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Can factorial moment scaling help to the detection of the QCD critical endpoint?

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Scaling of factorial moments in the multiplicity distribution of hadrons produced in high-energy ion collisions has been proposed as a possible signal of proximity to the QCD critical endpoint. We present an analysis showing that this is not the case. The reason is twofold. Firstly, experimentally only a few higher order moments are measurable which may display a transient scaling. This transient behaviour is common for many different distributions not related to criticality. Secondly, even if the asymptotic regime of very high order factorial moments was accessible, hypercontractivity restricts the possible scaling behaviour to a rather universal form for all distributions with finite moments. Based on these observations we argue that the recently observed anomalous factorial moment scaling in STAR experiment at RHIC is a transient behaviour. The unexpected value of the scaling exponent is related to the measure used in this analysis to quantify particle-particle correlations.

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1. Introduction

One of the main goals of modern heavy-ion collision experiments, such as STAR at RHIC (BNL) [1] and NA61/SHINE at SPS (CERN) [2], is the detection of the QCD critical endpoint. There is a large list of theoretical works, during the last decades, pinpointing experimentally measurable observables that signal proximity to the critical point [3]. Among others, factorial moment analysis of the momentum distribution of hadrons, produced in high-energy ion collisions, is a promising approach [4, 5]. However, in such an analysis, to reveal the critical point proximity, one should consider density fluctuations connected to the order parameter of the chiral transition [6]. In the spectrum of experimentally recorded hadrons the particles related to the chiral order parameter are the dipions ($(\pi^+,\pi^-)-pairs$) and the protons. Then, factorial moment analysis of distributions in small momentum-space cells, for these particles, is sensitive to critical fluctuations and intermittency effects, akin to critical opalescence in strongly interacting matter [6].

In this context, the term intermittency refers to the scaling law

$$F_q(M) \sim M^{(q-1)d_F}, \quad M \gg 1 \tag{1}$$

for the appropriately normalized factorial moments $F_q(M)$, defined as:

$$F_q(M) = \frac{\langle n(n-1)\ldots(n-q+1)\rangle}{\langle n\rangle^q} \tag{2}$$

occurring when the neighbourhood of the critical point is reached. In Eqs. (1, 2), M represents the number of cells in momentum space, q is the order of the factorial moment, and d_F is the fractal dimension associated with the geometry of particle clusters in momentum space, formed at the critical point. In Eq. (2), the numerator estimates the average number of q-plets of particles within a cell of very small volume $\propto \frac{1}{M}$, with averaging performed over cells and events. Notice that the fractal dimension d_F is determined by the isothermal critical exponent δ of the transition's universality class [7].

The scaling law in Eq. (1) implies another scaling relation connecting the q-th factorial moment to the second one:

$$F_q(M) \sim F_2(M)^{(q-1)} \tag{3}$$

Eq. (3) is usually called *Factorial Moment Scaling* (FMS) and it has been proposed as an an alternative approach to detect the QCD critical endpoint [8]. The authors in [8] argued that all experimental data, linking to the critical endpoint, refer to hadron states, i.e. to strong interacting matter in the spontaneous broken phase of chiral symmetry. Thus, they studied FMS employing a Ginzburg-Landau free energy describing the phase in which the chiral symmetry is broken. They found that in this case the following FMS relation holds:

$$F_q(M) \sim F_2(M)^{\beta_q} \quad , \quad \beta_q = (q-1)^\nu, \ \nu \approx 1.304 \tag{4}$$

As a consequence, they suggested searching for the scaling in Eq. (4) as a signature of proximity to the critical point from the hadronic phase. The primary task in this case is to measure the exponent ν, expecting a value close to 1.3. This contrasts with the behavior in Eq. (3), which suggests $\nu_c = 1$ under the assumption that the particle source at the critical point has a monofractal structure.

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The STAR Collaboration at RHIC recently applied this approach to charged particle momenta in Au+Au collisions across different beam energies and centralities [9]. They unexpectedly found a scaling behavior as in Eq. (4), with a ν-value well below 1, slightly varying with energy and centrality. Since the STAR analysis used all charged particles the link to the order parameter is missing in their treatment. Furthermore, the observed power-law behaviour occurred for all energies and centralities and therefore it is unlikely to be related to criticality. Another important issue is that, due to limited statistics, the analysis was performed using factorial moments up to sixth order ($q = 6$). Thus, there is no evidence that the observed scaling would also hold for $q > 6$. As we will argue in the following the observed scaling behaviour can only be a transient FMS holding for relatively low q-values.

2. Transient FMS

The FMS in the strict sense is defined only asymptotically for $q \to \infty$. Unfortunately, this regime is inaccessible by experiment due to limited statistics. Our first claim is that the scaling relation in Eq. (4) cannot hold asymptotically. This is due to a fundamental phenomenon in multivariate probability theory called *concentration of measure*, predicting that the average of bounded independent random variables is tightly concentrated around its expectation [10]. When applied to factorial moments this phenomenon is expressed through the notion of *hypercontractivity* [11] providing bounds on the asymptotic behaviour of FMS. In fact, it can be shown that the factorial moments of a distribution belonging to the very general class of *sub-Weibull distributions* satisfy:

$$F_q(M) < \frac{L^{\theta \cdot q \cdot \ln(q)}}{\langle n \rangle^q} \tag{5}$$

where θ is a classification parameter and L is an independent of q positive, real constant [12]. Inequality (5) dictates that the general FMS scaling, for distributions with finite moments, is bounded by $q \ln q$ which grows much slower than $(q-1)^{1.3}$ predicted in [8]. Thus, the observed scaling behaviour in [8] can only hold for a finite range of q: $2 < q < q_{max}$.

The above argumentation leads to the conclusion that the FMS scaling in Eq. (4) is transient, holding for $q \leq q_{max}$. Then, it can be shown by construction of different examples, that such a transient FMS is not a unique property of a system close to a critical point, but it can be achieved by a large variety of distributions, which differ from each other and they do not have any connection to critical behaviour. As a semi-analytical example one can use a mixture of Poisson distributions which are universal approximators of most positive discrete random variables [13]. The qth factorial moments of a Poisson distribution with parameter λ take the simple form of λ^q. To construct a mixture of Poisson distributions leading to FMS of Eq. (4) for $q \leq q_{max}$ one has to use q_{max} different Poisson distributions with parameters $\lambda_1, \lambda_2, \ldots, \lambda_{q_{max}}$ satisfying the equations:

$$\langle n(n-1)\ldots(n-q+1)\rangle = (\mu_2)^{(q-1)^{1.3}} \quad , \quad \frac{1}{q_{max}} \sum_{j=1}^{q_{max}} \lambda_j^q = (\mu_2)^{(q-1)^{1.3}} \tag{6}$$

with $\langle n \rangle = 1$ and $\frac{1}{q_{max}} \sum_{j=1}^{q_{max}} \lambda_j = 1$. As a concrete example let us assume $q_{max} = 6$ and $\mu_2 = 1.1$.

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Then, using the λ_i given as:

$$\lambda_1 = 0.3568, \quad , \quad \lambda_2 = 0.9048, \quad , \quad \lambda_3 = 1.1248,$$
$$\lambda_4 = 1.1248, \quad , \quad \lambda_5 = 1.1248, \quad , \quad \lambda_6 = 1.3635.$$

one obtains a mixture of 6 Poisson distributions satisfying the FMS in Eq. (4) with a relative error less than 10^{-4}.

3. FMS in STAR data

The recently published FMS analysis in STAR experiment at RHIC led to a surprising result [9]. Using factorial moments up to sixth order in momentum space of all charged particles in the detector, they found a scaling relation of the type given in Eq. (4) with an exponent ν significantly less than 1. This rather unexpected result was obtained for all compiled data at different energies and centralities. In fact, the quantity showing this anomalous FMS was the correlator $\Delta F_q(M)$ defined as:

$$\Delta F_q(M) = F_{q,d}(M) - F_{q,m}(M) \tag{7}$$

where the indices d, m refer to data and mixed events respectively. The subtraction in Eq. (7) removes a part of the background, namely that part which can be simulated by the mixed event factorial moments. Assuming the presence of a transient FMS in both data and mixed events, expressed through the relations:

$$F_{q,A} = a(q) F_{2,A}^{\beta_q} \quad , \quad A = \mathrm{d}, \mathrm{m} \tag{8}$$

and additionally that $\Delta F_2(M)$ is very small (second factorial moment of data very close to the second factorial moment of mixed events) one can arrive at the relation:

$$\Delta F_q = \bar{a}(q) \Delta F_2^{\bar{\beta}_q} \quad , \quad \bar{\beta}_q = (q-1)^{\bar{\nu}} \tag{9}$$

Notice that in Eq. (8) the amplitude $a(q)$ as well as the exponent $\beta(q)$ are taken to be the same for data and mixed events. Thus, the difference between data and mixed events is encoded in $\Delta F_2 = F_{2,d} - F_{2,m}$ which is small but nonzero. In such a scenario, when the inequality:

$$\beta_q F_{2,m}^{\beta_q - 1} < d(q) \Delta F_2 \quad , \quad d(q) = \bar{a}(q)/a(q) \tag{10}$$

holds, the exponent $\bar{\nu}$ in Eq. (9) turns out to be $\bar{\nu} < 1$. Thus, we have found some conditions which lead to transient FMS similar to that observed in STAR analysis [9].

Going one step further, we generated simulated data to reproduce behavior (9) with $\bar{\nu} < 1$ by modifying the cluster model from [14] for our needs in transverse momentum space. In each event, particle transverse momenta are normally distributed around a center $(p_{x,c}, p_{y,c})$ with variance σ_p, while the center is itself normally distributed around $(0,0)$ with variance σ_c. We then paired data sets that shared the same mean multiplicity and σ_c, differing only in σ_p, and computed $\Delta F_q(M) = F_q^{(1)}(M) - F_q^{(2)}(M)$ for each pair.

Our analysis shows that the difference $\delta\sigma_p = \sigma_p^{(1)} - \sigma_p^{(2)}$ is the crucial parameter influencing $\bar{\nu}$, which starts from values ≈ 0.69 for $\delta\sigma_p \approx 0.001$ and tends to 1 as $\delta\sigma_p$ increases. This result

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is consistent as long as the event multiplicity is high enough (we used values between 250 and 10^4 particles per event). For example, achieving accurate $\bar{\nu}$ estimates (within 1% error) requires fewer events at higher multiplicities: 10^3 particles per event need about 10^6 events compared to $2 \cdot 10^4$ events for 10^4 particles per event. At a low multiplicity of 250 particles per event, even 10^7 events yield large fluctuations in $\bar{\nu}$. Interestingly, for $\delta\sigma_p$ values between 0.001 and 0.01, $\bar{\nu}$ remains nearly constant at about $\bar{\nu} \approx 0.69$, suggesting a plateau characteristic for normally distributed transverse momenta.

4. Conclusions

In conclusion, we have demonstrated that the power-law behavior in Eq. (4) for $q \leq q_{max}$, with $q_{max} \lesssim 10$, is a transient property of a multitude of distributions and cannot serve as a signal of proximity to a critical point. For $q \to \infty$, such scaling is theoretically excluded by hypercontractivity and remains experimentally inaccessible. Consequently, the factorial moment analysis in [9] does not indicate critical behavior. Instead, the observed results can be fully explained by the scaling of factorial moment differences, where $\nu < 1$ naturally emerges under specific conditions. We derived a sufficient criterion for this property and confirmed our reasoning through a simple simulation of particle transverse momenta in ion collisions. While the precise value of ν may offer insights into higher-order statistical properties, it does not serve as evidence of criticality.

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