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The evolution of the parton distribution functions to percent accuracy

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I present analytic results for the scale evolution of the ten lowest moments of the Parton Distribution Functions (PDFs) through four loops in QCD. I discuss the evolution of the PDFs to approximate N^{3}LO accuracy, which is obtained by combining input from the computed moments and from physical constraints. The contribution to the scale evolution of the PDFs of the approximate N^{3}LO corrections is of the order of 1% or less for $x \gtrsim 10^{-4}$. The uncertainty of the approximations is estimated below 1% for $x \gtrsim 10^{-4}$, thus they are relevant for precision phenomenology at the LHC.

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1. Introduction

The forthcoming experiments at the High-Luminosity Large Hadron Collider (HL-LHC) and at the Electron-Ion Collider (EIC) are expected to push the precision of several measurements towards the level of 1% [1, 2]. It is important that the theoretical predictions for such processes match the same precision goal, requiring the calculation of radiative corrections at Next-to-Next-to-Next-to-Leading Order (N^{3}LO) in QCD. This talk focuses on the N^{3}LO contributions to the DGLAP equations [3–5]

$$\frac{d}{d\log\mu^2} q^i_{\rm ns} = \sum_{n=0} a_{\rm s}^{n+1}\, P^{(n)i}_{\rm ns} \otimes q^i_{\rm ns}, \tag{1a}$$

$$\frac{d}{d\log\mu^2} \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} = \begin{pmatrix} P_{\rm qq} & P_{\rm qg} \\ P_{\rm gq} & P_{\rm gg} \end{pmatrix} \otimes \begin{pmatrix} q_{\rm s} \\ g \end{pmatrix} \quad \text{with} \quad P_{\rm ik}(x,\alpha_{\rm s}) = \sum_{n=0} a_{\rm s}^{n+1}\, P^{(n)}_{\rm ik}(x)\,, \tag{1b}$$

which describe the scale evolution of the non-singlet (1a) and singlet (1b) Parton Distribution Functions (PDFs). The analytic calculation of the four-loop splitting functions, $P^{(3)}_{\rm ns}(x)$ and $P^{(3)}_{\rm ik}(x)$, is highly challenging. Currently only partial results are available, notably in the large-n_f limit [6–9]. In recent years, there has been significant progress in the calculation of the non-singlet splitting functions at four loops: the planar limit was computed in [10], the terms of $O(n_f)$ proportional to the Casimir invariant C_F^3 were published in [11] and the complete contribution at $O(n_f)$ has been determined very recently [12]. Despite the significant progress, the available results concern only selected colur factors and they cannot be used for phenomenology. In fact, by computing eight even *moments* (up to moment number $N = 16$) of the corresponding N^{3}LO splitting functions, the evolution of the non-singlet PDFs has been parameterised with a level of precision of the order of 1% [10]. Until recently, the status of the singlet calculation has been less advanced. Beyond the large-n_f limit [7–9], the contributions proportional to n_f^2 in $P^{(3)}_{\rm ps}$ and in $P^{(3)}_{\rm gq}$ have been published in [13] and [14], respectively. Regarding the calculation of fixed moments, the six lowest even moments ($N \leq 12$) of $P^{(3)}_{\rm ps}$ and the five lowest moments ($N \leq 10$) of $P^{(3)}_{\rm qg}$, $P^{(3)}_{\rm gq}$ and $P^{(3)}_{\rm gg}$ were published in [15, 16]. Since only a few moments of the singlet splitting functions were available, it was not possible to approximate the scale evolution of the singlet PDFs with the same precision as the non-singlet PDFs. In this talk, I report on the results of refs. [17–20], which provide the ten lowest moments ($N \leq 20$) of all the singlet splitting functions at four loops. In turn, these moments provide input for highly precise approximations of the N^{3}LO scale evoluion of the flavour singlet combination of quark PDFs and of the gluon PDF.

2. Methodology

The computational method used in refs. [17–20] is based on the operator product expansion of the structure functions in the Deep Inelastic Scattering (DIS) [21, 22]. Indeed, the moments of the splitting functions are the anomalous dimensions of the gauge invariant operators of leading twist. These are computed by renormalising *off-shell* two-point functions with one insertion of the leading twist operators. One complication in applying this method stems from the fact that, upon renormalisation, the gauge invariant operators mix with unphysical operators [21, 23–30],

also known as *aliens*. The work of [17–20] relies on the classification of the aliens operators that appear up to four loops and on the alien operator basis determined in ref. [29].

The Feynman diagrams of the two-point functions, with either quarks or gluons as external states and the insertion of one gauge invariant operator, were generated, up to four-loop order, with `QGRAF` [31]. The processing of the Feynman rules, the colour algebra and the projection of the open Lorentz or Dirac indices to form factors was performed using in-house code written in `FORM` [32–34]. The reduction of the scalar integrals to the master integrals of refs. [35, 36] was performed with `Forcer` [37]. The same procedure was applied to compute the two-point Feynman diagrams with external either quarks or gluons and with one insertion of the alien operators [29], which are required up to three loops. The moments of the splitting functions were extracted from the ultraviolet poles of the two-point functions featuring the insertion of the gauge invariant operators, after the subtraction of the *alien* counterterms, i.e. the correlators involving the alien insertion multiplied by the appropriate renormalisation constants giving the *mixing* between the gauge invariant operators and the aliens. These constants have been determined for the operators up to $N = 20$ in [17, 19]. In this conference, S. Van Thurenhout described how to determine the one-loop renormalisation constants for *every* value of N [38], by solving the set of constraints given in ref. [29]. The results obtained with this method [39] agree with the direct calculation of the counterterms, which has been performed recently [40].

3. Results

The ten lowest even moments of all the flavour singlet splitting functions are computed in refs. [17–20]. Such moments provide crucial input to construct highly precise approximations of the N^{3}LO splitting functions directly in the space of momentum fraction x. These consist in families of functions, which match the moments computed up to $N = 20$ and with prescribed asymptotic limits at the endpoints $x \to 0$ [41–48], and $x \to 1$ [49–56]. In [17–20], each splitting function $P_{\rm ij}^{(3)}$ is approximated with a family of 80 functions, obeying the properties above. Taking the envelope of these curves as an estimate of the uncertainty of the approximation, one finds that the error is negligible at large x, while it grows[1] for $x \lesssim 3 \cdot 10^{-3}$. The error on the four-loop coefficient of each splitting function, $P_{\rm ij}^{(3)}$, becomes $O(20\%)$ for $x \sim 10^{-4}$.

Neverthless, the precision on the *scale evolution* of the PDFs is much higher. Indeed, in the *convolution* of the splitting functions with the PDFs themselves, see eqs. (1a) and (1b), the low-x limit of the splitting functions, which suffers from large uncertainties, multiplies the PDFs evaluated at $x \to 1$, which vanish. In practice, the convolution suppresses the uncertainties at small x. The approximate N^{3}LO corrections to the evolution of both the singlet quark PDF and of the gluon PDF are below 1% for $x > 10^{-4}$ and they reach 2% at $x \simeq 10^{-5}$, showing good convergence of the perturbative expansion [18, 20]. The error of the approximation is negligible for $x \gtrsim 10^{-4}$ and it grows up to $O(1\%)$ for $x = 10^{-5}$. The uncertainty associated to variations of the renormalisation scale reduces significantly by going from NNLO to N^{3}LO, where scale uncertainties are below 2% for $x = 10^{-4}$ and below 3% at $x = 10^{-5}$.

[1] In fact, by including only the five lowest moments [15, 16], the uncertainty starts increasing already around $x \lesssim 10^{-1}$, thus compromising the application of the approximate splitting functions in precision phenomenology.

4. Conclusion and outlook

The series of papers [17–20] provides the ten lowest moments of all the flavour singlet splitting functions at four loops, which describe the scale evolution of the corresponding PDFs' *moments* to N^{3}LO accuracy in QCD. The N^{3}LO evolution of the singlet PDFs is approximated with an estimated precision higher than 1% for $x \gtrsim 10^{-4}$ and of $O(2\%)$ at $x = 10^{-5}$ and therefore it can be used for precision phenomenology at the LHC. Further checks on the approximations, such as the calculation of the higher moments and the comparison with the values predicted by the approximated splitting functions, are currently been carried out [57].

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