Postcarrollian gravity

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We construct postcarrollian gravity models in two, three, and four spacetime dimensions by applying algebraic expansion methods. As a byproduct, we present the most general postcarrollian 2d dilaton gravity model, construct its solutions and discuss some boundary aspects, including Schwarzian-type boundary actions. In 3d, we propose Brown–Henneaux-like boundary conditions, generalizing a corresponding Carrollian analysis, and derive the postcarrollian asymptotic symmetry algebra with its central extensions.

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1. Introduction

Carroll symmetries are spacetime symmetries dual to Galilean symmetries, leading to absolute space and relative time. Carroll boosts leave invariant space but linearly transform time, $t \to t + b_i x^i$, where b_i are the Carroll boost parameters. The Carroll algebra infinitesimally generating these symmetries formally emerges as the limit of vanishing speed of light ($c \to 0$) from an İnönü–Wigner contraction of the Poincaré algebra [1, 2].

A distinguishing feature of the Carroll algebra (as opposed to the Poincaré or Galilei algebras) is that the Hamiltonian H commutes with the boost generators B_a,

$$[H, B_a] = 0. \tag{1.1}$$

Physically, this is remarkable since it implies that Carroll boosting any state does not change its energy, in contrast to our relativistic and non-relativistic intuition. The vanishing commutator (1.1) is reflected in physical properties of systems with Carroll symmetries, such as pp-waves [3, 4], zero-signature spacetimes [5], cosmology [6–8], tachyon condensates [9], flat space holography [10–19], fluid-gravity correspondence [20, 21], null hypersurfaces [22–27], null infinity [28–31], fractons [32–34], current-current deformations [35], flat bands [36], tensionless strings [37], near horizon soft hair [38], near horizon strings [39], Carroll black holes [40], spacetime subsystem symmetries [41, 42], swiftons [43], open null strings [44], shallow water waves [45], phase separation [46], etc. (see [47–49] for additional Refs.)

Gauging the Carroll algebra leads to Carroll gravity [50]. Simple examples of Carroll gravity theories are Carroll dilaton gravity models in two dimensions (2d) [51–53] such as the Carroll Jackiw–Teitelboim (JT) model or the Carroll Callan–Giddings–Harvey–Strominger (CGHS) model. The key property (1.1) holds for all these models.

In some applications, it can be of interest to deviate from the Carroll limit without fully restoring the Poincaré symmetries. The purpose of this work is to do this in a systematic way in the context of Carroll gravity. Thus, our main goal is to construct postcarrollian gravity theories. To achieve this goal, we focus on the algebraic perspective, i.e., we allow the commutator (1.1) to be non-zero but in a minimalistic way to make sure we are not fully back to Poincaré symmetries. In the remainder of this introduction, we make this notion precise.

An expansion for small speed of light was previously considered in [54] using the second-order formulation of general relativity. In that case, the leading order theory is of "electric" character and the corrections refer to that theory. In this work, by contrast, we work in the first-order formulation, which is known to produce the "magnetic" theory at leading order. Thus, the postcarrollian expansion we perform in this work has to be thought of as introducing corrections to the magnetic Carroll theory.

Before addressing the general situation in arbitrary dimensions, we glean some intuition from 2d, where the commutator (1.1) simplifies to $[H, B] = 0$. From a purely algebraic perspective, we can make this commutator less Carrollian by relaxing the condition that its right-hand side vanishes. Instead, we allow the appearance of a new generator, M, but then demand that this new generator be central. The commutation relations $[H, B] = M$ and $[M, \cdot] = 0$ are one step closer to the Poincaré algebra, in the sense that the Hamiltonian and boost generators no longer commute with each other, but instead, they commute into a generator that commutes with everything.

While some aspects of this intuition remain apposite in higher dimensions, the analog M_a can no longer be central, simply because it carries a spatial vector index and thus must not commute trivially with rotations. We explain now how this works in detail.

The (A)dS Carroll algebra can be extended by allowing postcarrollian corrections in the relation between the (A)dS gravitational fields and the Carrollian fields. We denote the Lorentzian vielbein by $E_\mu^A = (E_\mu^0, E_\mu^a)$ and the spin connection by $\Omega_\mu^{AB} = (\Omega_\mu^{0a}, \Omega_\mu^{ab})$. The Carrollian fields consist of the clock 1-form τ_μ, the spatial vielbein e_μ^a, the Carroll boost connection ω_μ^a, and the spatial

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rotation connection ω_μ^{ab}. Here, A, B denote anholonomic spacetime indices, a, b anholonomic spatial indices and 0 an anholonomic time index. The holonomic components are denoted by μ for spacetime indices and i, j for spatial indices. The expansion we perform reads

$$
\begin{aligned}
E_\mu^0 &= c\,\tau_\mu + c^3\,l_\mu & E_\mu^a &= e_\mu^a + c^2\,m_\mu^a \\
\Omega_\mu^{0a} &= c\,\omega_\mu^a + c^3\,\sigma_\mu^a & \Omega^{ab} &= \omega_\mu^{ab} + c^2\,\sigma_\mu^{ab}
\end{aligned}
\tag{1.2}
$$

where c is our expansion parameter and all the relations are supplemented by corrections. Terms of $O(c^4)$ have been neglected. By introducing generators of Lorentz transformations $\mathcal{J}_{AB}$ and (A)dS translations $\mathcal{P}_A$, an (A)dS valued gauge connection 1-form

$$
A = E^A \otimes \mathcal{P}_A + \frac{1}{2}\Omega^{AB} \otimes \mathcal{J}_{AB}
\tag{1.3}
$$

can be expanded around the Carroll limit, $c \to 0$, by inserting (1.2) into (1.3) and subsequently renaming

$$
\begin{aligned}
H &= c \otimes \mathcal{P}_0 & P_a &= c^0 \otimes \mathcal{P}_a & B_a &= c \otimes \mathcal{J}_{0a} & J_{ab} &= c^0 \otimes \mathcal{J}_{ab} \\
L &= c^3 \otimes \mathcal{P}_0 & M_a &= c^2 \otimes \mathcal{P}_a & K_a &= c^3 \otimes \mathcal{J}_{0a} & S_{ab} &= c^2 \otimes \mathcal{J}_{ab}\,.
\end{aligned}
\tag{1.4}
$$

The construction (1.4) is a particular case of a Lie algebra expansion [55–58] and can be obtained as a particular quotient of an infinite-dimensional extension of the (A)dS Carroll algebra [59] (see also [60]). The first row of generators closes into the standard (A)dS Carroll algebra [61, 62] in the limit $c \to 0$, and is associated with the leading order in the gauge field expansion (1.2). The second row of generators is identified as the postcarrollian corrections of the (A)dS Carroll generators.

After these definitions, the postcarrollian gauge field components read

$$
A_\mu = \tau_\mu H + e_\mu^a P_a + \omega_\mu^a B_a + \frac{1}{2}\omega_\mu^{ab} J_{ab} + l_\mu L + m_\mu^a M_a + \sigma_\mu^a K_a + \frac{1}{2}\sigma_\mu^{ab} S_{ab}
\tag{1.5}
$$

and the postcarrollian generators (1.4) satisfy the commutation relations[1]

$$
[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]}
\tag{1.6a}
$$

$$
[J_{ab}, P_c] = 2\delta_{c[b}P_{a]}
\tag{1.6b}
$$

$$
[J_{ab}, B_c] = 2\delta_{c[b}B_{a]}
\tag{1.6c}
$$

$$
[P_a, P_b] = -\Lambda J_{ab}
\tag{1.6d}
$$

$$
[B_a, P_b] = \delta_{ab} H
\tag{1.6e}
$$

$$
[P_a, H] = \Lambda B_a
\tag{1.6f}
$$

$$
[B_a, B_b] = S_{ab}
\tag{1.6g}
$$

$$
[B_a, H] = M_a
\tag{1.6h}
$$

$$
[J_{ab}, S_{cd}] = 4\delta_{[a[c}S_{d]b]}
\tag{1.6i}
$$

$$
[J_{ab}, M_c] = 2\delta_{c[b}M_{a]} = [S_{ab}, P_c]
\tag{1.6j}
$$

$$
[J_{ab}, K_c] = 2\delta_{c[b}K_{a]} = [S_{ab}, B_c]
\tag{1.6k}
$$

$$
[P_a, M_b] = -\Lambda S_{ab}
\tag{1.6l}
$$

$$
[K_a, P_b] = \delta_{ab} L = [B_a, M_b]
\tag{1.6m}
$$

$$
[M_a, H] = \Lambda K_a = [P_a, L]
\tag{1.6n}
$$

commutators not displayed vanish.

The commutators in the left half form the Carroll algebra provided we set to zero the right-hand sides in the last two lines. The commutators in the right half all involve one subleading generator on each side of the equalities that mirror the corresponding leading generators from the left half.

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[1]If we set the cosmological constant to zero, $\Lambda = 0$, the algebra (1.4) describes postcarrollian corrections to the flat Carroll symmetries. In this case, additionally the commutators (1.6d), (1.6f), (1.6l), and (1.6n) vanish.

Gravitational theories that admit a gauge formulation in terms of Lie algebra-valued fields, such as (1.5), allow for a postcarrollian expansion that is equivalently obtained by expanding either the (A)dS Carroll generators or the corresponding gauge fields. On the other hand, the expansion (1.2) allows to construct postcarrollian gravitational models even in the case that a formulation in terms of Lie algebra-valued objects is not available. Examples of the former case include Chern–Simons and BF theories, whereas the latter situation arises in deformed BF models and Macdowell–Mansouri-type Lagrangians.

This paper is organized as follows. In section 2, we start with postcarrollian BF theories, with a focus on the postcarrollian JT and CGHS models. In section 3, we generalize to generic postcarrollian 2d dilaton gravity models. In section 4, we construct all solutions for all these models. In section 5, we focus on boundary aspects, propose specific boundary conditions, and discuss Schwarzian-type boundary actions for specific postcarrollian 2d dilaton gravity models. In section 6, we generalize to postcarrollian gravity in three dimensions (3d) and four dimensions (4d), including an asymptotic symmetry discussion for Brown–Henneaux-like boundary conditions in 3d. In section 7, we conclude with an outlook.

2. Postcarrollian BF gravity theories in 2d

In this section, we construct simple postcarrollian gravitational theories in 2d. Consider a family of BF-models

$$I_{\text{BF}}[\mathcal{X}, A] = \frac{k}{2\pi} \int \langle \mathcal{X}, F \rangle \tag{2.1}$$

based on a Lie algebra $\mathfrak{g}$ with an invariant metric $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. While generalizations to Lie algebras without such a metric are possible (see, e.g., [51] and section 3), we restrict to this case in the current section. The field $A = A_\mu^I \mathbf{e}_I \, dx^\mu$ is a $\mathfrak{g}$-valued one-form and $\mathcal{X} = \mathcal{X}^I \mathbf{e}_I$ is a $\mathfrak{g}$-valued scalar where we chose a particular basis $\{\mathbf{e}_I\}$ for $\mathfrak{g}$ satisfying $[\mathbf{e}_I, \mathbf{e}_J] = c_{IJ}{}^K \mathbf{e}_K$ with structure constants $c_{IJ}{}^K$. From A we define the curvature two-form $F = dA + A \wedge A$ and for gauge parameters $\lambda = \lambda^I \mathbf{e}_I$, one can show that the action is invariant under the gauge variations

$$\delta_\lambda \mathcal{X} = [\mathcal{X}, \lambda] \qquad\qquad \delta_\lambda A = d\lambda + [A, \lambda] \,. \tag{2.2}$$

The finite versions of the gauge transformations are obtained by integrating λ to the group element $g = \exp(\lambda)$ and act as

$$\mathcal{X} \to g^{-1} \mathcal{X} g \qquad\qquad A \to g^{-1}(d + A) g \,. \tag{2.3}$$

The equations of motion

$$F = 0 \qquad\qquad d\mathcal{X} + [A, \mathcal{X}] = 0 \tag{2.4}$$

make it evident that on-shell the field A is locally pure gauge, $A = g^{-1} dg$, while $\mathcal{X}$ is an element of the stabilizer for this configuration, $\delta_{\mathcal{X}} A = 0$. We shall also need the quadratic Casimir,

$$C = \langle \mathcal{X}, \mathcal{X} \rangle \tag{2.5}$$

another gauge-invariant quantity that is moreover conserved on-shell, $dC = 0$.

So far, these models are just gauge theories on a 2d manifold. To identify them as gravitational theories, we have to relate the field variables to geometric variables once we choose an algebra $\mathfrak{g}$ together with an invariant metric. We shall consider five-dimensional algebras $\mathfrak{g} = \{B, H, P, M, Z\}$ here and choose the gauge field A as

$$A = \omega B + \tau H + e P + m M + z Z \tag{2.6}$$

where e is the spatial einbein, τ is the clock one-form, ω is the boost connection, and m is a one-form we shall interpret as a postcarrollian correction to e. The generator Z is a central term that we need to obtain a non-degenerate bilinear form in the case of the postcarrollian CGHS model; correspondingly, z is an abelian 1-form that goes along for the ride. This is in analogy to a construction already used in [63, 64] for the case of the Poincaré algebra in 2d. In terms of the gauge field variables, the Carrollian metric is given by $h_{\mu\nu} = e_\mu e_\nu$ and the Carrollian vector field v^μ is determined by $v^\mu \tau_\mu = -1$, $v^\mu e_\mu = 0$.

For writing down the action it is convenient to express the field $\mathcal{X}$ in terms of its dual version $\mathcal{X}^* = \langle \mathcal{X}, \cdot \rangle = \mathcal{X}_I^* E^I$ where $E^I(e_J) = \delta_J^I$. Explicitly, we write

$$\mathcal{X}^* = X B^* + X_{\text{H}} H^* + X_{\text{P}} P^* + X_{\text{M}} M^* + X_{\text{Z}} Z^* \tag{2.7}$$

with the dual algebra basis elements satisfying $B^*(B) = H^*(H) = P^*(P) = M^*(M) = Z^*(Z) = 1$. Under a gauge transformation λ this field transforms coadjointly as $\delta_\lambda \mathcal{X}^*(\cdot) = -\mathcal{X}^*([\cdot, \lambda])$. We relate the gauge transformations to spacetime diffeomorphisms generated by a vector field ξ and internal transformations generated by $\{B, M, Z\}$ by defining, without loss of generality

$$\lambda = \xi^\mu A_\mu + \bar{\lambda} \qquad\qquad \bar{\lambda} = \lambda_{\text{B}} B + \lambda_{\text{M}} M + \lambda_{\text{Z}} Z \tag{2.8}$$

such that ($\mathcal{L}_\xi$ denotes Lie derivatives along ξ and i_ξ contraction with ξ from the left)

$$\delta_\lambda A = \delta_{\bar{\lambda}} A + \mathcal{L}_\xi A + i_\xi F \qquad\qquad \delta_\lambda \mathcal{X} = \delta_{\bar{\lambda}} \mathcal{X} + \mathcal{L}_\xi \mathcal{X} - i_\xi \left(\mathrm{d}\mathcal{X} + [A, \mathcal{X}] \right) . \tag{2.9}$$

Since the last terms vanish once we evaluate on the equations of motion (2.4), we see that ξ indeed generates diffeomorphisms on-shell.

2.1 Postcarrollian JT model

Our first example starts with the postcarrollian algebra

$$[B, P] = H \qquad\qquad [B, H] = M \qquad\qquad [H, P] = -\Lambda B \tag{2.10}$$

with cosmological constant Λ. The $\mathfrak{u}(1)$-sector associated to Z does not feature in these brackets and could therefore be truncated away. We will, however, keep it for now since it will play a role in taking the limit $\Lambda \to 0$. An invariant metric on $\mathfrak{g}$ is given by

$$\langle H, H \rangle = \mu \Lambda \qquad \langle M, P \rangle = -\mu \Lambda \qquad \langle B, B \rangle = \mu \qquad \langle Z, Z \rangle = \nu \tag{2.11}$$

where we are free to choose the two real constants $\mu, \nu \neq 0$.

Without the Z-sector, the algebra (2.10) was already studied in [51, 52] where the central extension M was introduced in order to obtain a non-degenerate bilinear form. Writing out the BF-action (2.1) for these choices, we obtain the postcarrollian JT-model

$$I_{\text{JT}} = \frac{k}{2\pi} \int \left[X \, \mathrm{d}\omega + X_{\text{H}} \left(\mathrm{d}\tau + \omega \wedge e \right) + X_{\text{P}} \, \mathrm{d}e + X_{\text{M}} \left(\mathrm{d}m + \omega \wedge \tau \right) - \Lambda X \tau \wedge e + X_{\text{Z}} \, \mathrm{d}z \right] . \quad (2.12)$$

The equations of motion (2.4) for this model read explicitly

δX	Constant Carroll curvature:	$\Omega = \mathrm{d}\omega = \Lambda \tau \wedge e$	(2.13a)
δX_{H}	No Carroll torsion:	$T = \mathrm{d}\tau + \omega \wedge e = 0$	(2.13b)
δX_{P}	No intrinsic torsion:	$\Theta = \mathrm{d}e = 0$	(2.13c)
δX_{M}	No postcarrollian torsion:	$\Xi = \mathrm{d}m + \omega \wedge \tau = 0$	(2.13d)
δX_{Z}	Vanishing electric field:	$E = \mathrm{d}z = 0$	(2.13e)
$\delta\omega$	Carroll metric:	$\mathrm{d}X + X_{\text{H}} \, e + X_{\text{M}} \, \tau = 0$	(2.13f)
$\delta\tau$	Carroll Casimir:	$\mathrm{d}X_{\text{H}} - X_{\text{M}} \, \omega - \Lambda X e = 0$	(2.13g)
δe	Auxiliary field:	$\mathrm{d}X_{\text{P}} - X_{\text{H}} \, \omega + \Lambda X \, \tau = 0$	(2.13h)
δm	Abelian Casimir I:	$\mathrm{d}X_{\text{M}} = 0$	(2.13i)
δz	Abelian Casimir II:	$\mathrm{d}X_{\text{Z}} = 0 .$	(2.13j)

We derive the general solution to these equations in section 4. The addition of the generator M makes it possible to solve for the boost connection ω uniquely in terms of the fields e, τ, and m. This is in contrast to the Carrollian case investigated in [40] and more generally in [65, 66] where a part of the boost connection stays undetermined on-shell. We also highlight the constant curvature $\Omega = \Lambda \tau \wedge e$ of all solutions, which is a characteristic feature of JT gravity. The action of internal gauge transformations generated by $\bar{\lambda} = \lambda_{\text{B}} B + \lambda_{\text{M}} M + \lambda_{\text{Z}} Z$ on the fields is given by

$$\delta_{\bar{\lambda}} \tau = -e\lambda_{\text{B}} \qquad \delta_{\bar{\lambda}} e = 0 \qquad \delta_{\bar{\lambda}} \omega = \mathrm{d}\lambda_{\text{B}} \qquad \delta_{\bar{\lambda}} m = -\tau\lambda_{\text{B}} \qquad \delta_{\bar{\lambda}} z = 0 \quad (2.14a)$$

$$\delta_{\bar{\lambda}} X_{\text{H}} = \lambda_{\text{B}} X_{\text{M}} \qquad \delta_{\bar{\lambda}} X_{\text{P}} = \lambda_{\text{B}} X_{\text{H}} \qquad \delta_{\bar{\lambda}} X = 0 \qquad \delta_{\bar{\lambda}} X_{\text{M}} = 0 \qquad \delta_{\bar{\lambda}} X_{\text{Z}} = 0 \quad (2.14b)$$

where one can recognize local Carroll boosts generated by λ_{B} (see, e.g., [40]). One can also check that besides the Carroll boost-invariant scalar fields $X, X_{\text{M}}, X_{\text{Z}}$ there is another boost-invariant combination, $\frac{1}{2} X_{\text{H}}^2 - X_{\text{P}} X_{\text{M}}$.

2.2 Postcarrollian CGHS model

Starting from the postcarrollian JT model introduced in the previous subsection, we perform the limit $\Lambda \to 0$ to obtain a postcarrollian version of the CGHS model [67] in the formulation of Cangemi and Jackiw [63]. For this, we first redefine the generator for Carroll boosts by

$$B \to B + \frac{1}{\Lambda} Z \quad (2.15)$$

which is just a shift by a trivial central term. However, once the limit has been taken, this central term becomes non-trivial and therefore cannot be transformed away by a change of basis. The algebra reads

$$[B, P] = H \qquad [B, H] = M \qquad [H, P] = Z \quad (2.16)$$

and corresponds to the doubly extended Carroll algebra in Table 1 of [51]. The limit on the level of the bilinear form is performed by choosing $\mu = 1/\Lambda$, $\nu = -\Lambda$, leading to

$$\langle H, H \rangle = 1 \qquad\qquad \langle M, P \rangle = -1 \qquad\qquad \langle B, Z \rangle = -1 \qquad (2.17)$$

with all the other products being zero. Here, it becomes evident why the introduction of the additional $\mathfrak{u}(1)$-sector associated to Z was necessary, as otherwise there was no way of making the bilinear form non-degenerate. This is in the spirit of the Poincaré case where a similar construction leads to the 2d Maxwell algebra in the limit $\Lambda \to 0$ [63, 64]. The action of the postcarrollian CGHS model

$$I_{\mathrm{CGHS}} = \frac{k}{2\pi} \int \left[X\, \mathrm{d}\omega + X_{\mathrm{H}} \left(\mathrm{d}\tau + \omega \wedge e \right) + X_{\mathrm{P}}\, \mathrm{d}e + X_{\mathrm{M}} \left(\mathrm{d}m + \omega \wedge \tau \right) + X_{\mathrm{Z}} \left(\mathrm{d}z + \tau \wedge e \right) \right] \quad (2.18)$$

produces the equations of motion

$$\delta X \qquad \text{No Carroll curvature:} \qquad \Omega = \mathrm{d}\omega = 0 \qquad (2.19\mathrm{a})$$

$$\delta X_{\mathrm{H}} \qquad \text{No Carroll torsion:} \qquad T = \mathrm{d}\tau + \omega \wedge e = 0 \qquad (2.19\mathrm{b})$$

$$\delta X_{\mathrm{P}} \qquad \text{No intrinsic torsion:} \qquad \Theta = \mathrm{d}e = 0 \qquad (2.19\mathrm{c})$$

$$\delta X_{\mathrm{M}} \qquad \text{No postcarrollian torsion:} \qquad \Xi = \mathrm{d}m + \omega \wedge \tau = 0 \qquad (2.19\mathrm{d})$$

$$\delta X_{\mathrm{Z}} \qquad \text{Constant electric field:} \qquad E = \mathrm{d}z = -\tau \wedge e \qquad (2.19\mathrm{e})$$

$$\delta \omega \qquad \text{Carroll metric:} \qquad \mathrm{d}X + X_{\mathrm{H}}\, e + X_{\mathrm{M}}\, \tau = 0 \qquad (2.19\mathrm{f})$$

$$\delta \tau \qquad \text{Carroll Casimir:} \qquad \mathrm{d}X_{\mathrm{H}} - X_{\mathrm{M}}\, \omega + X_{\mathrm{Z}}\, e = 0 \qquad (2.19\mathrm{g})$$

$$\delta e \qquad \text{Auxiliary field:} \qquad \mathrm{d}X_{\mathrm{P}} - X_{\mathrm{H}}\, \omega - X_{\mathrm{Z}}\, \tau = 0 \qquad (2.19\mathrm{h})$$

$$\delta m \qquad \text{Abelian Casimir I:} \qquad \mathrm{d}X_{\mathrm{M}} = 0 \qquad (2.19\mathrm{i})$$

$$\delta z \qquad \text{Abelian Casimir II:} \qquad \mathrm{d}X_{\mathrm{Z}} = 0 \,. \qquad (2.19\mathrm{j})$$

Similar to the Lorentzian counterpart, this model only admits locally flat solutions. The action of internal gauge transformations is the same as in (2.14).

We shall come back to this model in section 5 when addressing boundary conditions and boundary actions.

3. Postcarrollian 2d dilaton gravity

In this section, we generalize the models of the previous section to the most general deformation thereof by analogy to the Lorentzian case [68]. This deforms the gauge symmetries of a theory but preserves the number of local physical and gauge degrees of freedom. Additionally preserving the dictionary between the gauge theoretic variables and the geometric variables, we obtain general theories of postcarrollian dilaton gravity. These models can be embedded in a much larger class of non-linear gauge theories [69] known as Poisson–Sigma models (PSM) [70]. They are given by actions of the form

$$I_{\mathrm{PSM}}[X^I, \mathcal{A}_I] = \frac{k}{2\pi} \int \left[X^I\, \mathrm{d}\mathcal{A}_I + \frac{1}{2} P^{IJ} \mathcal{A}_I \wedge \mathcal{A}_J \right] \qquad (3.1)$$

with the target space being a Poisson manifold with Poisson tensor $P^{IJ}(X^K)$ and target space coordinates X^I. A consistent Poisson structure requires the non-linear Jacobi identities

$$P^{IL}\partial_L P^{JK} + P^{JL}\partial_L P^{KI} + P^{KL}\partial_L P^{IJ} = 0 \,. \tag{3.2}$$

The fields $\mathcal{A}_I$ are one-forms on the base manifold and one-forms on the target space.

With these properties, one can show that, up to boundary terms, the action (3.9) is invariant under gauge transformations generated by parameters $\lambda_I = (\lambda_\mathrm{B}, \lambda_\mathrm{H}, \lambda_\mathrm{P}, \lambda_\mathrm{M}, \lambda_\mathrm{Z})_I$

$$\delta_\lambda X^I = \lambda_J \, P^{JI} \qquad\qquad \delta_\lambda \mathcal{A}_I = \mathrm{d}\lambda_I + \partial_I P^{JK} \mathcal{A}_J \lambda_K \,. \tag{3.3}$$

If the Poisson tensor is linear in the target space coordinates, then the gauge field transforms like in Yang–Mills or BF theory (see section 2); otherwise, we have non-linear gauge symmetries.

3.1 Postcarrollian dilaton gravity action and equations of motion

To make contact with the postcarrollian gravity models from before, we identify the fields as

$$X^I = (X, X_\mathrm{H}, X_\mathrm{P}, X_\mathrm{M}, X_\mathrm{Z})^I \qquad\qquad \mathcal{A}_I = (\omega, \tau, e, m, z)_I \tag{3.4}$$

and choose the Poisson tensor

$$P^{IJ} = \begin{pmatrix} 0 & X_\mathrm{M} & X_\mathrm{H} & 0 & 0 \\ -X_\mathrm{M} & 0 & \mathcal{V} & 0 & 0 \\ -X_\mathrm{H} & -\mathcal{V} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{IJ} \tag{3.5}$$

where

$$\mathcal{V} = \mathcal{V}(X, X_\mathrm{M}, X_\mathrm{Z}, X_\mathrm{H}^2 - 2X_\mathrm{P}X_\mathrm{M}) \tag{3.6}$$

is any function of Carroll boost invariant arguments. For the special choices $\mathcal{V} = X_\mathrm{Z}$ we recover the postcarrollian CGHS model (2.18) while for $\mathcal{V} = \Lambda X$ we get the postcarrollian JT model (2.12).

Using BRST arguments, it can be shown along the lines of [68] that models with Poisson tensor (3.5) are the most general deformation. The equations of motion for these models are richer as compared to Carrollian 2d dilaton gravity, cf. Eqs. (8) in [40].

δX	Carroll curvature:	$\Omega = \mathrm{d}\omega = -\partial_X \mathcal{V}\, \tau \wedge e$	(3.7a)
δX_H	Carroll torsion:	$T = \mathrm{d}\tau + \omega \wedge e = -\partial_{X_\mathrm{H}} \mathcal{V}\, \tau \wedge e$	(3.7b)
δX_P	Intrinsic torsion:	$\Theta = \mathrm{d}e = -\partial_{X_\mathrm{P}} \mathcal{V}\, \tau \wedge e$	(3.7c)
δX_M	Postcarrollian torsion:	$\Xi = \mathrm{d}m + \omega \wedge \tau = -\partial_{X_\mathrm{M}} \mathcal{V}\, \tau \wedge e$	(3.7d)
δX_Z	Electric field:	$E = \mathrm{d}z = -\partial_{X_\mathrm{Z}} \mathcal{V}\, \tau \wedge e$	(3.7e)
$\delta\omega$	Carroll metric:	$\mathrm{d}X + X_\mathrm{H}\, e + X_\mathrm{M}\, \tau = 0$	(3.7f)
$\delta\tau$	Carroll Casimir:	$\mathrm{d}X_\mathrm{H} - X_\mathrm{M}\, \omega + \mathcal{V}\, e = 0$	(3.7g)
δe	Auxiliary field:	$\mathrm{d}X_\mathrm{P} - X_\mathrm{H}\, \omega - \mathcal{V}\, \tau = 0$	(3.7h)
δm	Abelian Casimir I:	$\mathrm{d}X_\mathrm{M} = 0$	(3.7i)
δa	Abelian Casimir II:	$\mathrm{d}X_\mathrm{Z} = 0$	(3.7j)

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In particular, one can see that the Carroll torsion equation in general implies non-vanishing torsion of ω. For our purposes it will be enough to restrict $\mathcal{V}$ to the form[2]

$$\mathcal{V} = V(X) - \left(\tfrac{1}{2}\, X_{\mathrm{H}}^2 - X_{\mathrm{M}} X_{\mathrm{P}}\right) U(X) \tag{3.8}$$

with two potential functions $V(X)$ and $U(X)$. This is inspired by the *UV*-family of 2d dilaton gravity [71].

Since the Poisson tensor is just another way of writing a Poisson bracket structure on the target manifold, we have for any two functions $\{F, G\} = P^{IJ} \partial_I F \partial_J G$, implying the Jacobi identities (3.2). For the case (3.8) we shall see in section 4 that there exist three Casimir functions C_i satisfying $\{C_i, \cdot\} = 0$ which corresponds to the Poisson tensor being rank 2. Two of them are just X_{M} and X_{Z}, see (3.5).

In summary, the generic postcarrollian 2d dilaton gravity bulk action is given by

$$I_{\mathrm{PC}} = \int \left[X\, \mathrm{d}\omega + X_{\mathrm{H}}\big(\mathrm{d}\tau + \omega \wedge e\big) + X_{\mathrm{P}}\, \mathrm{d}e + X_{\mathrm{M}}\big(\mathrm{d}m + \omega \wedge \tau\big) + X_{\mathrm{Z}}\, \mathrm{d}z + \mathcal{V}\, \tau \wedge e \right] \tag{3.9}$$

with a generic potential of the form (3.6) that we shall restrict to (3.8) in this work. Compared to Carroll dilaton gravity [40, 51], the main change in the action (3.9) — apart from the presence of additional subleading fields — is the presence of a term proportional to $\omega \wedge \tau$.

3.2 Postcarrollian dilaton gravity as a limit

Before delving into a derivation of the general solutions to the models (3.9) with (3.8) we show how they are obtained as a limit from Lorentzian theories. This motivates the interpretation as postcarrollian gravity theories, along the lines of our introduction. Following [71], the Lorentzian first-order dilaton gravity action for *UV*-models is given by

$$I_{\mathrm{Lorentz}} = \frac{\kappa}{2\pi} \int \left[X\, \mathrm{d}\varpi + X_a\big(\mathrm{d}E^a + \varpi \wedge E^b \epsilon^a{}_b\big) - \left(V(X) + \frac{U(X)}{2} X^c X_c\right)\frac{1}{2}\epsilon_{ab} E^a \wedge E^b \right] \tag{3.10}$$

where $\epsilon^{01} = 1$ and $X^c X_c = X^c X^d \eta_{cd} = -(X_0)^2 + (X_1)^2$ with $\eta_{00} = -1 = -\eta_{11}$. The fields E^a denote the Lorentzian zweibein and ϖ is the dual of the spin connection $\varpi^a{}_b = \epsilon^a{}_b \varpi$. These models can also be written as PSMs with a three-dimensional target space spanned by (X, X^a).

Let us first look at how to perform the limit without postcarrollian corrections. We change the variables to

$$E^0 = c\tau \qquad\qquad E^1 = e \qquad\qquad \varpi = c\omega \tag{3.11}$$

$$X = X \qquad\qquad X_0 = X_{\mathrm{H}} \qquad\qquad X_1 = cX_{\mathrm{P}} \tag{3.12}$$

such that we get

$$I_{\mathrm{Lorentz}} = \frac{c\kappa}{2\pi} \int \left[X\, \mathrm{d}\omega + X_{\mathrm{H}}(\mathrm{d}\tau + \omega \wedge e) + X_{\mathrm{P}}\, \mathrm{d}e + \left(V - \frac{U}{2}X_{\mathrm{H}}^2\right)\tau \wedge e \right] + O(c^3) \tag{3.13}$$

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[2]We could allow additionally arbitrary X_{M}-dependence in the potential. However, since on-shell X_{M} is constant, this generalization almost does not change anything. Its only effect is to alter the postcarrollian torsion equation, which obtains new sources.

which after rescaling the coupling constant $\kappa \to c^{-1}k$ and taking the limit $c \to 0$ is the magnetic Carroll dilaton gravity action [40, 51].

In order to get the postcarrollian action, one needs to introduce two additional Abelian BF-sectors before doing the expansion (this is analogous to similar constructions in 3d, see [72]). The extended Lorentzian action then reads

$$\bar{I}_{\text{Lorentz}} = I_{\text{Lorentz}} + \int \left[\Phi\,\mathrm{d}\eta + \Psi\,\mathrm{d}\zeta + \frac{U(X)}{4}\Phi^2 \epsilon_{ab} E^a \wedge E^b \right] \tag{3.14}$$

where η, ζ are a one-forms and Φ, Ψ are scalar fields. This can also be written as an extended PSM with a five-dimensional target space

$$\bar{I}_{\text{Lorentz}} = \frac{\kappa}{2\pi} \int \left[X^I\,\mathrm{d}\mathcal{A}_I + \frac{1}{2}P^{IJ}\mathcal{A}_I \wedge \mathcal{A}_J \right] \tag{3.15}$$

with $\mathcal{A}_I = (\varpi, E^0, E^1, \eta, \zeta)$, $X^I = (X, X^0, X^1, \Phi, \Psi)$, $\bar{\mathcal{V}} = V(X) + \frac{U}{2}\left(X^a X_a - \Phi^2\right)$, and

$$P^{IJ} = \begin{pmatrix} 0 & X^1 & -X^0 & 0 & 0 \\ -X^1 & 0 & \bar{\mathcal{V}} & 0 & 0 \\ X^0 & -\bar{\mathcal{V}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{IJ} \tag{3.16}$$

satisfying the non-linear Jacobi identities (3.2).

We again make a change of variables that prepares us for the limit,

$$E^0 = c\tau \qquad\qquad E^1 = e + \frac{c^2}{2}m \qquad\qquad \varpi = c\omega \tag{3.17}$$

$$\eta = e - \frac{c^2}{2}m \qquad\qquad \zeta = cz \qquad\qquad \Psi = X_{\text{Z}} \tag{3.18}$$

$$X_0 = X_{\text{H}} \qquad\qquad X_1 = \frac{1}{c}X_{\text{M}} + \frac{c}{2}X_{\text{P}} \qquad\qquad \Phi = -\frac{1}{c}X_{\text{M}} + \frac{c}{2}X_{\text{P}}\,. \tag{3.19}$$

Like before, the dilaton X does not expand. The action (3.15) then reduces to

$$\begin{aligned} \bar{I}_{\text{Lorentz}} = \frac{c\kappa}{2\pi} \int \Big[& X\,\mathrm{d}\omega + X_{\text{H}}\big(\mathrm{d}\tau + \omega \wedge e\big) + X_{\text{P}}\,\mathrm{d}e + X_{\text{M}}\big(\mathrm{d}m + \omega \wedge \tau\big) + X_{\text{Z}}\,\mathrm{d}z \\ & + \Big(V(X) - \frac{U(X)}{2}\big(X_{\text{H}}^2 - 2X_{\text{M}}X_{\text{P}}\big)\Big)\tau \wedge e \Big] + O(c^3) \end{aligned} \tag{3.20}$$

which for $\kappa \to c^{-1}k$ and $c \to 0$ is the postcarrollian action (3.9) with the potential (3.8).

4. All classical solutions

In this section, we derive all solutions to the classical equations of motion (3.7) for the family of models (3.8). These models automatically also include possible extensions of the potential (3.8) by X_{Z}-dependent shifts since X_{Z} is a Casimir, $\mathrm{d}X_{\text{Z}} = 0$ on-shell. Such dependencies can therefore always be interpreted as a constant shift of the function $V(X)$. The postcarrollian CGHS model in section 2.2 is such an example as it is given by $\mathcal{V} = X_{\text{Z}} = \text{cst}$.

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Another Casimir is X_M, which we shall use to classify solutions depending on whether it vanishes or not. In either of the cases, it is useful to define integrated versions of the potentials,

$$Q(X) := \int^X U(y)\,dy \qquad\qquad w(X) := \int^X e^{Q(y)} V(y)\,dy \qquad (4.1)$$

in terms of which the last Casimir function is given by

$$C = w(X) - e^{Q(X)}\left(\tfrac{1}{2}X_H^2 - X_M X_P\right). \qquad (4.2)$$

On-shell it satisfies $dC = 0$ and we recast (4.2) as

$$\tfrac{1}{2}X_H^2 - X_M X_P = e^{-Q(X)}\left(w(X) - C\right) := K(X). \qquad (4.3)$$

4.1 Vanishing X_M

We assume first $X_M = 0$. In this case, the target space coordinate X_H is Carroll boost invariant, and the classical solutions are quite similar to Carrollian black holes. Thus, we follow the procedure explained in [40] and obtain again two sectors. In the constant dilaton sector we have $X_H = 0$ and constant X, subject to the constraint $V(X) = 0$. For the remainder of this section, we focus on the linear dilaton sector, where $X_H \neq 0$ generically (though we could have $X_H = 0$ at isolated loci, Carroll extremal surfaces).

Since (3.7c) implies vanishing intrinsic torsion, $de = 0$, we choose our spatial coordinate r such that $e = dr$ and find that (3.7f) implies

$$e = dr = -\frac{dX}{X_H} = \frac{dX}{\mp\sqrt{2K(X)}} \qquad (4.4)$$

where we fixed the Carroll boost freedom λ_B by setting $X_P = 0$. Using moreover the gauge freedom λ_H to set $\tau = \tau_t\,dt$ we arrive at the solutions

$$e = dr \qquad \tau = -e^{Q(X)} X_H\,dt \qquad \omega = e^{Q(X)}\big(V(X) - K(X)U(X)\big)\,dt \qquad (4.5)$$

$$X_P = 0 \qquad X_H = \pm\sqrt{2K(X)} \qquad X(r): \text{ by integrating (4.4).} \qquad (4.6)$$

The remaining one-form m solves a trivial postcarrollian torsion condition,

$$dm = 0 \qquad (4.7)$$

and X_Z is any constant. Thus, as far as the bulk geometry is concerned, the postcarrollian sector $X_M = 0$ is indistinguishable from Carrollian solutions, and essentially all the results of [40] apply. For comparison with the sector $X_M \neq 0$, we stress that the dilaton field is here a spatial coordinate such that these solutions are interpreted as static postcarrollian spacetimes.

4.2 Non-vanishing X_M

We assume here $X_M \neq 0$. Using the Carroll vector field v^μ defined by $v^\mu \tau_\mu = -1$, $v^\mu e_\mu = 0$, the Carroll metric equation (3.7f) implies $v^\mu \partial_\mu X \neq 0$. The dilaton describes a temporal direction in contrast to the $X_M = 0$ sector.

We define a new 1-form Z by

$$e = e^{Q(X)} Z \tag{4.8}$$

and use the Carroll metric equation (3.7f) to obtain the clock 1-form

$$\tau = -\frac{1}{X_{\mathrm{M}}} \left(\mathrm{d}X + e^{Q(X)} X_{\mathrm{H}} Z \right) . \tag{4.9}$$

The volume form is therefore given by

$$\tau \wedge e = -\frac{e^{Q(X)}}{X_{\mathrm{M}}} \, \mathrm{d}X \wedge Z \tag{4.10}$$

which reduces the intrinsic torsion equation to

$$\mathrm{d}Z = 0 \qquad \Rightarrow \qquad Z = \mathrm{d}r . \tag{4.11}$$

Like before, we introduced a spatial coordinate r.

Since the value of X_{M} is not very important (as long as it is non-zero), we pick $X_{\mathrm{M}} = -1$, without loss of generality. We fix the Carroll boost freedom λ_{B} by choosing $X_{\mathrm{H}} = 0$. As a consequence, the results above simplify to

$$X := t \qquad e = e^{Q(t)} \, \mathrm{d}r \qquad \tau = \mathrm{d}t . \tag{4.12}$$

Thus, the dilaton field is the postcarrollian time in this solution sector. Moreover, there are no Carroll extremal surfaces. So this solution sector is qualitatively quite different from the vanishing X_{M}-sector.

Solving the postcarrollian Casimir equation for the boost connection yields

$$\omega = e^{Q(t)} \bigl(K(t)U(t) - V(t) \bigr) \, \mathrm{d}r . \tag{4.13}$$

The result (4.13) is automatically compatible with the Carroll curvature and torsion conditions.

The remaining target space coordinate is obtained from the Casimir relation (4.3),

$$X_{\mathrm{P}} = e^{-Q(t)} \left(w(t) - C \right) \tag{4.14}$$

which leaves as the only remaining equation of motion the postcarrollian torsion equation (3.7d). It simplifes to

$$\mathrm{d}m = \mathrm{d}r \wedge \mathrm{d}w(t) \tag{4.15}$$

and a possible solution of (4.15) is

$$m = -w(t) \, \mathrm{d}r . \tag{4.16}$$

In summary, the gauge-fixed solutions for the sector $X_{\mathrm{M}} \neq 0$ read

$$e = e^{Q(X)} \, \mathrm{d}r \qquad \tau = \mathrm{d}t \qquad \omega = e^{Q(X)} \bigl(K(X)U(X) - V(X) \bigr) \, \mathrm{d}r \tag{4.17}$$

$$X_{\mathrm{P}} = K(X) \qquad X_{\mathrm{H}} = 0 \qquad X = t \tag{4.18}$$

$$X_{\mathrm{M}} = -1 \qquad m = -w(X) \, \mathrm{d}r . \tag{4.19}$$

The time dependence makes these solutions similar to cosmologies rather than black holes.

5. Boundary actions

In this section, we shall investigate a few possibilities of boundary conditions for 2d postcarrollian gravity theories, restricting to those that admit formulations as BF-theories for simplicity. This requires the Poisson tensor to be linear in the target space coordinates. After recalling some generalities from the literature, we shall restrict to the examples of the postcarrollian JT and CGHS models introduced in section 2.

Since the spacetime manifold $\mathcal{M}$ is assumed to have a boundary $\partial\mathcal{M}$, the bulk action in general needs to be supplemented by a boundary action

$$\Gamma[\chi, A] = \frac{k}{2\pi} \int_{\mathcal{M}} \langle \chi, F \rangle + I_{\partial\mathcal{M}} \,. \tag{5.1}$$

This ensures that the variational principle is well-defined for a given set of boundary conditions of the dynamical variables, i.e., $\delta\Gamma$ vanishes on the solutions of the equations of motion,

$$\delta\Gamma = \text{(equations of motion)} + \frac{k}{2\pi} \int_{\partial\mathcal{M}} \langle \chi, \delta A \rangle + \delta I_{\partial\mathcal{M}} \overset{!}{=} 0 \,. \tag{5.2}$$

The variation of $I_{\partial\mathcal{M}}$ therefore has to be fixed as

$$\delta I_{\partial\mathcal{M}} = -\frac{k}{2\pi} \int_{\partial\mathcal{M}} \langle \chi, \delta A \rangle \,. \tag{5.3}$$

Functionally integrating this expression to obtain $I_{\partial\mathcal{M}}$ requires imposing boundary conditions on χ and A at $\partial\mathcal{M}$ [73]. A general choice to integrate (5.3) is

$$\chi\big|_{\partial\mathcal{M}} \,\mathrm{d}f = A\big|_{\partial\mathcal{M}} \tag{5.4}$$

where f is some monotonous function on the boundary. Calling the boundary coordinate t, we choose $f = t$, although generalizations are possible. The boundary action integrates to

$$I_{\partial\mathcal{M}} = -\frac{k}{4\pi} \int_{\partial\mathcal{M}} \mathrm{d}t \, \langle \chi, \chi \rangle \tag{5.5}$$

where we implicitly assumed that $\partial\mathcal{M}$ is connected and either a line or a circle. This shows that, as usual for 2d dilaton gravity models [73, 74], the on-shell action is essentially given by the quadratic Casimir (2.5) of the theory.

Since the boundary relation (5.4) fixes the components of χ in terms of A projected onto the boundary, we focus on the behavior of the latter and are assured that the variational principle is well defined by invoking (5.4).

From the equation of motion obtained by varying χ in (5.1), we find $F = 0$ and therefore know that the field A has to be pure gauge on-shell, $A = \tilde{g}^{-1} \,\mathrm{d}\tilde{g}$. We choose a boundary condition for A that is compatible with this solution,

$$A = b^{-1}\big(\mathrm{d}+a\big)b \qquad\qquad b = e^{rP} \tag{5.6}$$

where the boundary gauge field a only has a leg in the t-direction,

$$a = \big(a_{\mathrm{B}}(t)B + a_{\mathrm{H}}(t)H + a_{\mathrm{P}}(t)P + a_{\mathrm{M}}(t)M + a_{\mathrm{Z}}(t)Z\big)\,\mathrm{d}t \tag{5.7}$$

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and all the dependence on the spatial coordinate r is absorbed into the fixed group element b. We take r as the direction normal to the boundary. A similar parametrization applies to the field χ,

$$\chi = b^{-1} x b \qquad\qquad x = x_\mathrm{B}(t)B + x_\mathrm{H}(t)H + x_\mathrm{P}(t)P + x_\mathrm{M}(t)M + x_\mathrm{Z}(t)Z . \qquad (5.8)$$

The equations of motion reduce to

$$\mathrm{d}a + a \wedge a = 0 \qquad\qquad \mathrm{d}x + [a, x] = 0 \qquad\qquad (5.9)$$

such that on-shell we have $a = g(t)^{-1}\,\mathrm{d}g(t)$ with some t-dependent group element $g(t)$.

In that case, the boundary condition (5.4) reduces the action (5.1) to a pure boundary term

$$\Gamma[g] = -\frac{k}{4\pi} \int \mathrm{d}t \,\langle g^{-1}g',\, g^{-1}g'\rangle \qquad\qquad (5.10)$$

which is the action of a particle on the group manifold generated by $\mathfrak{g}$ [51, 73, 75, 76]. The integrand features the components of the Maurer–Cartan form $g^{-1}\,\mathrm{d}g$, where we denote ∂_t by prime. The equations of motion generated by this action are compatible with the bulk equations of motion (5.9). The global symmetries of the action (5.10) are given by left- or right-multiplication with constant group elements,

$$g \to g\,h \qquad\qquad g \to \tilde{h}\,g \qquad\qquad h,\, \tilde{h} = \mathrm{cst} . \qquad\qquad (5.11)$$

The left and right global symmetries (5.11) are not all independent from each other if some g, h, and $\tilde{h}$ commute, so the total number of global symmetries of the boundary action always lies between N and $2N$, where N is the rank of the gauge group.

5.1 Loop group boundary conditions for postcarrollian JT and CGHS

We start with the postcarrollian JT model (2.12) with $\Lambda = -1$. Since the abelian 1-form z is trivial for this model, we set it to zero from the start and also set to zero the corresponding target space coordinate X_Z. Then we are left with the Carroll boost connection ω, the clock 1-form τ, the spatial einbein e, the postcarrollian einbein m, and the associated target space coordinates $X, X_\mathrm{H}, X_\mathrm{P}, X_\mathrm{M}$.

It turns out to be convenient to change the basis, $L_\pm = H \pm B$, implying

$$[L_+, L_-] = 2M \qquad [L_\pm, P] = \pm L_\pm \qquad \langle L_+, L_-\rangle = -2 \qquad \langle M, P\rangle = 1 \quad (5.12)$$

for the choice $\mu = 1$ in (2.11). Assuming all components of the boundary connection

$$a = \left(\mathcal{L}_+(t)\,L_+ + \mathcal{L}_-(t)\,L_- + \mathcal{P}(t)\,P + \mathcal{M}(t)\,M\right)\,\mathrm{d}t \qquad\qquad (5.13)$$

are allowed to fluctuate, the boundary fields a, x translate into the bulk variables by using (5.6) and (5.8),

$$A = P\,\mathrm{d}r + \left(e^r\,\mathcal{L}_+\,L_+ + e^{-r}\,\mathcal{L}_-\,L_- + \mathcal{P}\,P + \mathcal{M}\,M\right)\,\mathrm{d}t \qquad (5.14)$$

$$\mathcal{X} = e^r\,x_+\,L_+ + e^{-r}\,x_-\,L_- + x_\mathrm{P}\,P + x_\mathrm{M}\,M . \qquad (5.15)$$

They are preserved by the gauge transformations $\lambda = b^{-1}\varepsilon b$ with

$$\varepsilon = \varepsilon_+(t)L_+ + \varepsilon_-(t)L_- + \varepsilon_\mathrm{P}(t)P + \varepsilon_\mathrm{M}(t)M \qquad\qquad (5.16)$$

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that act on the state-dependent functions as

$$\delta_\varepsilon \mathcal{L}_\pm = \varepsilon'_\pm \pm \mathcal{L}_\pm \, \varepsilon_\mathrm{P} \mp \mathcal{P} \, \varepsilon_\pm \tag{5.17}$$

$$\delta_\varepsilon \mathcal{P} = \varepsilon'_\mathrm{P} \tag{5.18}$$

$$\delta_\varepsilon \mathcal{M} = \varepsilon'_\mathrm{M} + 2\mathcal{L}_+ \, \varepsilon_- - 2\mathcal{L}_- \, \varepsilon_+ \,. \tag{5.19}$$

Choosing the boundary group element

$$g = e^{y \, L_+} e^{y_- \, L_-} e^{p \, P} e^{m \, M} \tag{5.20}$$

one finds the Maurer–Cartan form by repeatedly using the Baker–Campbell–Hausdorff formula (see appendix A)

$$g^{-1} g' = y' e^p \, L_+ + y'_- e^{-p} \, L_- + p' \, P + \left(m' + 2y' y_-\right) M = a_t \tag{5.21}$$

and can, as an alternative to (5.13), express the field A in terms of the fields $\{y, y_-, p, m\}$. The global symmetries (5.11) act on the functions in the boundary group element (5.20) as[3]

$$y \to c \, y + a + \alpha \, e^{-p} \qquad\qquad y_- \to \frac{1}{c} \, y_- - \frac{b}{2c} + \beta \, e^p \tag{5.22}$$

$$p \to p + \gamma \qquad\qquad m \to m + b \, y + d - 2\alpha \, e^{-p} \, y_- \tag{5.23}$$

which shows they are Goldstone-like fields as no derivatives of the transformation parameters appear. The boundary action (5.10) reads

$$\Gamma[y, y_-, p, m] = \frac{k}{2\pi} \int \mathrm{d}t \, \left(2y' y'_- - p'(m' + 2y' \, y_-)\right) \tag{5.24}$$

and has the seven global symmetries (5.22). The general solution to the equations of motion descending from the boundary action (5.24),

$$y = y_0 + y_1 \, e^{-p_1 t} \qquad y_- = y_2 + y_3 \, e^{p_1 t} \qquad p = p_0 + p_1 \, t \qquad m = m_0 + m_1 \, t - 2y_1 y_2 \, e^{-p_1 t} \tag{5.25}$$

contains the eight integration constants $y_{0,1,2,3}$, $p_{0,1}$, and $m_{0,1}$.

Let us turn to the postcarrollian CGHS model (2.18). We pick a specific $\mathfrak{g}$ and a bilinear form on it given by (2.16), (2.17). In this case the boundary fields a, x translate into the bulk variables by using (5.6) and (5.8), $A_r = P$, and

$$A_t = a_\mathrm{P}(t) \, P + a_\mathrm{B}(t) \, B + \left(r \, a_\mathrm{B}(t) + a_\mathrm{H}(t)\right) H + a_\mathrm{M}(t) \, M + \left(r \, a_\mathrm{H}(t) + a_\mathrm{Z}(t) + \frac{r^2}{2} a_\mathrm{B}(t)\right) Z \tag{5.26}$$

$$X = x_\mathrm{B}(t) \, B + (x_\mathrm{H}(t) + x_\mathrm{B}(t)r) \, H + x_\mathrm{P}(t) \, P + x_\mathrm{M}(t) \, M + \left(x_\mathrm{Z}(t) + x_\mathrm{H}(t)r + \frac{r^2}{2} x_\mathrm{B}(t)\right) Z \,. \tag{5.27}$$

The gauge transformations preserving (5.7) and (5.8) are restricted to the form $\lambda = b^{-1} \varepsilon(t) b$ where

$$\varepsilon = \varepsilon_\mathrm{B}(t) \, B + \varepsilon_\mathrm{H}(t) \, H + \varepsilon_\mathrm{P}(t) \, P + \varepsilon_\mathrm{M}(t) \, M + \varepsilon_\mathrm{Z}(t) \, Z \,. \tag{5.28}$$

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[3]After the Hamiltonian reduction in section 5.2, only the parameters a, b, c, d generate global symmetries of the reduced boundary action, while α, β, γ are no longer present. This motivates our notation. Also, we require $c \neq 0$.

This gives five free functions at the boundary. These boundary fields infinitesimally transform as

$$\delta_\varepsilon a_{\mathrm B} = \varepsilon_{\mathrm B}' \qquad\qquad\qquad \delta_\varepsilon x_{\mathrm B} = 0 \tag{5.29}$$

$$\delta_\varepsilon a_{\mathrm H} = \varepsilon_{\mathrm H}' + a_{\mathrm B}\varepsilon_{\mathrm P} - a_{\mathrm P}\varepsilon_{\mathrm B} \qquad\qquad \delta_\varepsilon x_{\mathrm H} = x_{\mathrm B}\varepsilon_{\mathrm P} - x_{\mathrm P}\varepsilon_{\mathrm B} \tag{5.30}$$

$$\delta_\varepsilon a_{\mathrm P} = \varepsilon_{\mathrm P}' \qquad\qquad\qquad \delta_\varepsilon x_{\mathrm P} = 0 \tag{5.31}$$

$$\delta_\varepsilon a_{\mathrm M} = \varepsilon_{\mathrm M}' + a_{\mathrm B}\varepsilon_{\mathrm H} - a_{\mathrm H}\varepsilon_{\mathrm B} \qquad\qquad \delta_\varepsilon x_{\mathrm M} = x_{\mathrm B}\varepsilon_{\mathrm H} - x_{\mathrm H}\varepsilon_{\mathrm B} \tag{5.32}$$

$$\delta_\varepsilon a_{\mathrm Z} = \varepsilon_{\mathrm Z}' + a_{\mathrm H}\varepsilon_{\mathrm P} - a_{\mathrm P}\varepsilon_{\mathrm H} \qquad\qquad \delta_\varepsilon x_{\mathrm Z} = x_{\mathrm H}\varepsilon_{\mathrm P} - x_{\mathrm P}\varepsilon_{\mathrm H} \,. \tag{5.33}$$

Choosing the boundary group element as

$$g(t) = e^{h(t)H} e^{p(t)P} e^{b(t)B} e^{m(t)M} e^{z(t)Z} \tag{5.34}$$

one finds the Maurer–Cartan form by repeatedly using the Baker–Campbell–Hausdorff formula (see appendix A)

$$g^{-1}g' = b'B + (h' - p'b)H + p'P + \left(m' - h'b + \frac{1}{2}p'b^2\right)M + (z' + h'p)Z = a_t \tag{5.35}$$

and can, as an alternative to (5.7), express the field a in terms of the fields $\{b, h, p, m, z\}$. The global symmetries (5.11) act on these as

$$b \to b + \varepsilon_{\mathrm B} \tag{5.36a}$$

$$h \to h + \varepsilon_{\mathrm H} + b\,\varepsilon_{\mathrm P} + p\varepsilon_{\mathrm B} \tag{5.36b}$$

$$p \to p + \varepsilon_{\mathrm P} \tag{5.36c}$$

$$m \to m + \varepsilon_{\mathrm M} + b\,\varepsilon_{\mathrm H} + \frac{b^2}{2}\,\varepsilon_{\mathrm P} + \frac{p}{2}\,\tilde\varepsilon_{\mathrm B}^2 \tag{5.36d}$$

$$z \to z + \varepsilon_{\mathrm Z} - p\,\varepsilon_{\mathrm H} - bp\,\varepsilon_{\mathrm P} - \frac{b}{2}\,\varepsilon_{\mathrm P}^2 - h\,\tilde\varepsilon_{\mathrm P} - p\,\tilde\varepsilon_{\mathrm B}\tilde\varepsilon_{\mathrm P} - \frac{p^2}{2}\,\tilde\varepsilon_{\mathrm B} \tag{5.36e}$$

which shows they are Goldstone-like fields as no derivatives of the gauge parameters appear. The boundary action (5.10) reads

$$\Gamma[b, h, p, m, z] = \frac{k}{2\pi} \int \mathrm{d}t \left(p'm' + b'\left(z' + ph'\right) - \frac{1}{2}\,h'^{\,2}\right) \tag{5.37}$$

and has the global symmetries (5.36). The field equations descending from the loop group boundary action (5.37) yield monomial expressions in time depending on ten integration constants,

$$b = b_0 + b_1 t \qquad m = m_0 + m_1 t + \frac{b_1 h_1}{2}t^2 - \frac{b_1^2 p_1}{6}t^3 \qquad\qquad h = h_0 + h_1 t - \frac{b_1 p_1}{2}t^2 \tag{5.38a}$$

$$p = p_0 + p_1 t \qquad z = z_0 + z_1 t + \frac{p_1(p_0 b_1 - h_1)}{2}t^2 + \frac{p_1^2 b_1}{3}t^3 \,. \tag{5.38b}$$

The loop group boundary actions (5.24) and (5.37) are the postcarrollian analogs of the WZW-action descending from Chern–Simons theories [77, 78]. Typically, one is interested in stricter boundary conditions that constrain the boundary dynamics in a geometrically interesting way. A famous example is the Drinfeld–Sokolov reduction of sl(2, ℝ) current algebras to Virasoro algebras,

which yields Brown–Henneaux boundary conditions [79] when applied to AdS_3 Einstein gravity [80].

For the case of JT gravity with a negative cosmological constant, this, for example, leads to the Schwarzian action [76, 81–83] as a reduction of the AdS_2-loop group boundary action [51, 52]. While there are several methods to perform this reduction, the one we employ here is the inverse Higgs mechanism [52, 84]. It amounts to fixing certain components of the Maurer–Cartan form and solving the resulting constraints for the remaining fields in the boundary action.

5.2 Schwarzian-type boundary actions for postcarrollian JT and CGHS

The postcarrollian JT algebra (2.10) has an alternative interpretation as centrally extended AdS–Carroll algebra (which in turn is isomorphic to the centrally extended Poincaré algebra) studied already in [51, 52]. Therefore, we simply copy the results from section 6.2 of [51] to get a Schwarzian-type boundary action. Below, we briefly summarize, simplify, and expand these results. As before, we set $\Lambda = -1$ and $\mu = 1$ in (2.11).

After the same basis change, $L_\pm = H \pm B$ [see (5.12)], we fix boundary conditions in highest-weight gauge with the ansatz (5.6) and the boundary connection

$$a = \left(L_+ - \mathcal{L}(t)\, L_- + \mathcal{T}(t)\, P\right)\,\mathrm{d}t\,. \tag{5.39}$$

The gauge parameters preserving these boundary conditions, $\lambda = b^{-1}\varepsilon b$ with $\varepsilon = \varepsilon_+ L_+ + \varepsilon_- L_- + \varepsilon_\mathrm{P} P + \varepsilon_\mathrm{M} M$, generate the conditions $\varepsilon_\mathrm{P} = \mathcal{T}\,\varepsilon_+ - \varepsilon'_+$, $\varepsilon_- = -\mathcal{L}\varepsilon_+ - \frac{1}{2}\varepsilon'_\mathrm{M}$, and the transformations

$$\delta_\varepsilon \mathcal{T} = \lambda_+ \mathcal{T}' + \lambda'_+ \mathcal{T} - \lambda''_+ \qquad \delta_\varepsilon \mathcal{L} = \lambda_+ \mathcal{L}' + 2\lambda'_+ \mathcal{L} + \frac{1}{2}\left(\lambda'_\mathrm{M} \mathcal{T} + \lambda''_\mathrm{M}\right) \tag{5.40}$$

on the state-dependent functions that are recognized as twisted warped transformations [64, 85–87].

In the second-order formulation, the boundary conditions (5.39) translate into the Carrollian metric and vector field

$$h_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(\mathrm{d}r + \mathcal{T}\,\mathrm{d}t\right)^2 \qquad v^\mu \partial_\mu = -\frac{1}{e^r - \mathcal{L}\,e^{-r}}\left(\partial_t - \mathcal{T}\,\partial_r\right) \tag{5.41}$$

which straightforwardly generalize to the Brown–Henneaux type conditions

$$h_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \left(\mathrm{d}r + O(1)\,\mathrm{d}t\right)^2 \left(1 + O(e^{-2r})\right) \qquad v^\mu \partial_\mu = -e^{-r}\left(\partial_t + O(1)\partial_r\right)\left(1 + O(e^{-2r})\right) \tag{5.42}$$

subject to the orthogonality constraint $h_{\mu\nu}v^\nu = 0$ to all orders. Applying the analysis of [40] to zero mode solutions (constant $\mathcal{T}, \mathcal{L}$), we deduce there is a Carroll extremal surface at $e^{2r} = \mathcal{L}$.

Parametrizing the postcarrollian JT group element as

$$g = e^{y\,L_+} e^{y_-\,L_-} e^{p\,P} e^{m\,M} \tag{5.43}$$

the boundary conditions (5.39) impose the constraints

$$g^{-1}g'\big|_{L_+} = 1 \qquad\qquad g^{-1}g'\big|_M = 0 \tag{5.44}$$

and yield the Schwarzian-type boundary action [51, 52]

$$\Gamma[y, m] = \frac{k}{2\pi} \int \mathrm{d}t \left(m' \ln' y' - m''\right)\,. \tag{5.45}$$

The boundary action (5.45) is invariant under the global postcarrollian transformations

$$y \rightarrow c\,y + a \qquad\qquad m \rightarrow m + b\,y + d\,. \tag{5.46}$$

The state-dependent functions

$$\mathcal{T} = -\ln' y' \qquad\qquad \mathcal{L} = -\frac{1}{2}\left(m'\ln' y' - m''\right) \tag{5.47}$$

are conserved on-shell, $\partial_t \mathcal{T} = 0 = \partial_t \mathcal{L}$, which is shown by varying the twisted warped boundary action (5.45) with respect to the time parametrization field y and the phase field m.

Let us turn to the postcarrollian CGHS model. One possibility for a set of constraints on the boundary variables is to fix

$$g^{-1}g'\big|_B = 1 \qquad\qquad g^{-1}g'\big|_Z = 0 \tag{5.48}$$

solved by

$$b = b_0 + t \qquad\qquad z' = -h'p\,. \tag{5.49}$$

Defining the time- and state-dependent functions

$$\mathcal{T} = p' \qquad \mathcal{P} = tp' - h' \qquad \mathcal{M} = m' - t\,h' + \frac{t^2}{2}\,p' \tag{5.50}$$

the Carrollian geometry is given by

$$h_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \left(\mathrm{d}r + \mathcal{T}\,\mathrm{d}t\right)^2 \qquad\qquad v^\mu\partial_\mu = \frac{1}{r - \mathcal{P}}\left(\partial_t - \mathcal{T}\,\partial_r\right) \tag{5.51}$$

and for zero mode solutions features a Carroll extremal surface at $r = \mathcal{P}$. The postcarrollian 1-form

$$m = \mathcal{M}\,\mathrm{d}t \tag{5.52}$$

is independent of the spatial coordinate r and only has a leg in the time direction.

The loop group boundary action (5.37) reduces to

$$\Gamma[h, p, m] = \frac{k}{2\pi}\int \mathrm{d}t\left(p'm' - \frac{1}{2}\,h'^2\right) \tag{5.53}$$

which resembles three free particles moving along the boundary, since on-shell one finds $h'' = p'' = m'' = 0$. Equivalently, the equations of motion are generalized conservation equations,

$$\partial_t\mathcal{T} = 0 \qquad \partial_t\mathcal{P} = \mathcal{T} \qquad \partial_t\mathcal{M} = \mathcal{P} - t\,\mathcal{T} \quad\Rightarrow\quad \partial_t^2\mathcal{P} = \partial_t^2\mathcal{M} = 0\,. \tag{5.54}$$

Imposing the constraints on the level of the transformation laws (5.36), one finds a reduced set of symmetries for the remaining fields

$$h \rightarrow h + \varepsilon_{\mathrm{H}} \qquad p \rightarrow p + \varepsilon_{\mathrm{P}} \qquad m \rightarrow m + \varepsilon_{\mathrm{M}}\,. \tag{5.55}$$

While these boundary conditions lead to quite simple dynamics in these variables, it might still be possible to impose different constraints such that one does not find just free particles. However,

such a choice of constraints would have to be more sophisticated than just imposing some of the Maurer–Cartan components to vanish, since all such combinations we investigated lead to similar conclusions.

As an alternative to the inverse Higgs mechanism, one can apply the technique presented in [51]. This amounts to constructing constraints as a consistent second-class system on the level of the algebra. The possible choices depend on the number of even-dimensional non-abelian subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ which have $\langle \mathfrak{h}, K \rangle \neq 0$ for some fixed algebra element $K \notin \mathfrak{h}$. For (2.16) the only 4d choices of $\mathfrak{h}$ are $\{B, H, M, Z\}$ and $\{P, H, M, Z\}$, both of which again lead to free particle actions. There are no 2d choices. However, this conclusion relies on the fixed element $K \in \mathfrak{g}$ being chosen in a state-independent manner, which at the same time suggests the direction in which this simple result could be circumvented.

6. Generalization to higher dimensions

In this section, we generalize postcarrollian gravity to higher dimensions. We start with 3d in section 6.1 and conclude with 4d in section 6.2.

6.1 Postcarrollian gravity in 3d

In $D = 2 + 1$ dimensions, it is possible to define a postcarrollian gravity theory by considering a Chern–Simons action

$$I_{3d} = \frac{k}{4\pi} \int \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle \tag{6.1}$$

where the connection one-form $A = A_\mu \, dx^\mu$ follows from (1.5), and $\langle \, , \, \rangle$ denotes a non-degenerate invariant bilinear form on the postcarrollian algebra (1.6). By defining the dual generators

$$J = \frac{1}{2}\epsilon^{ab} J_{ab} \qquad S = \frac{1}{2}\epsilon^{ab} S_{ab} \qquad C_a = -\epsilon_a{}^b B_b \qquad G_a = -\epsilon_a{}^b K_b \,. \tag{6.2}$$

where ϵ_{ab} is the 2d spatial Levi-Civita symbol ($\epsilon_{12} = 1$), the commutation relations (1.6) yields

$$[J, P_a] = \epsilon^b{}_a P_b \tag{6.3a}$$
$$[J, C_a] = \epsilon^b{}_a C_b \tag{6.3b}$$
$$[P_a, P_b] = -\Lambda \, \epsilon_{ab} \, J \tag{6.3c}$$
$$[P_a, C_b] = -\epsilon_{ab} \, H \tag{6.3d}$$
$$[P_a, H] = -\Lambda \, \epsilon^b{}_a C_b \tag{6.3e}$$
$$[C_a, C_b] = \epsilon_{ab} \, S \tag{6.3f}$$
$$[C_a, H] = \epsilon^b{}_a M_b \tag{6.3g}$$

$$[J, M_a] = \epsilon^b{}_a M_b = [S, P_a] \tag{6.3h}$$
$$[J, G_a] = \epsilon^b{}_a G_b = [S, C_a] \tag{6.3i}$$
$$[P_a, M_b] = -\Lambda \, \epsilon_{ab} \, S \tag{6.3j}$$
$$[P_a, G_b] = -\epsilon_{ab} \, L = [M_a, C_b] \tag{6.3k}$$
$$[P_a, L] = -\Lambda \, \epsilon^b{}_a G_b = [M_a, H] \tag{6.3l}$$

commutators not displayed vanish.

Up to dualization with the Levi-Civita symbol, the algebra above is the specialization of the general postcarrollian algebra (1.6) to 3d. Again, the commutators in the left half form the Carroll algebra provided we set to zero the right-hand sides in the last two lines. The commutators in the right half all involve one subleading generator on each side of the equalities that mirror the corresponding leading generators from the left half.

In $2 + 1$ dimensions, the postcarrollian algebra admits the invariant bilinear form

$$\langle J, H \rangle = \rho_0 \qquad\qquad \langle S, H \rangle = \rho_1 = \langle J, L \rangle \tag{6.4a}$$

$$\langle P_a, C_b \rangle = \rho_0\, \delta_{ab} \qquad\qquad \langle P_a, G_b \rangle = \rho_1\, \delta_{ab} = \langle M_a, C_b \rangle \tag{6.4b}$$

where ρ_0, $\rho_1 \in \mathbb{R}$ and $\rho_1 \neq 0$ for $\langle \cdot, \cdot \rangle$ to be non-degenerate. By defining dual gauge fields compatible with (6.2), the postcarrollian connection one-form is written as[4]

$$A = \tau H + e^a P_a + \omega^a C_a + \omega J + lL + m^a M_a + \sigma^a G_a + \sigma S \,. \tag{6.5}$$

From $\delta_\lambda A = \mathrm{d}\lambda + [A, \lambda]$ and

$$\lambda = \xi H + \xi^a P_a + \theta^a C_a + \theta J + \zeta L + \zeta^a M_a + \eta^a G_a + \eta S \tag{6.6}$$

one can infer the gauge-transformation behaviour of the various components

$$\delta_\lambda \tau = \mathrm{d}\xi + \epsilon_{ab}(\xi^a \omega^b - e^a \theta^b) \tag{6.7a}$$

$$\delta_\lambda e^a = \mathrm{d}\xi^a + \epsilon^a{}_b(\omega\xi^b - \theta e^b) \tag{6.7b}$$

$$\delta_\lambda \omega = \mathrm{d}\theta - \Lambda e^a \epsilon_{ab}\xi^b \tag{6.7c}$$

$$\delta_\lambda \omega^a = \mathrm{d}\theta^a + \epsilon^a{}_b(\omega\theta^b - \theta\omega^b + \Lambda\tau\xi^b - \Lambda\xi e^b) \tag{6.7d}$$

$$\delta_\lambda l = \mathrm{d}\zeta - \epsilon_{ab}(e^a \eta^b - \xi^a \sigma^b + m^a \theta^b - \zeta^a \omega^b) \tag{6.7e}$$

$$\delta_\lambda m^a = \mathrm{d}\zeta^a + \epsilon^a{}_b(\omega\zeta^b - \theta m^b + \sigma\xi^b - \eta e^b - \tau\theta^b + \xi\omega^b) \tag{6.7f}$$

$$\delta_\lambda \sigma = \mathrm{d}\eta + \epsilon_{ab}(\omega^a \theta^b - \Lambda e^a \zeta^b + \Lambda\xi^a m^b) \tag{6.7g}$$

$$\delta_\lambda \sigma^a = \mathrm{d}\eta^a + \epsilon^a{}_b(\omega\eta^b - \theta\sigma^b + \sigma\theta^b - \eta\omega^b + \Lambda l\xi^b - \Lambda\zeta e^b + \Lambda\tau\zeta^b - \Lambda\xi m^b) \,. \tag{6.7h}$$

The curvature two-form $F = \mathrm{d}A + A \wedge A$ is given by

$$F = TH + T^a P_a + \left(R^a(C) + \Lambda\epsilon^a{}_b\tau \wedge e^b \right) C_a + \left(R(J) - \frac{\Lambda}{2}\epsilon_{ab}e^a \wedge e^b \right) J$$
$$+ \Theta L + \Theta^a M_a + \left(R^a(G) + \Lambda\epsilon^a{}_b\tau \wedge m^b + \Lambda\epsilon^a{}_b l \wedge e^b \right) G_a + \left(R(S) - \Lambda\epsilon_{ab}e^a \wedge m^b \right) S \tag{6.8}$$

where we have defined the Carrollian and postcarrollian torsion two-forms

$$T = \mathrm{d}\tau - \epsilon_{ab}\,\omega^a \wedge e^b \tag{6.9a}$$

$$T^a = \mathrm{d}e^a + \epsilon^a{}_b\,\omega \wedge e^b \tag{6.9b}$$

$$\Theta = \mathrm{d}l - \epsilon_{ab}\left(\sigma^a \wedge e^b - \omega^a \wedge m^b \right) \tag{6.9c}$$

$$\Theta^a = \mathrm{d}m^a + \epsilon^a{}_b\left(\omega \wedge m^b + \sigma \wedge e^b - \tau \wedge \omega^b \right) \tag{6.9d}$$

and curvature two-forms

$$R^a(C) = \mathrm{d}\omega^a + \epsilon^a{}_b\,\omega \wedge \omega^b \tag{6.10a}$$

$$R(J) = \mathrm{d}\omega \tag{6.10b}$$

$$R^a(G) = \mathrm{d}\sigma^a + \epsilon^a{}_b\,\omega \wedge \sigma^b + \epsilon^a{}_b\,\sigma \wedge \omega^b \tag{6.10c}$$

$$R(S) = \mathrm{d}\sigma + \frac{1}{2}\epsilon_{ab}\,\omega^a \wedge \omega^b \,. \tag{6.10d}$$

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[4]The generators for subleading boosts and rotations G_a and S are the dualized versions, but we still use the same letter for the corresponding gauge field components.

Using these expressions, the Chern-Simons action (6.1) takes the form

$$I_{3d} = \rho_0\, I_{(A)dSC} + \rho_1\, I_{PC} \tag{6.11}$$

where the first term corresponds to the magnetic (A)dS$_3$ Carroll gravity action [62, 88, 89]

$$I_{(A)dSC} = \frac{k}{2\pi} \int \left(e_a \wedge R^a(C) + \tau \wedge R(J) - \frac{\Lambda}{2}\epsilon_{ab}\tau \wedge e^a \wedge e^b \right) \tag{6.12}$$

whereas the second is given by the following postcarrollian gravity action.

$$I_{PC} = \frac{k}{2\pi} \int \left(m_a \wedge R^a(C) + \tau \wedge R(S) + l \wedge R(J) + e_a \wedge R^a(G) - \frac{\Lambda}{2}\epsilon_{ab}\left(2\tau \wedge m^a \wedge e^b + l \wedge e^a \wedge e^b\right) \right) \tag{6.13}$$

The field equations that follow from varying the (A)dS Carroll action (6.12) read

$$T = 0 = T^a \qquad R^a(C) = -\Lambda\epsilon^a{}_b\tau \wedge e^b \qquad R(J) = \frac{\Lambda}{2}\epsilon_{ab}e^a \wedge e^b \tag{6.14}$$

The variations of the postcarrollian action (6.13) with respect to the postcarrollian fields reproduce field equations associated with the (A)dS Carroll action (6.12). Varying the action with respect to Carrollian gauge fields yields a new set of field equations,

$$\Theta = 0 = \Theta^a \qquad R^a(G) = -\Lambda\epsilon^a{}_b\left(\tau \wedge m^b + l \wedge e^b\right) \qquad R(S) = \Lambda\epsilon_{ab}e^a \wedge m^b\,. \tag{6.15}$$

This nested structure of the equations of motion is typical for theories arising from algebra expansions and, e.g., also appears in non-relativistic expansions of general relativity [90].

6.1.1 Reduction from 3d to 2d

Among the solutions of the 3d postcarrollian theory (6.13) we just presented are the spherically symmetric ones, which admit an effective description in terms of the postcarrollian JT model we described in section 2.1. This is in direct analogy to the Lorentzian case [76, 91]. Let us see which conditions spherical symmetry implies on the frame variables. For this, we use adapted coordinates (x^α, ϕ) where we denote the 2d indices by $\alpha, \beta = 0, 1$. We also split 3d frame components into constituents adapted to the symmetry like $e^a = (e, e^{\underline{2}})$, $m^a = (m, m^{\underline{2}})$ and denote the values of anholonomic indices by $a, b = \underline{1}, \underline{2}$ with $\underline{0}$ denoting the component in the direction of τ. In the ansatz for the spherical reduction,

$$e = e_\alpha\, dx^\alpha \qquad\qquad e^{\underline{2}} = Y\, d\phi \tag{6.16}$$

$$m = m_\alpha\, dx^\alpha \qquad\qquad m^{\underline{2}} = X\, d\phi \tag{6.17}$$

$$\tau = \tau_\alpha\, dx^\alpha \qquad\qquad l = l_\alpha\, dx^\alpha \tag{6.18}$$

all the components are independent of the coordinate direction ϕ. It follows that some of the equations of motion reduce to kinematical constraints and only a part of them carries dynamical information. This dynamical part should be recovered as equations of motion of the reduced 2d theory. The kinematical part will be solved directly and inserted into the action.

As a first step, we therefore expand the torsion constraints

$$T^a = T = \Theta^a = \Theta = 0 \tag{6.19}$$

in terms of their frame components like

$$T^a = T^a_{\underline{01}} \tau \wedge e + T^a_{\underline{02}} \tau \wedge e^{\underline{2}} + T^a_{\underline{12}} e \wedge e^{\underline{2}} \tag{6.20}$$

and similarly for the other torsion two-forms. The connection and vielbein components are likewise expanded as $\omega^a = \tau \omega^a_{\underline{0}} + e \omega^a_{\underline{1}} + e^{\underline{2}} \omega^a_{\underline{2}}$ and $\sigma^a = \tau \sigma^a_{\underline{0}} + e \sigma^a_{\underline{1}} + e^{\underline{2}} \sigma^a_{\underline{2}}$. The kinematical part of the first half in (6.19) implies

$$\omega_{\underline{0}} = 0 \qquad \omega_{\underline{1}} = 0 \qquad \omega^{\underline{1}}_{\underline{1}} = -\omega^{\underline{2}}_{\underline{2}} \qquad \omega^{\underline{1}}_{\underline{0}} = 0 \tag{6.21}$$

while the second half yields the constraints

$$\omega_{\underline{0}} m^{\underline{2}}_{\underline{2}} + \sigma_{\underline{0}} - \omega^{\underline{2}}_{\underline{2}} = \omega_{\underline{1}} m^{\underline{2}}_{\underline{2}} + \sigma_{\underline{1}} = \omega_{\underline{0}} m_{\underline{1}} - \omega_{\underline{1}} m_{\underline{0}} + \sigma_{\underline{0}} - \omega^{\underline{1}}_{\underline{1}}$$

$$= \sigma^{\underline{1}}_{\underline{1}} + \sigma^{\underline{2}}_{\underline{2}} + \omega^{\underline{1}}_{\underline{1}} m^{\underline{2}}_{\underline{2}} + \omega^{\underline{2}}_{\underline{2}} m_{\underline{1}} = \sigma^{\underline{1}}_{\underline{0}} + \omega^{\underline{1}}_{\underline{0}} m^{\underline{2}}_{\underline{2}} - \omega^{\underline{2}}_{\underline{2}} m_{\underline{1}} = 0 \,. \tag{6.22}$$

If used together with (6.21) these equations imply

$$\sigma^{\underline{0}} = 0 \qquad \sigma^{\underline{1}} = 0 \qquad \sigma^{\underline{1}}_{\underline{1}} = -\sigma^{\underline{2}}_{\underline{2}} \qquad \sigma^{\underline{1}}_{\underline{0}} = 0 \qquad \omega^{\underline{1}}_{\underline{1}} = 0 = \omega^{\underline{2}}_{\underline{2}} \,. \tag{6.23}$$

The rest of the equations of motion (6.12)-(6.15) do not give any additional constraints but are either trivially satisfied or carry dynamical information. As usual for a Carrollian connection the torsion constraints in the first half of (6.19) do not determine ω^a uniquely but only up to the shift $\omega^a \to \omega^a + \epsilon^{ab} W_{bc} e^c$ where W_{ab} is an arbitrary symmetric tensor [65, 66]. The torsion constraints in the second half of (6.19) have a similar structure, σ^a is only determined up to $\sigma^a \to \sigma^a + \epsilon^{ab} V_{bc} e^c$. Before looking at the full postcarrollian theory, let us see how things work in the (A)dS Carroll theory as a warm-up.

We solve the constraints (6.21),

$$\omega = \omega_\phi \, \mathrm{d}\phi \qquad \omega^{\underline{1}} = -\omega^{\underline{2}}_{\underline{2}} e + \omega^{\underline{1}}_\phi \, \mathrm{d}\phi \qquad \omega^{\underline{2}} = \Omega + \omega^{\underline{2}}_{\underline{2}} X \, \mathrm{d}\phi \tag{6.24}$$

where Ω is some one-form with legs only in the τ and e directions. The ambiguity W_{ab} acts on these components by

$$\omega^{\underline{2}}_{\underline{2}} \to \omega^{\underline{2}}_{\underline{2}} - W_{\underline{12}} \qquad \omega^{\underline{1}}_\phi \to \omega^{\underline{1}}_\phi + W_{\underline{22}} \qquad \Omega \to \Omega - W_{\underline{11}} e \,, \tag{6.25}$$

which allows to parametrize them as

$$\omega = \omega_\phi \, \mathrm{d}\phi \qquad \omega^{\underline{1}} = W_{\underline{12}} e + W_{\underline{22}} \, \mathrm{d}\phi \qquad \omega^{\underline{2}} = \Omega - W_{\underline{12}} X \, \mathrm{d}\phi \,. \tag{6.26}$$

The action (6.12) then reads

$$I_{\text{(A)dSC}} = \frac{k}{2\pi} \int \mathrm{d}\phi \wedge \left[Y \, \mathrm{d}\Omega + \omega_\phi \left(\mathrm{d}\tau + \Omega \wedge e \right) + W_{\underline{22}} \, \mathrm{d}e - Y \Lambda \tau \wedge e \right] \tag{6.27}$$

which after integrating over ϕ becomes the action of the Carrollian JT model [51] upon identifying

$$X_{\text{H}} = \omega_\phi \qquad X_{\text{P}} = W_{\underline{22}} \qquad \Omega = \omega \,. \tag{6.28}$$

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Note that $W_{\underline{12}}$ does not feature in the action. Moreover, the component $W_{\underline{11}}$ can always be absorbed into Ω and therefore does not feature in the action either.

The postcarrollian theory is obtained by solving all the kinematical equations, i.e., solving additionally (6.22). This yields the general solution

$$\sigma = \sigma_\phi \, d\phi \qquad\qquad \sigma^{\underline{1}} = V_{\underline{12}}e + V_{\underline{22}} \, d\phi \qquad\qquad \sigma^{\underline{2}} = B - V_{\underline{12}}Y \, d\phi \qquad (6.29)$$

$$\omega = \omega_\phi \, d\phi \qquad\qquad \omega^{\underline{1}} = W_{\underline{22}} \, d\phi \qquad\qquad \omega^{\underline{2}} = \Omega \qquad (6.30)$$

where we introduced two one-forms $B = B_\alpha \, dx^\alpha$ and $\Omega = \Omega_\alpha \, dx^\alpha$ and all the ambiguous components were already parametrized by V_{ab} and W_{ab}.

Inserting this into the postcarrollian action (6.13) we get up to total derivatives

$$I_{\mathrm{PC}} = \frac{k}{2\pi} \int d\phi \wedge \Big[X \, d\Omega + \sigma_\phi \big(d\tau + \Omega \wedge e \big) + V_{\underline{22}} \, de + W_{\underline{22}} \big(dm + \Omega \wedge \tau \big) - \Lambda X \tau \wedge e$$

$$+ \omega_\phi \big(dl + B \wedge e \big) + Y \big(dB - \Lambda (\tau \wedge m + l \wedge e) \big) \Big]. \qquad (6.31)$$

One can see that the 2d theory from previous sections appears as a subsector by solving the constraints associated with ω_ϕ and Y, effectively setting the last line to zero. If we then identify

$$\sigma_\phi = X_{\mathrm{H}} \qquad\qquad V_{\underline{22}} = X_{\mathrm{P}} \qquad\qquad W_{\underline{22}} = X_{\mathrm{M}} \qquad\qquad \Omega = \omega \qquad (6.32)$$

we obtain the postcarrollian JT model (2.12) after integrating out the angular part.

6.1.2 Lifting Carrollian boundary conditions to postcarrollian ones

In 3d gravity theories, there is a menagerie of boundary conditions for a given bulk theory, see [92, 93] and Refs. therein for examples within Einstein gravity. It is not the intention of this work to generalize such a discussion to postcarrollian gravity theories. Instead, we confine ourselves to one example to show that the same methods that worked for 3d Einstein gravity and 3d Carroll gravity continue to work for 3d postcarrollian gravity.

Our example is based on the Carrollian boundary conditions proposed in [72]. We adopt their ansatz and extend it to the postcarrollian gravity theory introduced at the beginning of section 6.1, fixing $\rho_0 = 0$ and $\rho_1 = 1$ in the bilinear form (6.4) and choosing the value $\Lambda = 0$ for the cosmological constant. Thus, we start with the bulk Chern–Simons connection

$$A = b^{-1}(r) \big(d + a(t, \varphi) \big) b(r) \qquad\qquad b(r) = e^{r \, P_2} \qquad (6.33)$$

where the boundary connection

$$a(t, \varphi) = \big(J + a_{\mathrm{H}} H + a_{\mathrm{P}}^b P_b + a_{\mathrm{C}}^b C_b + a_{\mathrm{L}} L + a_{\mathrm{M}}^b M_b + a_{\mathrm{G}}^b G_b + a_{\mathrm{S}} S \big) \, d\varphi + \mu H \, dt \qquad (6.34)$$

contains all the Carrollian and postcarrollian generators introduced above. The quantities with letters a are all state-dependent functions that can depend on the postcarrollian time t and the angular coordinate $\varphi \sim \varphi + 2\pi$ but not on the radial coordinate r. The quantity μ is a chemical potential, i.e., an arbitrary state-independent function of t, φ.

The choices (6.33), (6.34) yield the bulk connection

$$A = a + P_2 \, dr + r \, [a, P_2] \qquad (6.35)$$

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that is linear in the radial coordinate r. The nonvanishing commutators with the translation generator P_2 are given by

$$[J, P_2] = P_1 \qquad\qquad [P_2, C_1] = H \qquad\qquad [S, P_2] = M_1 \qquad\qquad [P_2, G_1] = L \,. \qquad (6.36)$$

The metric deduced from (6.35),

$$h_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = \mathrm{d}r^2 + 2a_\mathrm{P}^2\, \mathrm{d}\varphi\, \mathrm{d}r + \left((r + a_\mathrm{P}^1)^2 + (a_\mathrm{P}^2)^2\right)\, \mathrm{d}\varphi^2 \qquad (6.37)$$

means that in the second-order formulation, we have imposed the boundary conditions

$$h_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = \mathrm{d}r^2 + r^2\, \mathrm{d}\varphi^2 + O(r)\, \mathrm{d}\varphi^2 + O(1)\, \mathrm{d}r\, \mathrm{d}\varphi \,. \qquad (6.38)$$

The clock 1-form is given by

$$\tau = \mu\, \mathrm{d}t + \left(a_\mathrm{H} - a_\mathrm{C}^1 r\right)\, \mathrm{d}\varphi \qquad (6.39)$$

which through the defining relations $v^\mu \tau_\mu = -1$, $v^\mu e_\mu^a = 0$ determines the Carrollian vector field

$$v^\mu\, \partial_\mu = -\frac{1}{\mu}\, \partial_t \,. \qquad (6.40)$$

The postcarrollian 1-form

$$m_\mu^a\, \mathrm{d}x^\mu = \left(a_\mathrm{S}\, r\, \delta_1^a + a_\mathrm{M}^a\right)\, \mathrm{d}\varphi \qquad (6.41)$$

has vanishing contraction with the Carrollian vector field (6.40).

Parametrizing the boundary condition preserving gauge transformations (6.7) by $\lambda = b^{-1}\varepsilon b$ one finds the conditions $\partial_t \varepsilon_I = 0$ except for ε_M which satisfies $\partial_t \varepsilon_\mathrm{M}^a - \mu\, \epsilon^a{}_b\, \varepsilon_\mathrm{C}^b = 0$. Additionally, there is the constraint $\partial_\varphi \varepsilon_\mathrm{J} = 0$ from the requirements in the boundary connection (6.34) that the J-component is fixed to 1.

The variation of the boundary charges

$$\delta Q[\varepsilon] = \frac{k}{2\pi}\, \oint\, \mathrm{d}\varphi\, \langle \varepsilon, \delta a_\varphi \rangle \qquad (6.42)$$

can be functionally integrated for the boundary conditions (6.34), yielding

$$Q[\varepsilon] = \frac{k}{2\pi}\, \oint\, \mathrm{d}\varphi\, \left(\varepsilon_\mathrm{S}\, a_\mathrm{H} + \varepsilon_{\mathrm{M}b}\, a_\mathrm{C}^b + \varepsilon_{\mathrm{G}b}\, a_\mathrm{P}^b + \varepsilon_\mathrm{J}\, a_\mathrm{L} + \varepsilon_{\mathrm{P}b}\, a_\mathrm{G}^b + \varepsilon_{\mathrm{C}b}\, a_\mathrm{M}^b + \varepsilon_\mathrm{H}\, a_\mathrm{S}\right) \,. \qquad (6.43)$$

The on-shell conditions $\partial_t a_\varphi - \partial_\varphi a_t + [a_t, a_\varphi] = 0$ imply

$$\partial_t a_\mathrm{H} = \partial_\varphi \mu \qquad\qquad \partial_t a_\mathrm{M}^a - \mu\, \epsilon^a{}_b\, a_\mathrm{C}^b = 0 \qquad\qquad \partial_t(a_\mathrm{P}^a, a_\mathrm{C}^a, a_\mathrm{L}, a_\mathrm{G}^a, a_\mathrm{S}) = 0 \qquad (6.44)$$

which in turn establishes the conservation of the canonical boundary charges

$$\partial_t Q[\varepsilon] = \frac{k}{2\pi}\, \oint\, \mathrm{d}\varphi\, \left(\partial_t a_\mathrm{H} \varepsilon_\mathrm{S} + \partial_t a_\mathrm{M}^b \varepsilon_{\mathrm{C}b} + \partial_t \varepsilon_{\mathrm{M}b} a_\mathrm{C}^b\right) = \frac{k}{2\pi}\, \oint\, \mathrm{d}\varphi\, \partial_\varphi \mu\, \varepsilon_\mathrm{S} \qquad (6.45)$$

in the sense that the right-hand side is state-independent, i.e., $\partial_t \delta Q = 0$ as a consequence of the on-shell conditions (6.44). For the asymptotic symmetry algebra, it is useful to absorb the time-dependence by redefining[5]

$$\varepsilon_\mathrm{M}^a(t, \varphi) \to \varepsilon_\mathrm{M}^a(\varphi) + \epsilon^a{}_b\, \varepsilon_\mathrm{C}^b(\varphi)\, v(t, \varphi) \qquad\qquad a_\mathrm{M}^a(t, \varphi) \to a_\mathrm{M}^a(\varphi) + \epsilon^a{}_b\, a_\mathrm{C}^b(\varphi)\, v(t, \varphi) \qquad (6.46)$$

---

[5]Algebraically, we exploit the automorphism $C_n^a \to C_n^a - v\, \epsilon^a{}_b\, M_n^b$ of the asymptotic symmetry algebra (6.48).

where $\nu(t, \varphi) := \int^t \mu(t', \varphi) \, dt'$.

Introducing Fourier modes for the state-dependent functions and transformation parameters

$$P_n^a = \frac{k}{2\pi} \oint d\varphi \, e^{in\varphi} \, a_{\mathrm{G}}^a(\varphi) \qquad\qquad C_n^a = \frac{k}{2\pi} \oint d\varphi \, e^{in\varphi} \, a_{\mathrm{M}}^a(\varphi) \tag{6.47a}$$

$$J = \frac{k}{2\pi} \oint d\varphi \, a_{\mathrm{L}}(\varphi) \qquad\qquad H_n = \frac{k}{2\pi} \oint d\varphi \, e^{in\varphi} a_{\mathrm{S}}(\varphi) \tag{6.47b}$$

$$M_n^a = \frac{k}{2\pi} \oint d\varphi \, e^{in\varphi} \, a_{\mathrm{C}}^a(\varphi) \qquad\qquad G_n^a = \frac{k}{2\pi} \oint d\varphi \, e^{in\varphi} \, a_{\mathrm{P}}^a(\varphi) \tag{6.47c}$$

$$S_n = \frac{k}{2\pi} \oint d\varphi \, e^{in\varphi} \, a_{\mathrm{H}}(\varphi) \qquad\qquad \varepsilon_{I\,n} = \frac{1}{2\pi} \oint d\varphi \, e^{in\varphi} \, \varepsilon_I(\varphi) \tag{6.47d}$$

yields the mode expansion of the postcarrollian asymptotic symmetry algebra via the relation $\delta_{\varepsilon_1} Q[\varepsilon_2] = \{Q[\varepsilon_1], Q[\varepsilon_2]\}$ (see, e.g., [94])

$$\{J, P_n^a\} = \epsilon_b{}^a \, P_n^b \qquad\qquad\qquad \{J, C_n^a\} = \epsilon_b{}^a \, C_n^b \tag{6.48a}$$

$$\{J, M_n^a\} = \epsilon_b{}^a \, M_n^b \qquad\qquad\qquad \{J, G_n^a\} = \epsilon_b{}^a \, G_n^b \tag{6.48b}$$

$$\{P_n^a, C_m^b\} = -\epsilon^{ab} \, H_{n+m} \qquad\qquad \{C_n^a, C_m^b\} = \epsilon^{ab} \, S_{n+m} \tag{6.48c}$$

$$\{P_n^a, G_m^b\} = \left(in\,\delta^{ab} - \epsilon^{ab}\right) L \, \delta_{n+m,0} = \{M_n^a, C_m^b\} \qquad \{S_n, C_m^a\} = \epsilon_b{}^a \, G_{n+m}^b \tag{6.48d}$$

$$\{C_n^a, H_m\} = \epsilon_b{}^a \, M_{n+m}^b = \{S_n, P_m^a\} \qquad\qquad \{H_n, S_m\} = in\,L\,\delta_{n+m,0} \,. \tag{6.48e}$$

The postcarrollian asymptotic symmetry algebra (6.48) features a central extension, which we labelled by the generator L so that the zero-mode subalgebra (which coincides with the wedge subalgebra) matches with the postcarrollian algebra (6.3).

6.2 Postcarrollian gravity in 4d

A 4d postcarrollian gravity action is defined as a MacDowell–Mansouri-like action,

$$I_{4d} = -\frac{\kappa}{2\Lambda} \int \langle F \wedge F \rangle \tag{6.49}$$

invariant under the postcarrollian extension of the homogeneous Carroll symmetry, namely, the subalgebra of (1.6) spanned by the generators $\{B_a, J_{ab}, K_a, S_{ab}\}$. This subalgebra admits the following non-degenerate invariant bilinear form ($\rho_1 \neq 0$)

$$\langle B_a, J_{cd} \rangle = \rho_0 \epsilon_{abc} \qquad\qquad \langle B_a, S_{cd} \rangle = \rho_1 \epsilon_{abc} = \langle K_a, J_{cd} \rangle \,. \tag{6.50}$$

In 4d, the field strength $F = dA + A \wedge A$ associated to the postcarrollian connection (1.5),

$$F = TH + T^a P_a + \left(R^a(B) - \Lambda \tau \wedge e^a\right) B_a + \frac{1}{2}\left(R^{ab}(J) - \Lambda e^a \wedge e^b\right) J_{ab} \tag{6.51}$$

$$+ \Theta L + \Theta^a M_a + \left(R^a(K) - \Lambda \tau \wedge m^a - \Lambda l \wedge e^a\right) K_a + \frac{1}{2}\left(R^{ab}(S) - 2\Lambda e^{[a} \wedge m^{b]}\right) S_{ab}$$

contains the Carrollian torsion forms T, T^a and the postcarrollian torsion forms Θ, Θ^a,

$$T = d\tau + \omega^a \wedge e_a \qquad\qquad T^a = De^a \tag{6.52a}$$

$$\Theta = dl + \omega^a \wedge m_a + \sigma^a \wedge e_a \qquad\qquad \Theta^a = Dm^a + \sigma^a{}_b \wedge e^b + \omega^a \wedge \tau \tag{6.52b}$$

whereas the Carrollian curatures $R^a(B)$, $R^{ab}(J)$ and the postcarrollian curvature two-forms $R^a(K)$, $R^{ab}(S)$ are given by

$$R^a(B) = D\omega^a \qquad\qquad R^{ab}(J) = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \tag{6.53a}$$

$$R^a(K) = D\sigma^a + \sigma^a{}_b \wedge \omega^b \qquad\qquad R^{ab}(S) = D\sigma^{ab} + \omega^a \wedge \omega^b \tag{6.53b}$$

where we have introduced an $SO(3)$-covariant derivative D acting on $SO(3)$-tensor-valued differential forms as

$$DX^{a_1\cdots a_s} = \mathrm{d}X^{a_1\cdots a_s} + \omega^{a_1}{}_b \wedge X^{ba_2\cdots a_s} + \omega^{a_2}{}_b \wedge X^{a_1 b a_3 \cdots a_s} + \cdots + \omega^{a_s}{}_b \wedge X^{a_1 \cdots a_{s-1} b} . \tag{6.54}$$

The torsion and curvature forms satisfy Bianchi identities $\mathrm{d}F + A \wedge F - F \wedge A = 0$,

$$\mathrm{d}T + \omega_a \wedge T^a = e_a \wedge R^a(B) \tag{6.55a}$$

$$DT^a = R^a{}_b(J) \wedge e^b \tag{6.55b}$$

$$\mathrm{d}\Theta + \omega_a \wedge \Theta^a + \sigma_a \wedge T^a = m_a \wedge R^a(B) + e_a \wedge R^a(K) \tag{6.55c}$$

$$D\Theta^a + \sigma^a{}_b \wedge T^b + \omega_a \wedge T = \tau \wedge R^a(B) + R^a{}_b(J) \wedge m^b + R^a{}_b(S) \wedge e^b \tag{6.55d}$$

$$DR^a(B) = R^a{}_b(J) \wedge \omega^b \tag{6.55e}$$

$$DR^{ab}(J) = 0 \tag{6.55f}$$

$$DR^a(K) = R^a{}_b(J) \wedge \sigma^b + R^a{}_b(S) \wedge \omega^b - \sigma^a{}_b \wedge R^b(B) \tag{6.55g}$$

$$DR^{ab}(S) = 2R^{[a}(B) \wedge \omega^{b]} + 2R_c{}^{[a}(J) \wedge \sigma^{b]c} . \tag{6.55h}$$

The gauge transformations for the Carrollian and postcarrollian gauge fields follow from the transformation rule of the connection one-form, $\delta A = \mathrm{d}\lambda + [A, \lambda]$, where λ is a local parameter taking values in the postcarrollian algebra (1.6), i.e.,

$$\lambda = \xi H + \xi^a P_a + \zeta L + \zeta^a M_a + \theta^a B_a + \frac{1}{2}\theta^{ab} J_{ab} + \eta^a K_a + \frac{1}{2}\eta^{ab} S_{ab} . \tag{6.56}$$

The gauge fields transform in the adjoint representation as

$$\delta_\lambda \tau = \mathrm{d}\xi + \omega^a \xi_a - \theta_a e^a \tag{6.57a}$$

$$\delta_\lambda e^a = D\xi^a - \theta^a{}_b e^b \tag{6.57b}$$

$$\delta_\lambda l = \mathrm{d}\zeta + \omega^a \zeta_a - \theta_a m^a + \sigma^a \xi_a - \eta_a e^a \tag{6.57c}$$

$$\delta_\lambda m^a = D\zeta^a - \theta^a{}_b m^b + \sigma^a{}_b \xi^b - \eta^a{}_b e^b + \omega^a \xi - \theta^a \tau \tag{6.57d}$$

$$\delta_\lambda \omega^a = D\theta^a - \theta^a{}_b \omega^b + \Lambda\left(e^a \xi - \xi^a \tau\right) \tag{6.57e}$$

$$\delta_\lambda \omega^{ab} = D\theta^{ab} - 2\Lambda e^{[a}\xi^{b]} \tag{6.57f}$$

$$\delta_\lambda \sigma^a = D\eta^a - \theta^a{}_b \sigma^b + \sigma^a{}_b \theta^b - \eta^a{}_b \omega^b + \Lambda\left(m^a \xi - \zeta^a \tau + e^a \zeta - \xi^a l\right) \tag{6.57g}$$

$$\delta_\lambda \sigma^{ab} = D\eta^{ab} - 2\theta^{c[a}\sigma^{b]}{}_c + 2\omega^{[a}\theta^{b]} - \Lambda\left(e^{[a}\zeta^{b]} - m^{[a}\xi^{b]}\right) . \tag{6.57h}$$

Using (6.50) and (6.53), up to boundary terms the action (6.49) takes the form

$$I_{4d} = \rho_0 \, I_{\text{(A)dSC}} + \rho_1 \, I_{\text{PC}} . \tag{6.58}$$

The term along ρ_0 is the magnetic (A)dS Carroll gravity action

$$I_{(A)dSC} = \kappa \int \epsilon_{abc} \left(\tau \wedge e^a \wedge R^{bc}(J) + e^a \wedge e^b \wedge R^c(B) - \Lambda \tau \wedge e^a \wedge e^b \wedge e^c \right) . \tag{6.59}$$

The $\Lambda = 0$ limit of this action was introduced in [65] by gauging the Carroll symmetry, and was later shown in [66] to be equivalent to magnetic Carroll gravity [95]. The term along ρ_1 is the magnetic postcarrollian gravity action

$$I_{PC} = \kappa \int \epsilon_{abc} \Big(\tau \wedge e^a \wedge R^{bc}(S) + \tau \wedge m^a \wedge R^{bc}(J) + l \wedge e^a \wedge R^{bc}(J) \tag{6.60}$$

$$+ e^a \wedge e^b \wedge R^c(K) + 2 e^a \wedge m^b \wedge R^c(B) + \Lambda \left(3\tau \wedge m^a + l \wedge e^a \right) \wedge e^b \wedge e^c \Big) . \tag{6.61}$$

This action is obtained from the Einstein–Cartan action for gravity with cosmological constant as the sub-leading correction in an expansion in powers of c^2, where the Lorentzian frame-fields and connections expand as in (1.2).

As in the 3d case, the field equations associated with the postcarrollian fields reproduce the field equations known from magnetic Carroll gravity. In particular, this leads to the vanishing of the Carrollian torsion forms (6.52a). These constraints allow for solving the spin connections in terms of the vielbein as

$$\omega_\mu^{ab} = 2 e^{[a|\nu} \partial_{[\mu} e_{\nu]}^{b]} - e^{a\nu} e^{b\rho} e_{c\mu} \partial_{[\nu} e_{\rho]}^c \qquad \omega_\mu^a = -e^{a\nu} \left(v^\rho \tau_\mu \partial_{[\nu} \tau_{\rho]} + \partial_{[\mu} \tau_{\nu]} \right) + W^a{}_b e_\mu^b \tag{6.62}$$

where $W^{ab} = \omega_\mu^{(a} e^{b)\mu}$ is a symmetric tensor encoding the part of the Carroll boost connection that is not determined by the Carrollian torsion constraints [65]. Here we used the definition of the inverse frame fields v^μ, e_a^μ

$$v^\mu \tau_\mu = -1 \qquad \delta_\nu^\mu = -v^\mu \tau_\nu + e_a^\mu e_\nu^a \qquad v^\mu e_\mu^a = 0 \qquad e_a^\mu \tau_\mu = 0 . \tag{6.63}$$

Similarly, the variations of the postcarrollian action with respect to the spin connections yield the vanishing of the postcarrollian torsion two-forms (6.52b). The postcarrollian torsion constraints allow to solve the fields σ^a and σ^{ab}, and fix the tensor W^{ab} in terms of the postcarrollian field m^a

$$\sigma_\mu^{ab} = 2 e^{[a|\nu} D_{[\mu} m_{\nu]}^{b]} - e^{a\nu} e^{b\rho} e_{c\mu} D_{[\nu} m_{\rho]}^c - e^{[a|\nu} \omega_\nu^{b]} \tau_\mu \tag{6.64a}$$

$$\sigma_\mu^a = e^{a\nu} \left(-v^\rho \tau_\mu + \delta_\mu^\rho \right) \left(\partial_{[\nu} l_{\rho]} + \omega_{a[\nu} m_{\rho]}^a \right) + V^a{}_b e_\mu^b \tag{6.64b}$$

$$W^{ab} = 2 D_{[\mu} m_{\nu]}^{(a} e^{b)\mu} v^\nu . \tag{6.64c}$$

As in the previous case, $V^{ab} = \sigma_\mu^{(a} e^{b)\mu}$ is a symmetric tensor containing the part of σ^{ab} that remains independent of the Carrollian frame-fields and their postcarrollian corrections. This situation resembles the Galilean case [96], where the Bargmann extension allows to fix the components of the spin connection that were not determined by the Galilean torsion constraints. Indeed, in the context of non-relativistic expansions, the mass generator in the Bargmann algebra corresponds to a postgalilean correction. Replacing the solutions (6.62) and (6.64) in the postcarrollian action (6.60) leads to a second order formulation of the theory. Similarly to what happens in the case of magnetic Carroll gravity [65], the contribution of the curvature two-form $R^a(K)$ yields term

$$\epsilon_{abc} e^a \wedge e^b \wedge D\sigma^c \sim V^{ab} \left(e_a^\mu e_b^\nu - \delta_{ab} e_c^\mu e^{c\nu} \right) K_{\mu\nu} \det(\tau_\alpha, e_\alpha^d) d^4 x \tag{6.65}$$

where $K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_\nu(e_\mu^c e_{c\nu})$ is the extrinsic curvature. Therefore V_{ab} is a Lagrange multiplier that enforces the extrinsic curvature constraint $K_{\mu\nu} = 0$.

7. Discussion and Outlook

We finish with a few remarks and further points worth pursuing. First and foremost, postcarrollian corrections render the Hamiltonian H no longer central so that the defining Carrollian property (1.1) no longer holds. Physically, this means that boosting a postcarrollian state, in general, changes its energy so that postcarrollian descendants of a parent state no longer are "soft". This can provide a controlled cutoff on the spectrum of soft excitations (see the discussion in [38] and Refs. therein for why such a cutoff can be desirable).

As stated initially, the postcarrollian expansion we performed is around the magnetic theory and as such should contain corrections to Carroll black holes [40] away from the strict $c \to 0$ limit. Once additional corrections are taken into account, there should be a resummation that leads back to the full Lorentzian theory. It could be interesting to see how this works exactly.

Remarkably, there are two distinct solution sectors for postcarrollian 2d dilaton gravity discussed in section 4, one of which leads to Carroll black hole solutions while the other yields cosmological solutions. Since this is qualitatively different from the leading order Carrollian theory, it could be rewarding to understand physically better why the postcarrollian corrections introduce this dichotomy.

Studies of asymptotic symmetries of 2d, 3d, and 4d Carroll gravity show that non-trivial Drinfeld–Sokolov reductions exist [51, 72, 97–99], which provides encouragement to search for such symmetries in postcarrollian gravity. In section 5 we found the twisted warped boundary action for the postcarrollian JT model and a more mundane boundary action for the postcarrollian CGHS model. It could be worthwhile to investigate alternatives to the boundary constraints we imposed.

Another open task is to holographically renormalize the postcarrollian action (3.9). We suspect that this can be done by analogy to the Carrollian case [100] and the Lorentzian case [101].

In 3d and 4d, so far we only wrote down the models but did not investigate the solution space apart from the spherically symmetric sector, which is effectively again a postcarrollian 2d dilaton gravity theory. While finding solutions as general as in 2d could turn out to be as difficult as in fully fledged Einstein gravity, one could take a first step away from spherical symmetry and consider axisymmetric configurations.

An explicit example for studying the postcarrollian corrections to a 3d Carroll geometry would be to take the recently found solution of the 3d AdS Carroll theory [99]. It already solves the leading order equations (6.14), so it remains to figure out how the postcarrollian corrections can be solved for by (6.15). It is interesting that the wedge subalgebra of the asymptotic symmetry algebra found in [99] is given by the subalgebra of postcarrollian symmetry (6.3) spanned by the generators $\{H, P_a, J, C_a, S, M_a\}$.

Finally, the boundary conditions imposed in section 6.1.2 are just one example of Brown–Henneaux-inspired boundary conditions in postcarrollian gravity. They led to the asymptotic symmetry algebra (6.48), which features two independent central extensions and has the postcarrollian algebra (6.3) as wedge subalgebra. It would be gratifying to analyze more comprehensively the menagerie of possible boundary conditions, their associated asymptotic symmetries, and their central extensions, some of which may engender cardyological microstate counting of postcarrollian black hole states.

Acknowledgements

A. Baker–Campbell–Hausdorff identities

For the calculations in this work, we found three Baker–Campbell–Hausdorff identities useful. The first one,

$$[X, Y] = \sigma Y \qquad \Rightarrow \qquad e^X e^Y = e^{e^\sigma Y} e^X \tag{A.1}$$

is a well-known braiding identity and implies, for instance,

$$e^{p P} e^{y L_+} = e^{e^{-p} y L_+} e^{p P} \tag{A.2}$$

for the commutator $[L_+, P] = L_+$.

The second one,

$$[X, Y] = Z \qquad [Z, \cdot] = 0 \qquad \Rightarrow \qquad e^X e^Y = e^Y e^X e^Z \tag{A.3}$$

is well known from the Heisenberg algebra and implies, for instance,

$$e^{y L_+} e^{y_- L_-} = e^{y_- L_-} e^{y L_+} e^{2yy_- M} \tag{A.4}$$

for the commutator $[L_+, L_-] = 2M$.

The third one applies to the postcarrollian CGHS commutation relations

$$[B, P] = H \qquad [B, H] = M \qquad [H, P] = Z \qquad [M, \cdot] = [Z, \cdot] = 0 \tag{A.5}$$

and yields the identity

$$e^{-b B} P e^{b B} = P - b H + \frac{b^2}{2} M . \tag{A.6}$$

Moreover, applying (A.3) yields

$$e^{-p P} H e^{p P} = H + p Z \qquad\qquad e^{-b B} H e^{b B} = H - b M . \tag{A.7}$$

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