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Data-driven methods for the hadronic contributions to the muon $g - 2$: status and perspectives

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After the final release of results from the Fermilab Muon $g - 2$ experiment, the experimental world average has reached a precision of 124 parts per billion, allowing for a Standard-Model precision test below 0.2 parts per million if the theory prediction could be evaluated at a similar level of precision as the experimental result. At present, this is not the case, diluting the sensitivity to potential beyond-the-Standard-Model degrees of freedom. Apart from improved lattice-QCD calculations, a number of ongoing efforts using data-driven methods aim at improving the current puzzling situation, to ideally arrive at two independent determinations of the critical hadronic contributions and thereby increase confidence in the resulting Standard-Model prediction. Here, we present a summary of the current status, both for hadronic vacuum polarization and hadronic light-by-light scattering, discuss the most pressing challenges, and give an overview over the various avenues for future improvements.

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1. Introduction

Ever since Schwinger's seminal prediction $a_\ell = (g_\ell - 2)/2 = \alpha/(2\pi) \simeq 1.16 \times 10^{-3}$ [1] for the leading-order (LO) QED correction to the anomalous magnetic moment of leptons and its subsequent confirmation in experiment [2] have a_ℓ, $\ell \in \{e, \mu\}$, been used to perform increasingly stringent precision tests of the Standard Model (SM). Experimentally, a precision of

$$\begin{aligned} a_e^{\text{exp}} &= 115{,}965{,}218{,}059(13) \times 10^{-14} && [112\,\text{ppt}] && \text{Northwestern 2023 [3]},\\ a_\mu^{\text{exp}} &= 116{,}592{,}071.5(14.5) \times 10^{-11} && [124\,\text{ppb}] && \text{Fermilab 2025 [4–11]}, \end{aligned} \tag{1}$$

has been achieved, setting the goal for the respective SM prediction to fully leverage the experimental sensitivity in the search for physics beyond the SM (BSM). For a wide range of scenarios, potential BSM effects scale with the square of the lepton mass, $m_\mu^2/m_e^2 \simeq 43{,}000$, by which measure a_μ^{exp} would be more sensitive by a factor of 40, but, importantly, the interplay between a_μ and a_e allows one to test as well the flavor and chirality structure of potential BSM contributions [12, 13].[1] Despite the precision reported in Eq. (1), at present BSM effects are only tested at $|a_e^{\text{BSM}}| \lesssim 1.0 \times 10^{-12}$ and $|a_\mu^{\text{BSM}}| \lesssim 63 \times 10^{-11}$, diluting the experimental precision by a factor eight and four, respectively. In the case of a_e, this is due to a 5.4σ tension in the fine-structure constant α [21, 22], while the SM prediction [23–30] would allow one to reach a precision of 3×10^{-14}, limited by hadronic vacuum polarization (HVP), the very same effect that limits the precision of the SM prediction for a_μ [3, 21, 22, 24–28, 31–88]. After a short review of the experimental result, Sec. 2, here we focus on the SM prediction, decomposed according to

$$a_\ell^{\text{SM}} = a_\ell^{\text{QED}} + a_\ell^{\text{EW}} + a_\ell^{\text{had}}, \qquad a_\ell^{\text{had}} = a_\ell^{\text{HVP}} + a_\ell^{\text{HLbL}}, \tag{2}$$

into QED, electroweak (EW), and hadronic contributions, the latter further separated into HVP and hadronic light-by-light (HLbL) scattering. QED and EW contributions are addressed in Sec. 3, the status of HLbL scattering and HVP is reviewed in Secs. 4 and 5, respectively, before concluding in Sec. 6. The presentation draws from the reviews [89–91], focusing on data-driven methods, while the lattice-QCD perspective is covered in Refs. [92, 93].

2. Measurement of a_μ

The lifetime of the muon, $\tau_\mu \simeq 2.2\,\mu$s, allows one to observe muons in a storage ring, in which their anomalous precession frequency $\boldsymbol{\omega}_a$, defined as the difference of spin precession frequency $\boldsymbol{\omega}_s$ and cyclotron frequency $\boldsymbol{\omega}_c$, follows the Bargmann–Michel–Telegdi equation [94]

$$\boldsymbol{\omega}_a = -\frac{q}{m_\mu}\left[a_\mu \mathbf{B} - a_\mu\left(\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\boldsymbol{\beta}\times\mathbf{E}\right], \tag{3}$$

where $\mathbf{B}$, $\mathbf{E}$ denote the magnetic and electric field, $\boldsymbol{\beta}$ the muon velocity, and $\gamma = 1/\sqrt{1-\boldsymbol{\beta}^2}$ the relativistic γ factor. All modern experiments since CERN-III [95] have exploited the fact that the

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[1] For the same reason, it would be very interesting to obtain competitive constraints on a_τ, but reaching a precision below 10^{-5}, at which point meaningful BSM constraints start to arise, is challenging, one possible avenue relying on asymmetry measurements in $e^+e^- \to \tau^+\tau^-$ with a polarized electron beam at a future SuperKEKB upgrade [14–20].

E-field correction vanishes at the "magic momentum" $\gamma = \sqrt{1+1/a_\mu} \simeq 29.3$, corresponding to a muon momentum $P_0 = 3.094\,\text{GeV}$, $\gamma\tau_\mu \simeq 64.4\,\mu\text{s}$. In addition, a pitch correction needs to be applied because of muon motion with non-vanishing projection onto **B**, and the **B** field is constantly monitored by nuclear-magnetic-resonance probes, supplemented by dedicated trolley runs to map the exact field encountered by the muon beam. In this way, ω_a is determined from a fit to the muon decay distribution ("wiggle plot"), while the **B** field is replaced by a reference frequency $\tilde{\omega}'_p$. The ratio $\mathcal{R}'_\mu = \omega_a/\tilde{\omega}'_p$ is reported by the experiment, to be converted to a_μ by a set of external constants such as m_μ/m_e [96]. Together, the sufficiently long lifetime of the muon, the existence of a magic momentum, the fact that $\omega_a \propto a_\mu$, that the muon is its own polarimeter, and the possibility to access the magnetic field via protons as co-magnetometers constitute five "miracles of nature" that make a high-precision measurement of a_μ possible, see Ref. [91] for a more detailed description of each of them, as well as their realization in the Fermilab E989 experiment. Their final result [4] by far dominates the world average given in Eq. (1), obtained by averaging with the final result from the Brookhaven E821 experiment [11].

In the future, the J-PARC E34 experiment [97] aims at a measurement using a different technique based on re-accelerated ultracold muons, see Ref. [98] for a comparison to the magic-momentum approach. Essentially, no **E** field is required to contain the muons and the transverse momentum becomes negligible, eliminating the two correction terms in Eq. (3), at the expense of much smaller γ, reduced lifetime, and thus the need for higher statistics. As important milestone the acceleration to 100 keV was demonstrated in Ref. [99], and strategies are under investigation to match the precision achieved in the Fermilab experiment. Finally, **E** and **B** fields can be tuned in such a way that $\boldsymbol{\omega}_a = \mathbf{0}$, and this "frozen-spin technique" will be used for a dedicated muon EDM experiment at PSI [100, 101].

3. QED and electroweak contributions

The QED contribution is known up to five-loop order in the fine-structure constant α, in particular, the mass-independent terms up to four-loop order are known in semi-analytic form [1, 23, 102–104], and a tension in the numerical calculation of the mass-independent five-loop coefficient has been resolved recently [24–26]. For a_e, the main uncertainty derives, at present, from the input for α

$$\begin{aligned}
a_e^{\text{SM}}[\text{Cs}] &= 115{,}965{,}218{,}159.0(23.0)_\alpha(0.4)_{\text{5-loop}}(0.4)_{\text{6-loop}}(3.0)_{\text{HVP}} \times 10^{-14},\\
a_e^{\text{SM}}[\text{Rb}] &= 115{,}965{,}218{,}023.8(8.2)_\alpha(0.4)_{\text{5-loop}}(0.4)_{\text{6-loop}}(3.0)_{\text{HVP}} \times 10^{-14},
\end{aligned} \tag{4}$$

taken from atom interferometry using Cs [21] or Rb [22] atoms. Uncertainties from higher orders in QED are below 10^{-14}, in such a way that currently the largest theory uncertainty arises from the HVP contribution, evaluated as in Ref. [90]. The discrepancy between the two measurements is illustrated in Fig. 1 directly at the level of α, together with the value of α extracted from a_e assuming SM theory. There are a number of ongoing developments to resolve this tension and further improve the precision [105], including another order of magnitude for the measurement of a_e. At this level, $\Delta a_e^{\text{exp}} \simeq 1\times 10^{-14}$, the current HVP uncertainty would become the limiting factor.

For a_μ^{QED}, enhancements beyond the perturbative expansion in α can occur due to logarithmically enhanced electron loops (as well as factors of π^2 from electron light-by-light topologies), and

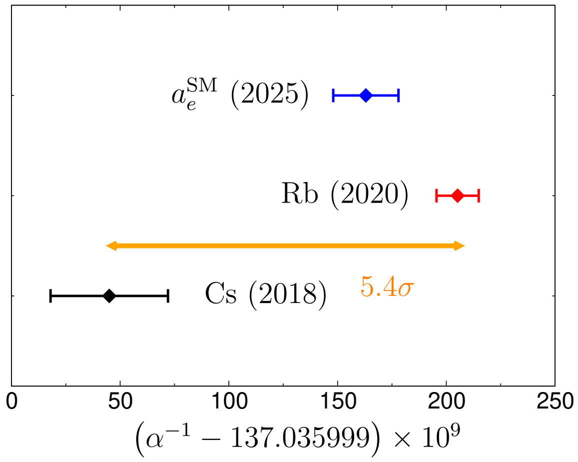

Figure 1: α extracted from a_e assuming SM theory, in comparison to direct atom-interferometry measurements using Cs [21] and Rb [22] atoms.

propagating the maximal such enhancement at six-loop order generates a similar uncertainty as the current Cs vs. Rb tension in α:

$$a_\mu^{\rm QED} = 116{,}584{,}718.8(1)_\alpha(1)_{\text{6-loop}}[2]_{\rm total} \times 10^{-11}. \tag{5}$$

The total QED uncertainty is well below the target (1).

For the EW contribution, the one-loop result reads [106–110]

$$a_\mu^{\rm EW}[\text{1-loop}] = \frac{G_F}{\sqrt{2}}\frac{m_\mu^2}{8\pi^2}\left[\frac{5}{3}+\frac{1}{3}(1-4s_{\rm W}^2)^2+\mathcal{O}\left(\frac{m_\mu^2}{M_W^2},\frac{m_\mu^2}{M_H^2}\right)\right] = 194.79(1)\times 10^{-11}, \tag{6}$$

with the Fermi constant G_F as defined in muon decay [111] and the weak mixing angle in the on-shell scheme. Two-loop effects lead to a sizable reduction [35]

$$a_\mu^{\rm EW} = 154.4(3)_{\rm had}(2)_{\rm NLL}[4]_{\rm total} \times 10^{-11}, \tag{7}$$

where the first uncertainty refers to non-perturbative effects in the $\gamma^*\gamma^*Z$ matrix elements [32, 34, 35, 112, 113] and the second one to an estimate of three-loop corrections [32, 114]. Again, the final uncertainty is well below the target (1).

4. Data-driven methods for hadronic light-by-light scattering

The basic idea of data-driven evaluations amounts to reconstructing the HLbL tensor from its singularities, which correspond to $\gamma^*\gamma^* \to$ hadrons matrix elements and are therefore accessible in experiment. In a dispersive approach [54, 57, 115–117], this reconstruction proceeds for a given intermediate state at a time, e.g., pseudoscalar poles require transition form factors (TFFs) as input, two-meson intermediate states the corresponding helicity partial-wave amplitudes, and higher multiplicities are most efficiently described by (narrow) resonances, necessitating axial-vector and tensor TFFs as input. For a number of intermediate states again dedicated programs have been carried out to determine the full virtuality dependence of the matrix elements, e.g.,

Region		Dispersive	hQCD	Regge	DSE/BSE
$Q_i > Q_0$		$6.2^{+0.2}_{-0.3}$	6.3(7)	4.8(1)	2.3(1.5)
Mixed	A, S, T	3.8(1.5)			
	OPE	10.9(0.8)			
	Effective pole	1.2			
	Sum	15.9(1.7)	13.5(2.4)	12.8(5)	10.1(3.0)
$Q_i < Q_0$	$A = f_1, f_1', a_1$	12.2(4.3)	13.1(1.5)	10.9(1.0)	8.6(2.6)
	$S = f_0(1370), a_0(1450)$	−0.7(4)			−0.8(3)
	$T = f_2, a_2$	−2.5(8)	2.9(4)		
	Other	2.0	8.0(9)	3.2(6)	2.8(6)
	Sum	11.0(4.4)	24.0(2.8)	14.1(1.2)	10.6(2.7)
Sum		33.2(4.7)	43.8(5.9)	31.7(1.6)	23.0(7.4)

Table 1: Contributions to the different momentum regions in a dispersive approach, holographic QCD (hQCD), Regge theory, and Dyson–Schwinger/Bethe–Salpeter equations (DSE/BSE), in units of 10^{-11}. Table taken from Ref. [90].

for the π^0 TFF [58, 59, 118–122], the η, η' TFF [79, 80, 123–128], and $\gamma^*\gamma^* \to \pi\pi/\bar{K}K/\pi\eta$ partial-wave amplitudes [56, 57, 67, 74, 129–136]. For axial-vector TFFs, less experimental information is available, mostly for the $f_1(1285)$, in which case a global analysis of the three TFFs is possible [71, 137], relying on data for $e^+e^- \to e^+e^-A$, $A = f_1, f_1'$ [138, 139], $e^+e^- \to f_1\pi^+\pi^-$ [140, 141], and radiative decays [71, 137, 142], as well as asymptotic constraints [143].

In addition, especially since in contrast to the HVP contribution there is no closed formula that accounts for all intermediate states, the matching to constraints from perturbative QCD and the operator product expansion is paramount. These short-distance constraints have been studied in detail in recent years [65, 66, 69, 76], and strategies developed to implement them in HLbL evaluations [62–64, 75, 144–147]. The combined evaluation of the critical subleading contributions from Ref. [90] is reproduced in Table 1, forming the basis for the phenomenological HLbL evaluation shown in Fig. 2. The data-driven and lattice-QCD averages are in slight tension of 1.5σ, so that the uncertainty of the overall average

$$a_\mu^{\text{HLbL}}[\text{phenomenology} + \text{lattice}] = 112.6(9.6) \times 10^{-11} \tag{8}$$

includes a scale factor $S = 1.5$. Crucially, for the HLbL contribution we are in a situation in which two independent methods can be validated against each other, increasing confidence in the robustness of the final average.

With a precision (8) barely below $\Delta a_\mu^{\text{exp}}$, it would be valuable to better understand the interplay between phenomenology and lattice QCD, to hopefully improve the precision further in the future. For the pseudoscalar TFFs such comparisons to lattice QCD [148–152] (in addition to direct comparisons to data, see, e.g., Ref. [153] for a recent measurement of the π^0 TFF) have already become common, and strategies are being developed to also compare the subleading contributions in more detail. On the data-driven side, ongoing developments include the study of a dispersive

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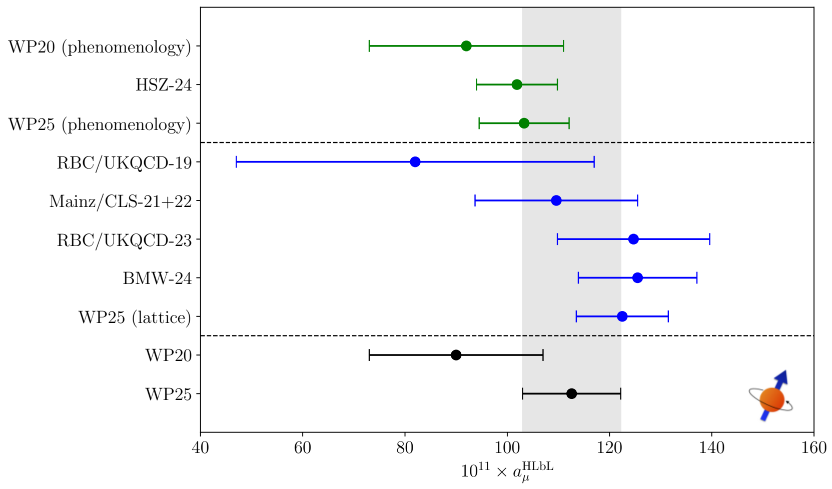

Figure 2: Summary of the current HLbL status. Figure taken from Ref. [90].

approach in triangle kinematics [154] to be able to fully address tensor contributions, as well as improved calculations of their TFFs and the interplay with $\gamma^*\gamma^* \to \pi\pi$ in the narrow-width limit.

5. Data-driven methods for hadronic vacuum polarization

5.1 $e^+e^- \to$ hadrons scattering

Traditionally, the LO HVP contribution has been inferred from $e^+e^- \to$ hadrons cross sections via the master formula [155, 156]

$$a_\mu^{\text{HVP, LO}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{s_{\text{thr}}}^{\infty} ds \frac{\hat{K}(s)}{s^2} R_{\text{had}}(s), \qquad R_{\text{had}}(s) = \frac{3s}{4\pi\alpha^2}\sigma\big[e^+e^- \to \text{hadrons}(+\gamma)\big], \tag{9}$$

where, by convention, the cross section is defined photon inclusively, thus with a threshold $s_{\text{thr}} = M_{\pi^0}^2$ due to the $e^+e^- \to \pi^0\gamma$ channel. The main challenge arises from the extraordinary precision requirements, since, with $a_\mu^{\text{HVP, LO}} \simeq 700 \times 10^{-10}$, matching the Fermilab experiment sets a target precision of 0.2%.[2] This is particularly acute for the dominant $e^+e^- \to \pi^+\pi^-$ channel, for which the current landscape is illustrated in Fig. 3, including the energy-scan experiments from SND06 [159], CMD-2 [160], SND20 [161], and CMD-3 [162, 163] and the initial-state-radiation measurements from BaBar [164], KLOE [165], and BESIII [166]. In each case, the percentage indicates the part of the phase space covered by the respective experiment, and the points are obtained from the CHKLS [167–170], DHMZ [171–173], and KNTW [27, 174, 175] methodologies. The long-standing tension between BaBar and KLOE is clearly visible, but the situation has become significantly worse after the CMD-3 result appeared. A preliminary new result from BaBar [176] almost coincides with Ref. [164], while preliminary new results from SND move closer to CMD-3 compared to Ref. [161]. In the future, major new results are expected from Belle II, BESIII, and KLOE [177].

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[2] Higher-order HVP insertions are also important, but sufficiently under control [27, 28, 53, 157, 158].

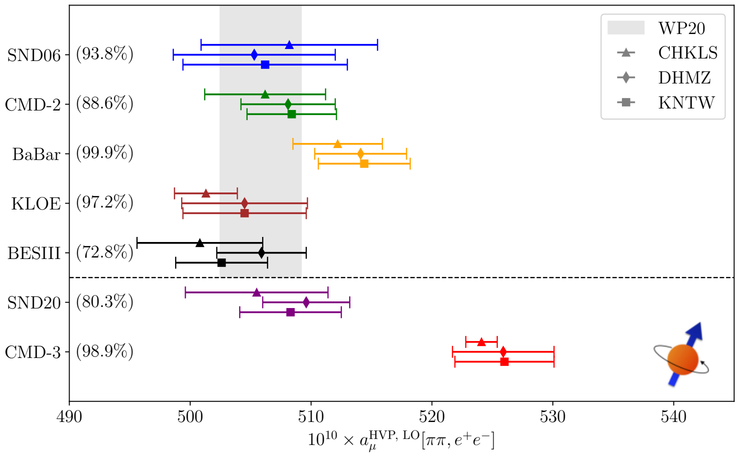

Figure 3: Summary of the current landscape of $e^+e^- \to \pi^+\pi^-$ experiments. Figure taken from Ref. [90].

The tensions in the $e^+e^- \to \pi^+\pi^-$ channel clearly constitute the main concern, but further improvements are required in other channels as well. In the $e^+e^- \to 3\pi$ channel, the Belle II result [178] disagrees with previous measurements [179, 180] by about 2σ, and is further disfavored by analyticity and unitarity constraints [181–184] (as well as the preliminary SND result [185]). In the $e^+e^- \to K^+K^-$ channel CMD-3 [186], BaBar [187], and SND [188] disagree outside the quoted errors, see Ref. [68] for a global analysis of the kaon electromagnetic form factors. Finally, the inclusive data from BESIII [189] in the range (2.2–3.7) GeV lie systematically above perturbative QCD, a 3σ effect that cannot be explained by the expected size of duality violations [190].

5.2 Radiative corrections

After the CMD-3 result appeared, the treatment of radiative corrections and their implementation in Monte-Carlo (MC) generators [191–199] have received increased attention and scrutiny, organized within the *RadioMonteCarLow 2* effort [200]. In particular, CMD-3 [162, 163] found as an ancillary measurement that the forward–backward asymmetry in $e^+e^- \to \pi^+\pi^-$ could not be described by a point-like approximation for the pion multiplied by external form factors, but that structure-dependent corrections within the loop integral became important [192], due to resonance enhancement from the $\rho(770)$. This conclusion can be put onto more solid theoretical grounds within a dispersive approach [193], confirming the conclusion that structure-dependent virtual corrections are crucial to describe the forward–backward asymmetry.

For energy scan experiments, this effect is C-odd and thus does not occur in the integrated cross section, but in the initial-state-radiation configuration there are radiative corrections that should display the same resonance enhancement and do contribute to the C-even cross section [194, 197]. This class of radiative corrections could therefore lead to sizable effects and is currently under investigation. In the meantime, a detailed comparison of MC generators was started in Ref. [197], which will further allow one to assess the precision of MC generators and their impact on the measured $e^+e^- \to \pi^+\pi^-$ cross sections.

		Refs. [173, 205]	Ref. [207]	Refs. [168, 208]	Ref. [90]
Phase space		-7.88	-7.52	–	$-7.7(2)$
S_{EW}		$-12.21(15)$	$-12.16(15)$	–	$-12.2(1.3)$
G_{EM}		$-1.92(90)$	$(-1.67)^{+0.60}_{-1.39}$	–	$-2.0(1.4)$
FSR		$4.67(47)$	$4.62(46)$	$4.42(4)$	$4.5(3)$
ρ–ω mixing		$4.0(4)$	$2.87(8)$	$3.79(19)$	$3.9(3)$
$\frac{F_\pi^V}{f_+}$ (w/o ρ–ω)	ΔM_ρ	$0.20(^{+27}_{-19})(9)$	$1.95^{+1.56}_{-1.55}$	–	
	$\Delta\Gamma_\rho(\Delta M_\pi)$	$4.09(0)(7)$	3.37	–	
	$\Delta\Gamma_\rho(\pi\pi\gamma)$	$-5.91(59)(48)$	$-6.66(73)$	–	
	$\Delta\Gamma_\rho(g_{\rho\pi\pi})$	–	–	–	
	Total	$-1.62(65)(63)$	$(-1.34)^{+1.72}_{-1.71}$	–	$-1.5(4.7)$
Sum		$-14.9(1.9)$	$(-15.20)^{+2.26}_{-2.63}$	–	$-15.0(5.1)$

Table 2: Summary of the different classes of isospin-breaking corrections to $\tau \to \pi\pi\nu_\tau$, expressed in terms of corrections to the HVP integral (in units of 10^{-10}). Table taken from Ref. [90].

5.3 Hadronic τ decays

The spectrum for hadronic τ decays, $\tau \to \pi\pi\nu_\tau$, can be expressed in the form

$$\frac{1}{K_\Gamma(s)}\frac{d\Gamma}{ds}[\tau \to \pi\pi\nu_\tau(\gamma)] = \underbrace{S_{\text{EW}}^{\pi\pi}}_{\text{short distance}} \times \underbrace{\left[\beta_{\pi\pi^0}\right]^3}_{\text{phase space}} \times \underbrace{|f_+(s)|^2}_{\langle\pi\pi^0|j_W^\mu|0\rangle} \times \underbrace{G_{\text{EM}}(s)}_{\text{radiative corrections}}, \tag{10}$$

providing an alternative access [201] to the electromagnetic form factor of the pion, $F_\pi^V(s)$, if the various sources of isospin breaking can be controlled [202–207], see Table 2 for a summary. This includes phase-space, short-distance, and long-distance radiative corrections as indicated in Eq. (10), but also radiative corrections in $e^+e^- \to \pi^+\pi^-$ (FSR and ρ–ω mixing) and, crucially, isospin breaking in the two-pion matrix element to relate $f_+(s)$ as measured in the weak decay back to $F_\pi^V(s)$. The estimates for the various corrections from Ref. [90] are collected in Table 2, showing that by far the largest uncertainty resides in the matrix element, but also S_{EW} and G_{EM} were assigned sizable uncertainties. The resulting total corrections are shown in Fig. 4.

In the meantime, an improved calculation of the long-range radiative corrections has become available [209, 210], revealing a shift of $-2.0(1.4) \to -5.4(5)$, significantly outside the previously quoted uncertainties. The origin can be traced back again to structure-dependent radiative corrections, similarly to the forward–backward asymmetry in $e^+e^- \to \pi^+\pi^-$, as well as a local contribution in the matching of chiral perturbation theory and short-distance corrections. This matching of short-distance effects with hadronic matrix elements was performed beyond leading-logarithmic accuracy in Ref. [211], using input from lattice QCD for the non-perturbative input [212, 213]. More details are provided in Ref. [214].

With these developments, the remaining obstacle concerns isospin breaking in the matrix element, which could be addressed using a combination of lattice QCD [215] and dispersive

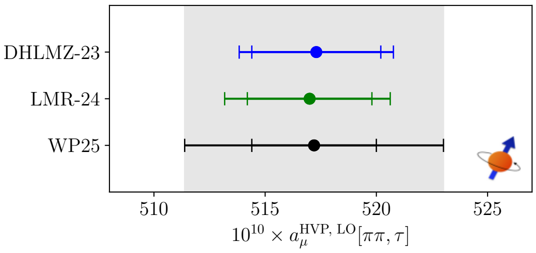

Figure 4: Summary of the $\pi\pi$ LO HVP contribution determined from τ data. Figure taken from Ref. [90].

techniques [216]. While the main limitation thus continues to arise from the theory corrections, also new data especially on the $\tau \to \pi\pi\nu_\tau$ spectrum would be extremely valuable, as could become available at Belle II [217].

5.4 Detailed comparisons to lattice QCD

Detailed comparisons to lattice QCD have been performed on the basis of Euclidean windows [36] in a number of works, including Refs. [208, 218–225]. In particular, Refs. [222, 224, 225] isolate the light-quark-connected (ud) and the strange + disconnected contributions from the hadronic cross sections, relying on G parity for the separation of the hadronic channels as well as isospin-breaking corrections, see Fig. 5 for the ud case. In this context, care has to be taken in the comparison between lattice QCD and phenomenology due to subtleties in the scheme conventions [226], as will receive renewed attention in future such comparisons once improved lattice-QCD calculations of the isospin-breaking contributions become available.

5.5 MUonE

The MUonE experiment aims at measuring the space-like HVP contribution in muon–electron scattering [227, 228], via the running of α, thereby yielding another independent data-driven determination with completely different systematics. A full proposal will be prepared during the CERN Long Shutdown 3, based on earlier test runs, including a 2025 campaign that aims at a precision of 20% (statistical and systematics each). On the theory side, the most challenging aspect concerns the development of MC generators at 10 ppm [229], as required for a competitive determination of the HVP contribution.

6. Conclusions and outlook

The Fermilab E989 experiment has measured a_μ with a precision of $\Delta a_\mu^{\text{exp}} = 14.5 \times 10^{-11}$, yet the uncertainty of the SM prediction from Ref. [90] is four times larger, $\Delta a_\mu^{\text{SM}} = 63 \times 10^{-11}$, dominated by the uncertainty in the HVP contribution. While lattice QCD has made remarkable progress in recent years, it is invaluable to have (at least) two independent methods to evaluate the hadronic contributions, to validate and potentially improve the precision. For HLbL scattering, data-driven and lattice-QCD evaluations agree at a level of $\lesssim 10 \times 10^{-11}$, just below $\Delta a_\mu^{\text{exp}}$, but further scrutiny of data-driven and lattice-QCD results is required to better understand the current slight tension and hopefully improve the precision to render the HLbL error negligible. Data-driven HVP

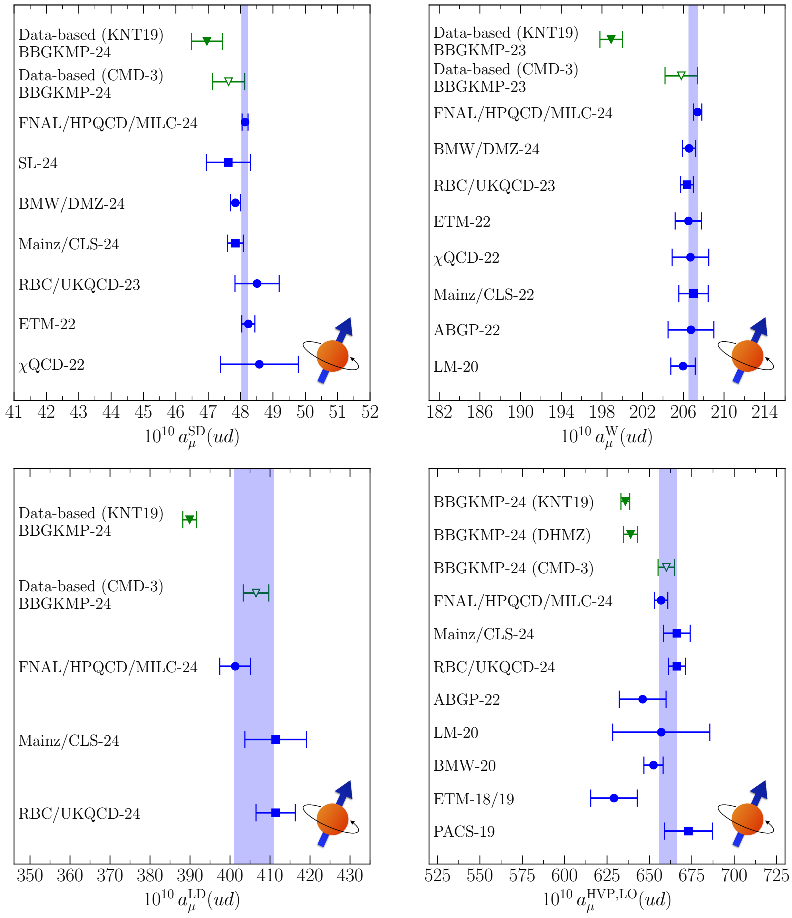

Figure 5: Comparison of the ud contributions between lattice QCD and phenomenology. Figure taken from Ref. [90].

evaluations do not paint a conclusive picture at the moment, here several avenues are being pursued to understand the current puzzles and be able to quote again a competitive data-driven evaluation in the future. This includes new data and analyses for the critical $e^+e^- \to \pi^+\pi^-$ channel, together with improved radiative corrections and MC generators. In addition, HVP evaluations based on hadronic τ decays may become possible in the future, if the the required isospin-breaking corrections can be controlled, and also new data from Belle II would be highly valuable in that regard. Finally, the MUonE project offers yet another opportunity for a data-driven HVP evaluation, with completely different systematics. All these efforts, together with improved lattice-QCD calculations, have the potential to improve the SM test by a factor of three, as soon as the precision achieved in experiment can be matched.

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