boldsymbol{\phi}. \tag{16}$$

which are illustrated in Fig. 3.

## 4. Unified tetraquark equations II

The unified tetraquark equations described above, utilise a kernel $V$ that is expressed as a perturbation series

$$V = V^{(0)} + V^{(1)} + V^{(2)} + \ldots \tag{17}$$

where each term $V^{(n)}$ is a $2\times 2$ matrix of $MM \longrightarrow MM$, $MM \leftrightarrow D\bar{D}$, and $D\bar{D} \longrightarrow D\bar{D}$, amplitudes, each consisting of $n$ rescatterings between 2 quarks. Although the first two terms of this series unify the Giessen and Moscow tetraquarks models, it is doubtful if the full series of Eq. (17) is convergent. This is because the multiple-scatterings occur via full two-quark t matrices $T_{ij}$ which themselves have meson, diquark, or antidiquark poles. Here we describe a way to overcome the

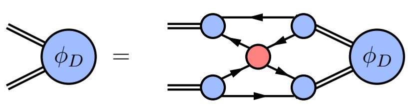



**Figure 2:** Tetraquark equations of the Moscow group [4]: $\phi_D$ is the transition amplitude for tetraquark $\longrightarrow D\bar{D}$ transition.

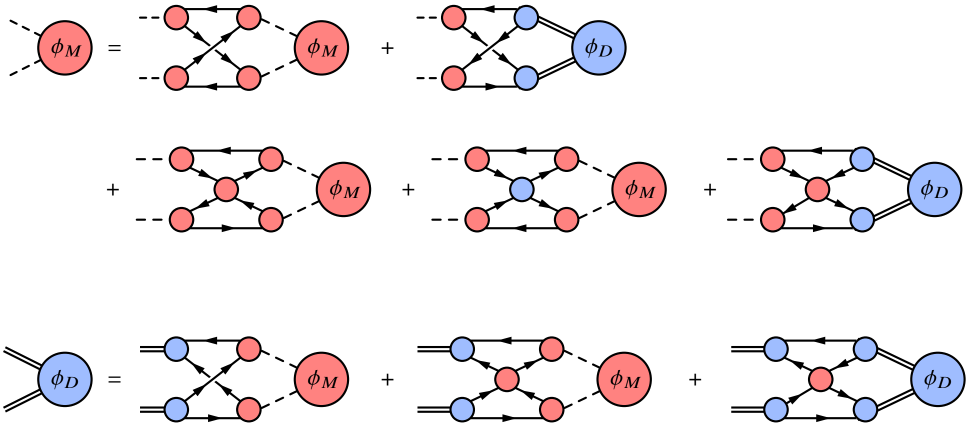



**Figure 3:** Illustration of the unified tetraquark equations, Eq. (16).

convergence problem posed by Eq. (17). The essential idea is to split off the pole parts of $T_{ij}$ and include their contributions non-perturbatively.

Thus we write *all* 2-body t matrices $T_{ij}$ as the sum of a "pole" part $T^P_{ij}$ and a "non-pole" part $T^{NP}_{ij}$ which, to simplify notation, we shall denote by the two-body kernel $K_{ij}$ (which is only one of the terms contributing to $T^{NP}_{ij}$):

$$T_{ij} = T^P_{ij} + K_{ij}. \tag{18}$$

This then enables us to write the t matrices $\mathcal{T}^\times$ and $\mathcal{T}^+$ of Eq. (5), as

$$\mathcal{T}^\times = \mathcal{T}^\times_P + \mathcal{T}^\times_{PK} + \mathcal{K}^\times \tag{19a}$$

$$\mathcal{T}^+ = \mathcal{T}^+_P + \mathcal{K}^+ \tag{19b}$$

in a self-explanatory notation. One can then write the sum of these, which constitutes the kernel of Eq. (5), as

$$\begin{aligned}\mathcal{T}^+ + \mathcal{T}^\times &= \mathcal{T}^+_P + \mathcal{T}^\times_P + \mathcal{T}^\times_{PK} + \mathcal{K}^+ + \mathcal{K}^\times \\ &\equiv \sum_{j=1}^{3} \mathcal{F}_j \mathcal{D}_j \bar{\mathcal{F}}_j + \mathcal{K}\end{aligned}$$

where $\mathcal{K} = \mathcal{K}^+ + \mathcal{K}^\times$ is made up of only non-pole terms while $\mathcal{T}^+_P + \mathcal{T}^\times_P + \mathcal{T}^\times_{PK}$ is expressed in terms of matrices $\mathcal{F}_j$, $\mathcal{D}_j$, and $\bar{\mathcal{F}}_j$ whose detailed forms will be given elsewhere [5]. The resulting tetraquark equations are formulated by analogy to the ones of the Giessen group, described above, and are expressed as

$$\boxed{\boldsymbol{\phi}_i = \sum_{j=1}^{3} \bar{\mathcal{F}}_i \mathcal{R}(1 - \mathcal{K}\mathcal{R})^{-1} \mathcal{F}_j \mathcal{D}_j \boldsymbol{\phi}_j} \tag{20}$$

where

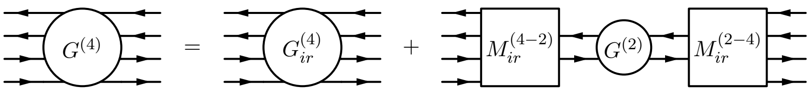
$$\boldsymbol{\phi}_j = \bar{\mathcal{F}}_j \mathcal{R} \tilde{\Phi}. \tag{21}$$

Although a detailed description will be presented elsewhere, here it is important to note that expanding the inverse term in Eq. (20) results in a perturbation series involving multiple scattering of two-quarks via their background (non-pole) t matrices only. By contrast, all the pole parts of the two-quark interactions are summed non-perturbatively.

## 5. Incorporating $q\bar{q}$ annihilation

All the above tetraquark models are based on the 4-body equations of KK for which the number of particles is conserved. However, a $2q2\bar{q}$ system couples to $q\bar{q}$ channels via $q\bar{q}$ annihilation, so the natural question is how to correctly incorporate $q\bar{q}$ annihilation into the above teteraquark models. We provided a correct but lengthy answer to this question in 2014 involving addition of a disconnected part to the usual (connected) 2-body $q\bar{q}$ t matrix $T_{q\bar{q}}$ [6]. More recently we found a much simpler way to derive the same equations as those of 2014, but with a bonus whereby the $q\bar{q}$ model-kernel can be supplemented with an amplitude $\Delta$ which is able to make the overall tetraquark description exact in QFT. The steps that lead to this description are as follows:

- **Step 1:** Recognise that the full $2q2\bar{q}$ Green function $G^{(4)}$ can be expressed in terms of its $q\bar{q}$-irreducible [$G^{(4)}_{ir}$] and $q\bar{q}$-reducible parts as




  where $G^{(2)}$ is the full $q\bar{q}$ Green function specified by a two-body kernel $K^{(2)}$ as $G^{(2)} = G^{(2)}_0 + G^{(2)}_0 K^{(2)} G^{(2)}$. This illustrates that the same tetraquark pole must be present in both $G^{(4)}$ and $G^{(2)}$: as $P^2 \to M^2$,

$$G^{(4)} \to i\frac{\Psi\bar{\Psi}}{P^2 - M^2}, \qquad G^{(2)} \to i\frac{G^{(2)}_0\Gamma^*\bar{\Gamma}^*G^{(2)}_0}{P^2 - M^2}, \tag{22}$$

  where $M$ is the tetraquark mass and $\Gamma^*$ is the $q\bar{q}$ bound state form factor (for the formation of the tetraquark) satisfying the two-body equation

$$\Gamma^* = K^{(2)}G^{(2)}_0\Gamma^*. \tag{23}$$

  So *all* poles in $G^{(2)}$ will appear in $G^{(4)}$, suggesting that a tetraquark should be defined in QFT as a pole in $G^{(4)}_{ir}$ at $P^2 = M_0^2$ where $M_0$ is the tetraquark mass *before* $q\bar{q}$ annihilation is taken into account.

- **Step 2:** Express the *exact* two-body ($q\bar{q}$) kernel $K^{(2)}$ as

$$\boxed{K^{(2)} = \Delta + A^{(2-4)}G^{(4)}_{ir}A^{(4-2)}} \tag{24}$$

  where $A^{(2-4)}$ and $A^{(4-2)}$ are $2 \leftarrow 4$ and $4 \leftarrow 2$ $q\bar{q}$-irreducible amplitudes, and $\Delta$ is defined as consisting of all the contributions missing from the last term of Eq. (24). Assuming that $G^{(4)}_{ir}$ has a "bare" tetraquark pole at $P^2 = M_0^2$, so that as $P^2 \to M_0^2$,

$$G^{(4)}_{ir} \;\to\; i\frac{\Psi_0\bar{\Psi}_0}{P^2 - M_0^2} + B \tag{25}$$

where $B$ is a background term, it is easy to show that $G^{(2)}$ has a "physical" tetraquark pole at $P^2 = M^2$ where [7]

$$M^2 = M_0^2 + i\bar{\Psi}_0 A^{(4-2)} \left[ G_0^{(2)-1} - \Delta - A^{(2-4)} B A^{(4-2)} \right]^{-1} A^{(2-4)} \Psi_0. \tag{26}$$

- **Step 3: Apply to a coupled-channels $MM - D\bar{D}$ tetraquark model**

  All the coupled-channels $MM - D\bar{D}$ models of a tetraquark discussed above (i.e., those based on the 4-body equations of KK), are described by a 2-body bound state equation of the form given by Eq. (10), with kernel $V$ being specific to the particular model. In any such model, Eq. (24) takes the form

$$K^{(2)} = \Delta + \bar{N} G N \tag{27}$$

  where $G$ is the 2×2 matrix Green function generated by iterating the kernel $V$, and $N$ (similarly $\bar{N}$) is a $2 \times 1$ matrix of transition amplitudes $q\bar{q} \to MM, D\bar{D}$. Since $G = D + GVD$, where $D$ is the matrix of $MM$ and $D\bar{D}$ propagators defined in Eq. (7), one has that

$$K^{(2)} = \Delta + \bar{N} D (1 - VD)^{-1} N. \tag{28}$$

  Then using this expression in Eq. (23) results in the coupled equations

$$\boldsymbol{\phi} = VD\boldsymbol{\phi} + NG_0^{(2)}\Gamma^*$$
$$\Gamma^* = \Delta G_0^{(2)}\Gamma^* + \bar{N}D\boldsymbol{\phi}$$

  which are the corresponding covariant tetraquark equations with $q\bar{q}$ annihilation included.

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