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Field line separation in 3D magnetostatic turbulence

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Studying magnetic field line separation, i.e. how initially close field lines spread from each other, gives important insights on the chaotic behaviour of a magnetic turbulence. This behavior has noticeable consequences, especially in the context of cosmic ray transport in the case of low energetic particles (small Larmor radii) around acceleration regions such as supernovae remnants (SNR). This separation depends on the distance along the field lines and the initial distance between neighbor field lines. In this proceeding, we run 3D synthetic turbulence simulations to characterize field line separation. We then try to account for the observed regimes with a theoretical model based on Corrsin's approximation. The model tends to reproduce the ballistic regime as well as the early stage of a super-ballistic regime.

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1. Introduction

Cosmic ray (CR) transport through the galaxy's turbulent magnetic field is still a major research topic. These charged particles, depending on their energy, are more or less sensitive to the electromagnetic fields. In particular particles with characteristic Larmor radii r_L much smaller than the turbulence coherence length L_c will tend to follow magnetic field lines. In a turbulent magnetic field, the field lines trajectories become chaotic for displacements beyond L_c; a behaviour commonly named Field Line Random Walk (FLRW). While pitch angle scattering dominates the transport of CR in the parallel direction with respect to the background field B_0, it was quite early understood that particle transport in the perpendicular direction is affected by FLRW [1] . A comprehensive microphysics characterization of both parallel and perpendicular transport in the wide range of astrophysical environments is still to be performed to better understand CR transport and acceleration. Most of the studies so far have focused on particle transport in the diffusive regime, i.e. when particles travel distances much larger than L_c. Few others were studying the transport of CR and its consequences at scales smaller than L_c [2–5]. This is of importance for particles with Larmor radii such that $r_L/L_c \ll 1$ and, as we will exemplify, could have important consequences on CR phenomenology, notably CR acceleration. In the interstellar medium, GeV to multi-TeV particles fulfill this condition To characterize this small scale behaviour we need to study field line displacements relative to the local magnetic field, or equivalently, field line separation.

As we follow field lines along their arc-length s, they separate very quickly through the inertial range of turbulence until they reach the diffusive regime. This phenomenon was firstly observed and analysed by Richardson [6] (also named *super-diffusion*) in hydrodynamic flows by following the progressive separation of two initially close fluid particles. One expects to find an analogous process for magnetic field lines: if the initial separation distance is small enough with respect to the coherence length, and large enough compared to dissipation scales, field lines in the inertial range lose the information of their initial separation distance and separate super-ballistically. In the case of Richardson super-diffusion, an Eulerian approach gives a particular scaling for this separation l as function of time as: $\langle l^2 \rangle \propto t^3$.
The consequences of this super-diffusion were studied in the context of non-ideal magnetohydrodynamics (MHD), and it was shown to lead to the so-called *stochastic flux-freezing* which explain fast reconnection rate in a non-resistive media [7–9].

In the context of CR transport, the particles may experience transverse super-diffusion, with respect to the local field, for displacements shorter than L_c. This could have some impacts on their acceleration at perpendicular shocks [2, 3]. Two regimes were identified, which depends on the behavior of cosmic rays under L_c. If CR gyrocenters have ballistic trajectories, they strictly follow magnetic field lines which are behaving super-diffusively. Thus CR experience the same super-diffusion with transverse displacement $\langle x^2 \rangle \propto t^3$. On the other hand, if CR diffuse along these field lines, super diffusion is less efficient $\langle x^2 \rangle \propto t^{3/2}$. The latter case applies, for example, if turbulence is relatively local around a shock. This behavior can enhance CR acceleration through first order Fermi acceleration because it allows an optimal balance between fast injection and sufficient confinement. However, in the first case, when CR gyrocenter trajectories are still ballistic, CR may strongly deviate from the shock, reducing the acceleration efficiency.

Some attempts to evidence these transport regimes were performed in magnetohydrodynamics

(MHD) simulations using very low energy particles [4]. The exploitation of MHD cubes shows that in order to get recovered the Richardson regime needs large dynamics in wavenumber in the inertial range. A method inherently limited by grid resolution. For these reasons we wanted to see if we can characterize field line separation in a clean setup: integrating field lines in synthetic turbulence simulations. Note that in the magnetostatic case studied here, the arc-length will be the analogue of time for fluid particles.

2. Numerical study of field line separation

The turbulent interstellar medium induces magnetic field line diffusion. However, there is a distinction between magnetic field diffusion and fluid particle diffusion in an hydrodynamic context. In a turbulent environment, stochastic perturbations cause the particle to diffuse after a certain time when its initial conditions memory is lost. This is the same behavior for magnetic field lines, but particle diffusion depends on space and time, as the particle evolves, its environment evolves with time too. For magnetic field lines, we do not study their transport strictly speaking, but we will refer to it for simplicity. In fact, from a certain magnetic configuration where turbulence is fully developed (including a fully developed inertial range), we select some snapshots of magnetic field lines in a *magnetostatic* state. From these snapshots, we reconstruct field lines "trajectories" by solving field lines equations and integrating them along s. These equations are:

$$\frac{d\mathbf{r}}{ds} = \frac{\mathbf{B}}{|\mathbf{B}|}, \quad \text{with} \quad \mathbf{r} = (x, y, z) \ , \tag{1}$$

and where $\mathbf{B}$ is the local total magnetic field. In our study the magnetic field is splitted into a perturbation $\delta\mathbf{B}$ and a background field $\mathbf{B}_0$, such as $\mathbf{B} = \delta\mathbf{B} + \mathbf{B}_0$. In order to characterize the diffusion, we will be interested in computing the diffusion coefficient transverse to the background field as:

$$\mathcal{D}_\perp(s) = \frac{d}{ds}\left(\langle (x - x_0^2)\rangle + \langle (y - y_0^2)\rangle\right) \text{ or } D_\perp(\tau) = \frac{d}{d\tau}\left(\langle (x - x_0^2)\rangle + \langle (y - y_0^2)\rangle\right) \ , \tag{2}$$

where (x_0, y_0) is the initial position, and $d\tau = ds/|\mathbf{B}|$, a suitable parametrization for analytical interpretation [10]. The symbol $\langle ... \rangle$ stands for the ensemble average over a large number of field lines. Initial positions of the field lines are taken randomly and homogeneously in space. As the turbulent magnetic field is statistically homogeneous, it allows to consider that space is ergodic at greater scales than the turbulent injection scale L. By choosing random initial points distributed in space with spacing larger than L, we ensure to probe different spatial realisations, ensuring robustness of computed means.

We study magnetic field line transport in synthetic magnetic turbulence realisations, with perturbation amplitude δB and background field B_0. These are generated by a superposition of plane waves following :

$$\boldsymbol{\delta}\mathbf{B}(\mathbf{r}) = \sum_{n=1}^{N_k} A_n \hat{\boldsymbol{\xi}}_n \cos[k_n \hat{\mathbf{k}_n}.\mathbf{r} + \beta_n] \ . \tag{3}$$

Details on this method can be found in [11], [12]. This formula corresponds to a superposition of N_k independent waves travelling in the directions $\hat{\mathbf{k}}_n$ with amplitudes $A_n(k)$, the polarization vector is $\hat{\boldsymbol{\xi}}_n$, wave numbers and phases are k_n and β_n respectively. We choose the amplitudes $A_n(k)$ so that the power spectrum is Kolmogorov like. A few parameters control the field lines behaviour. First, the ratio $\delta B/B_0$ quantifies the turbulence level. For low turbulence level, the mean magnetic field dominates the behavior of field lines transport which tends to stretch along this mean field. On the other hand, for strong turbulence levels ($\delta B/B_0 \geq 1$), field lines have more excursions in the transverse plane thus get stronger diffusion. To ensure that the field lines probe multiple scales within the inertial range, the ratio of the maximal to minimal scales $\Delta_L = L_{max}/L_{min}$ or k_{min}/k_{max}, has to be sufficiently large. As L_{min} decreases, smaller magnetic perturbations on smaller scales are captured which can be of interest especially for studying field line separation. The last parameter is the number of modes N_k. It controls, in combination with Δ_L, the density of modes, and therefore the accuracy of field line reconstruction.

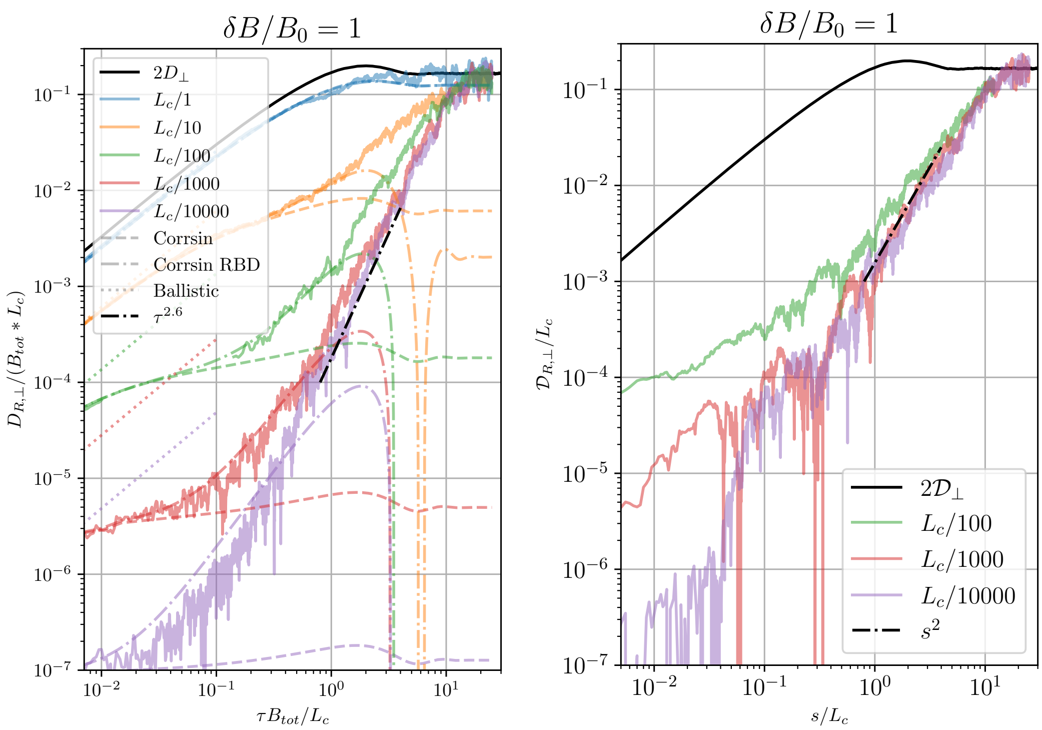

Figure 1: Field lines separation transverse running diffusion coefficients in $\delta B/B_0 = 1$ synthetic turbulence. **Left panel**: Diffusion coefficient in the τ variable. The different colored lines correspond to different initial separation R_0. Dashed lines highlight the initial ballistic regimes which can be predicted from Eq. 16. Dashed-dotted black lines display slopes of the superballistic regime. The dotted black line shows $2D_\perp$ (cf Eq. 2) in the same simulation. **Right panel**: Same figure as the left pannel but with the diffusion coefficient in the s variable.

To study field lines separation we consider two field lines initially close to each other with initial coordinates $\mathbf{r}_1(0)$ and $\mathbf{r}_2(0)$. Their initial coordinates are on the same transverse plane. The separation distance between field lines is $\mathbf{R} = \Delta\mathbf{R} - \mathbf{R}_0 = \mathbf{r}_1(\tau) - \mathbf{r}_2(\tau)$. The associated transverse and parallel diffusion coefficient are defined as follows:

$$D_{R,\perp} = \frac{1}{4}\frac{d}{d\tau}\left(\langle(\Delta x_1(\tau) - \Delta x_2(\tau))^2\rangle + \langle(\Delta y_1(\tau) - \Delta y_2(\tau))^2\rangle\right) \tag{4}$$
$$= \frac{1}{2}\left(\langle\left(B_{1,x}(\tau) - B_{2,x}(\tau)\right)\left(\Delta x_1(\tau) - \Delta x_2(\tau)\right) + \left(B_{1,y}(\tau) - B_{2,y}(\tau)\right)\left(\Delta y_1(\tau) - \Delta y_2(\tau)\right)\rangle\right) ,$$
$$D_{R,\parallel} = \frac{1}{2}\frac{d}{d\tau}\left(\langle(\Delta z_1(\tau) - \Delta z_2(\tau))^2\rangle\right) \tag{5}$$
$$= \langle\left(B_{1,z}(\tau) - B_{2,z}(\tau)\right)\left(\Delta z_1(\tau) - \Delta z_2(\tau)\right)\rangle ,$$

In Fig. 1 we display field line separation transverse diffusion coefficients (Eq. 4) computed in harmonic simulations with $\delta B/B_0 = 1$. The different colors show different initial separation distances R_0. When the initial separation is of the order or larger than L_c, field lines are initially uncorrelated and the diffusion coefficient $D_{R,\perp}$ approaches to $2\,D_\perp$. Indeed, the cross-terms $\langle\Delta x_2(\tau)\Delta x_2(\tau)\rangle$ are close to 0 in Eq. 4. In this case the running diffusion coefficient shows at low τ values a ballistic regime which flattens out at large τ to the asymptotic diffusive regime. On the other hand, for initial separation R_0 smaller that L_c, $D_{R,\perp}$ is initially suppressed for low τ values: the field lines being initially correlated, $\mathbf{B}_1 \approx \mathbf{B}_2$ in Eq. 4. A ballistic regime is followed by a super-ballistic regime, during which field lines separate very fast, with $D_{R,\perp} \propto \tau^\alpha$, with $\alpha > 1$. In this process the modes driving the separation at scale R have wave number $k \sim 1/R$. The smaller R_0, the larger the slope α, up to an asymptotic value that we numerically found to be around 2.4. The super-ballistic regime eventually flattens out to the same plateau as $2\,D_\perp$, at about the same τ value $\approx 10L_c/B_{\text{tot}}$. We observe a similar regime for $\mathcal{D}_{\perp,R}$ the diffusion coefficient studied as a function of the s variable. In that case the asymptotic value of the super-ballistic regime is close to 2, characterising the so-called Richardson regime with $\langle\Delta R^2(s)\rangle \propto s^3$, or $D_{\perp,R} \propto s^2$.

In the following, we try to model the behaviour of the running diffusion coefficient $D_{R,\perp}$ with respect to τ using the Corrsin's hypothesis.

3. The Corrsin's approach

The Corrsin's hypothesis [13], is often used in quasi-linear theories, and has shown to successfully reproduce the scaling and normalisation of the diffusion coefficients of particle and field line in specific regimes [10], [14], [15], or [16].

The form of the diffusion coefficient Eq. 4, requires to study Lagrangian correlators of two types : $\langle B_{1,x}(\mathbf{0})B_{1,x}(\mathbf{r}(\Delta\tau))\rangle_L$ and $\langle B_{1,x}(\mathbf{0})B_{2,x}(\mathbf{r}(\Delta\tau))\rangle_L$. As a theory of Lagrangian magnetic field correlator (or speed correlator in fluid particle tracking) is lacking, we use the Corrsin's hypothesis to relate the Lagrangian correlators $\langle\rangle_L$ to the Eulerian ones $\langle\rangle_E$ as follow:

$$\langle B_{1,x}(\mathbf{0})B_{1,x}(\mathbf{r}(\Delta\tau))\rangle_L = \int \langle B_{1,x}(\mathbf{r})B_{1,x}(\mathbf{r}+\Delta\mathbf{r})\rangle_E \mathcal{P}(\Delta\mathbf{r}|\Delta\tau)\Delta\mathbf{r} \tag{6}$$

where $\Delta\mathbf{r}(\tau) = \mathbf{r} - \mathbf{r}_0 = (x - x_0, y - y_0, z - z_0)$, and $\mathcal{P}(\Delta\mathbf{r}|\Delta\tau)$ is the probability to find a displacement $\Delta\mathbf{r}$ at a given $\Delta\tau$. This probability is assumed to follow a 3D gaussian along each (x, y, z) axes, with means $(0, 0, B_0\tau)$ and variances $(\sigma_x, \sigma_y, \sigma_z)$. For axi-symmetric turbulence, as we consider here, $\sigma_x = \sigma_y$. In the same way, we use the Corrsin's hypothesis to approximate the correlator between two field lines:

$$\langle B_{1,x}(\mathbf{0})B_{2,x}(\mathbf{r}(\Delta\tau))\rangle_L = \int \langle B_{1,x}(\mathbf{r})B_{1,x}(\mathbf{r}+\mathbf{R})\rangle_E \mathcal{P}(\mathbf{R}|\Delta\tau)d\mathbf{R} \tag{7}$$

where $\mathbf{R}(\tau) = \mathbf{r}_1(\tau) - \mathbf{r}_2(\tau) = \mathbf{\Delta R} + \mathbf{R}_0$, and $\mathcal{P}$ is assumed to follow a 3D Gaussian with means $(0, 0, B_0\tau)$ and variances $(\sigma_X, \sigma_Y, \sigma_Z)$, with again for axisymmetric turbulence, $\sigma_X = \sigma_Y$. Following the computations of [15] but in the τ variable, we end up with the following set of ODE :

$$\begin{aligned}
\frac{d(D_{R,\perp})}{d\tau} &= \int Ng(k)\frac{1}{2}(\chi^2+1)e^{-k_\perp^2\sigma_x^2/2}e^{-k_\parallel^2\sigma_z^2/2}\cos(k_zB_0\Delta\tau) \\
&\quad\times\left[1 - e^{-k_\perp^2\sigma_{R,x}^2/2}e^{-k_\parallel^2\sigma_{R,z}^2/2}\cos(k_x\mu_X + k_y\mu_Y + k_xX_0 + k_yY_0)\right]d\mathbf{k} &(8)\\
\frac{d(D_{R,\parallel})}{d\tau} &= 2\int Ng(k)\frac{1}{2}(1-\chi^2)e^{-k_\perp^2\sigma_x^2/2}e^{-k_\parallel^2\sigma_z^2/2}\cos(k_zB_0\Delta\tau) \\
&\quad\times\left[1 - e^{-k_\perp^2\sigma_{R,x}^2/2}e^{-k_\parallel^2\sigma_{R,z}^2/2}\cos(k_x\mu_X + k_y\mu_Y + k_xX_0 + k_yY_0)\right]d\mathbf{k} &(9)\\
\frac{d\sigma_{R,x}^2}{d\tau} &= 2D_{R,\perp} &(10)\\
\frac{d\sigma_{R,z}^2}{d\tau} &= 2D_{R,\parallel} &(11)
\end{aligned}$$

where $D_{R,\perp} = \frac{1}{2}(\langle(x_1(\tau) - x_2(\tau))^2\rangle + \langle(y_1(\tau) - y_2(\tau))^2\rangle)$ and $g(k) = (k/k_0)^{-5/3}$ for Kolmogorov turbulence. We also have to add the one field line diffusion coefficients to relate σ_x^2 and σ_z^2:

$$\begin{aligned}
\frac{d(D_\perp)}{d\tau} &= \frac{1}{2}\int Ng(k)\frac{1}{2}(1+\chi^2)e^{-k_\perp^2\sigma_x^2/2}e^{-k_\parallel^2\sigma_y^2/2}\cos(k_zB_0\Delta\tau)k^3d(\ln(k))d\phi d\chi &(12)\\
\frac{d(D_\parallel)}{d\tau} &= \int Ng(k)\frac{1}{2}(1-\chi^2)e^{-k_\perp^2\sigma_x^2/2}e^{-k_\parallel^2\sigma_y^2/2}\cos(k_zB_0\Delta\tau)k^3d(\ln(k))d\phi d\chi &(13)\\
\frac{d\sigma_x^2}{d\tau} &= 2D_\perp &(14)\\
\frac{d\sigma_z^2}{d\tau} &= 2D_\parallel &(15)
\end{aligned}$$

Note that, at variance with [15], we are not using any closure hypothesis (such as the random ballistic decorellation (RBD) or the diffusive decorrelation (DD) hypothesis), and solve this system numerically with a RK4 scheme.

The results of the numerical integration of the ODE system are shown with dashed lines in the left panel of Fig 1. The different colors correspond to the same initial separations R_0 as the simulations. The model reproduces the ballistic regime but eventually converges at $\tau B_{tot}/l_c \sim 10^{-2}$ and fails to reproduce the following superdiffusive and diffusive regimes of field line separation. In numerical solving, the issue stems from the correlator associated to the separation (Eq. 7) which does not vanish at large τ values. In order to make it vanish, one can accelerate the growth of σ_X^2 by using RBD closure; the result is shown in a dash-dotted line. The transition to the super-ballistic regime is well reproduced but it also eventually drops around $\tau B_{tot}/L_c \sim 2$. For completeness we also display the ballistic regimes with dotted lines, that can be predicted by solving numerically the following equation:

$$\frac{dD_{R,\perp}}{d\tau} = 2\pi \int Ng(k)\frac{1}{2}(1+\chi^2)[1 - J_0(k\sqrt{1-\chi^2}R_0)]k^2 dk d\chi \tag{16}$$

where we supposed $\Delta\tau \to 0$, so that in first order $\cos(k_z B_0 \Delta\tau) \sim 1$, $\sigma_x^2 = \langle(x-x_0)^2\rangle \sim 0$, $\sigma_z^2 = \langle(z-z_0)^2\rangle \sim 0$, $\sigma_X^2 = \langle\Delta X^2\rangle = \langle(\Delta x_1 - \Delta x_2)^2\rangle \to 0$ and $\sigma_Z^2 = \langle\Delta Z^2\rangle \to 0$.

4. CONCLUSION

This work can be summarized with the following points:

- A characteristic super-diffusion regime of magnetic field lines in the inertial range of turbulence is observed in synthetic turbulence simulations.
- Whereas Corrsin's tends to be a good approximation for modeling field line diffusion, it fails to model field lines separation. Using the RBD closure, the model manages to predict the onset of the super-diffusive regime. One of the open questions is whether a specific closure can perform better.

If successful, such a model of magnetic field-line separation could be used to improve the modelling of cosmic-ray transport around SNR shocks.

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