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# Critical behavior of the Schwinger model via gauge-invariant variational uniform matrix product states



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We numerically analyze the critical behaviors of the lattice Schwinger model by applying the variational uniform matrix product state (VUMPS) algorithm combined with a gauge-invariant matrix product ansatz that locally enforces the Gauss law constraint. Both the continuum and lattice versions of the Schwinger model with $\theta = \pi$ are known to exhibit first-order phase transitions for the values of the fermion mass above a critical value, where a second-order phase transition occurs. Our algorithm enables a precise determination of the critical point in the continuum theory. We further analyze the scaling in the simultaneous critical and continuum limits and confirm that the data collapse aligns with the Ising universality class with high precision.

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## 1. Introduction

The Schwinger model—$(1+1)$-dimensional QED—is a paradigmatic gauge theory that shares with QCD several important features such as confinement and the chiral anomaly [1, 2]. With a topological $\theta$ term, Euclidean Monte Carlo simulations suffer from a sign problem, which makes Hamiltonian-based approaches particularly attractive. In 1 + 1 dimensions, matrix product state (MPS) methods provide a highly efficient description of low-energy states, and have been widely applied to the Schwinger model and related gauge theories (see e.g. [3–5]).

At $\theta = \pi$, the massive Schwinger model exhibits a first-order phase transition at large $m/g$ (mass $m$ divided by the coupling $g$ to make it dimensionless), which ends at a second-order critical point $(m/g)_c$ [3, 6]. Improving the precision of $(m/g)_c$ is not only a benchmark for tensor network algorithms that tackle infinite entanglement at criticality by increasing bond dimensions, but also a clean setting to test scaling in a theory with both ultraviolet (lattice spacing $a$) and infrared (finite bond dimension) cutoffs.

In these proceedings, we summarize the results of [7], where we introduced a gauge-invariant VUMPS algorithm: VUMPS [8] is applied directly to an MPS ansatz that solves Gauss's law locally [5], keeping the gauge field degrees of freedom explicit. This enables high-precision extraction of the correlation length from the MPS transfer matrix and, in turn, a sharp determination of the critical point. (See [9] for a concurrent work that reported a consistent value of $(m/g)_c$ with comparable precision.)

## 2. Continuum and lattice Schwinger models

The massive Schwinger model with the $\theta$ term is defined by the action

$$S = \int \mathrm{d}^2x \left[ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{g\theta}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu} + i\bar{\psi}(\gamma^\mu\partial_\mu + igA_\mu)\psi - m\bar{\psi}\psi \right]. \tag{1}$$

Working in temporal gauge $A_0 = 0$ gives a Hamiltonian with the Gauss law constraint $(\partial_1 F_{01} - g\bar{\psi}\gamma^0\psi)|\mathrm{phys}\rangle = 0$.

We discretize space with lattice spacing $a$ in the Kogut-Susskind Hamiltonian formulation [10]. After the standard Jordan-Wigner map to spins, the Hamiltonian up to additive constants takes the form

$$H_\theta = \sum_{n\in\mathbb{Z}} \left[ \frac{g^2a}{2}\left(L(n) + \frac{\theta}{2\pi}\right)^2 + \frac{m_{\mathrm{lat}}}{2}(-1)^n\sigma_z(n) + \frac{1}{2a}(\sigma_+(n)U(n)\sigma_-(n+1) + \mathrm{h.c.}) \right], \tag{2}$$

with integer electric flux $L(n) \in \mathbb{Z}$ and link variable $U(n)$ obeying $[L(m), U(n)] = \delta_{mn}U(n)$. Physical states satisfy Gauss's law

$$G(n) \equiv L(n) - L(n-1) - \frac{\sigma_z(n) + (-1)^n}{2}, \qquad G(n)|\mathrm{phys}\rangle = 0. \tag{3}$$

Following [11], we use the improved mass parameter

$$m \equiv m_{\mathrm{lat}} + \frac{g^2a}{8}, \tag{4}$$

which suppresses $O(a)$ effects in the approach to the continuum.

At $\theta = \pi$ the theory has a $CT$ symmetry (naive charge conjugation with a one-site translation) that can be spontaneously broken. A convenient local order parameter is the mean electric field

$$\phi \equiv \frac{1}{2}\langle L(n) + L(n+1) + 1 \rangle, \tag{5}$$

which is odd under $CT$.

## 3. Gauge-invariant uniform MPS and VUMPS

We work directly in infinite volume using a two-site translationally invariant uniform MPS (uMPS) ansatz,

$$|\Psi\rangle = \sum_{\{s_n, p_n\}} \Big(\prod_{j\in\mathbb{Z}} \boldsymbol{A}_1^{s_{2j-1}, p_{2j-1}} \boldsymbol{A}_2^{s_{2j}, p_{2j}}\Big) |\{s_n, p_n\}\rangle, \tag{6}$$

where $s_n = \pm 1$ (eigenvalue of $\sigma_z$) and $p_n \in \mathbb{Z}$ label the local matter and gauge basis states.

To enforce Gauss's law (3) *locally*, we use the gauge-invariant MPS structure of [5]: each virtual bond is decomposed into $(q, \alpha_q)$, where $q$ is the eigenvalue of the electric field $L$ and $\alpha_q \in \{1, \dots, D_q\}$ is a degeneracy index. The MPS tensors take the block-sparse form

$$(A_n^{s,p})_{(q\alpha_q; r\beta_r)} = (a_n^{s,q})_{\alpha_q, \beta_r} \delta_{p, q+(s+(-1)^n)/2} \delta_{p,r}, \quad n = 1, 2. \tag{7}$$

The total bond dimension is $D \equiv \sum_q D_q$, where $D_q$ vanishes for large enough $|q|$. In practice, contributions from sectors with large $|q|$ are strongly suppressed in the Schmidt spectrum for $0 \leq \theta < 2\pi$, allowing systematic truncation as part of the VUMPS optimization [7].

We optimize the variational parameters $a_n^{s,q}$ using VUMPS [8], which iteratively solves effective eigenvalue problems for the uMPS tensors and environments.

## 4. Transfer matrix and correlation length

Given the optimized uMPS tensors $\boldsymbol{A}^K$, the MPS transfer matrix is defined as

$$(\mathcal{T}_A)_{(\beta_1\beta_2; \gamma_1\gamma_2)} \equiv \sum_K A^K_{\beta_1\gamma_1} \bar{A}^K_{\beta_2\gamma_2} \tag{8}$$

with eigenvalues $\lambda_j$ satisfying $1 = \lambda_0 > |\lambda_1| \geq |\lambda_2| \geq \cdots$. Defining $\epsilon_j \equiv -\ln|\lambda_j|$, the leading correlation length in lattice units is

$$\xi = \frac{1}{\epsilon_1}. \tag{9}$$

A key point is that a finite bond dimension discretizes the transfer matrix spectrum and acts as an infrared cutoff. Following [12, 13], we quantify this cutoff by

$$\delta(D) \equiv \epsilon_2 - \epsilon_1, \tag{10}$$

which empirically provides a clean extrapolation parameter. For fixed $(ga, m/g)$ and large enough bond dimension, we observe an approximately linear relation

$$\epsilon_1(D) = \epsilon_{1,\infty} + c_1 \delta(D), \tag{11}$$

enabling an estimate of $\epsilon_{1,\infty}$ corresponding to the $D \to \infty$ limit at fixed lattice spacing. See Figure 1.

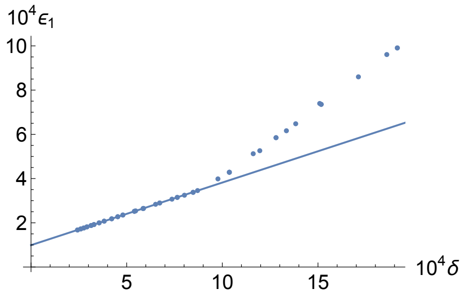


**Figure 1:** Plot of the simulation data for $(\delta = \delta(D), \epsilon_1)$ with $ga = 0.1$ and $m/g = 0.333$. For $\delta < 9 \times 10^{-4}$ $(D > 220)$, $\delta$ and $\epsilon_1$ show a linear relationship. The corresponding fitting line is displayed in the figure. For fixed $ga$ and $m/g < (m/g)_*$, the simulations with large $\delta$ (small $D$) prepare $CT$-breaking states rather than approximating the true $CT$-preserving ground state.

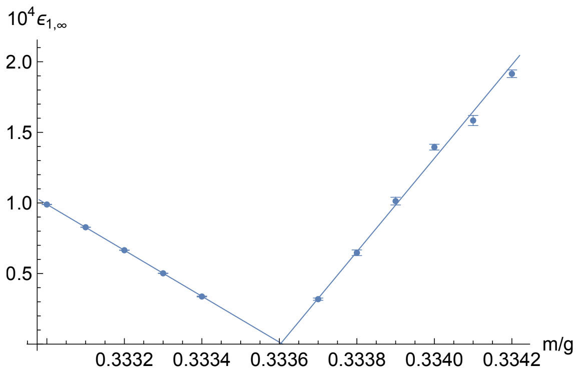


**Figure 2:** Plot of the data for $(m/g, \epsilon_{1,\infty})$ with $ga = 0.1$. The data are linearly fitted separately in the two regions $m/g < 0.3336$ and $m/g > 0.3336$ as in (12). Fitting in the former region gives higher precision because the data points there align more closely with a straight line and have smaller error bars.

## 5. Locating the critical point

For each lattice spacing (made dimensionless by multiplying with the coupling) $ga$ we compute $\epsilon_{1,\infty}$ as a function of $(m/g, ga)$ using (10). The lattice critical mass $(m/g)_*$ as a function of $ga$ is then identified by the vanishing of the inverse correlation length, $\epsilon_{1,\infty} \to 0$, which is well described by a piecewise linear behavior around the transition:

$$\epsilon_{1,\infty} \simeq \begin{cases} c_-(m_* - m)/g, & m < m_*, \\ c_+(m - m_*)/g, & m > m_*, \end{cases} \tag{12}$$

with $c_\pm > 0$. See Figure 2. Finally, we take the limit

$$(m/g)_c = \lim_{ga \to 0} (m/g)_* \tag{13}$$

by extrapolation. See Figure 3. Our best estimate is

$$(m/g)_c = 0.333556(5), \tag{14}$$

improving the long-standing benchmark value $(m/g)_c = 0.3335(2)$ from DMRG [3] by roughly two orders of magnitude.

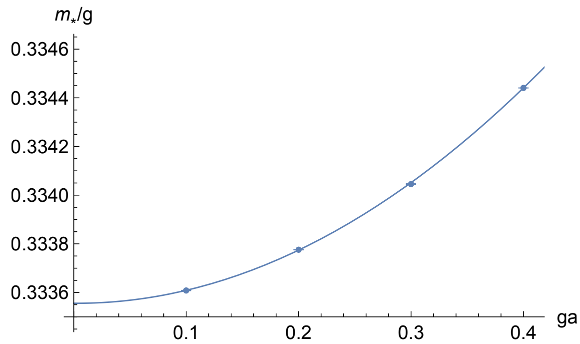


**Figure 3:** Plot of the data for $(ga, (m/g)_*)$, which are quadratically fitted to obtain the critical mass $(m/g)_c = \lim_{ga\to 0}(m/g)_*$ in the continuum limit.

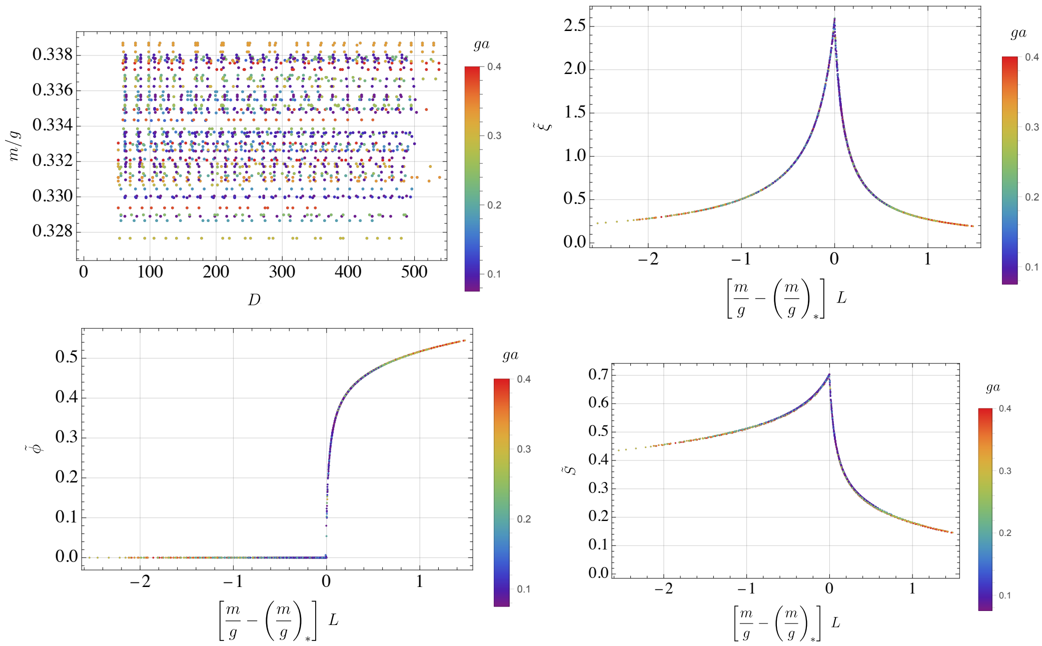


**Figure 4:** Generated data points used in the double collapse on $(m/g, D)$-plane (top left). Double collapse of the correlation length (top right), the local order parameter (bottom left) and the entanglement entropy (bottom right). The UV and IR exponents are fixed. See the main text for details.

## 6. Double data collapse and Ising universality

Beyond locating the critical point, we test the scaling of the data in the *simultaneous* limits: criticality ($m \to m_*$), continuum ($ga \to 0$), and infinite bond dimension ($\delta \to 0$). Interpreting the finite-$D$ effect as an effective system size $L = ga/\delta$ and the lattice spacing as a UV cutoff $\Lambda = 1/(ga)$, we can construct scale-invariant combinations of observables. Assuming Ising universality in the IR and a $c = 1$ CFT in the UV, we perform a data-collapse analysis for $\xi$, the order parameter $\phi$ in (5), and the entanglement entropy (Figure 4). We find an excellent collapse of randomly sampled data points in the vicinity of criticality, consistent with Ising exponents ($\Delta_t^{\mathrm{IR}} = 1$, $\Delta_\phi^{\mathrm{IR}} = 1/8$ and $c_{\mathrm{IR}} = 1/2$) and $c_{\mathrm{UV}} = 1$ [7].

## 7. Conclusion and outlook

Gauge-invariant VUMPS provides a precise and systematically controllable framework for Hamiltonian lattice gauge theories in $(1 + 1)$ dimensions. For the Schwinger model at $\theta = \pi$ we obtained the continuum critical mass (14) and confirmed Ising universality through a double data collapse. Natural next targets include multi-flavor Schwinger models and non-abelian gauge theories in $(1+1)$ dimensions, where the ability to impose Gauss's law locally while keeping gauge fields explicit may be advantageous.

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