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Understanding thermalization of the magnetic gluons in a non-Abelian gauge theory

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We explore chaotic dynamics in SU(2) gauge theory by extracting the Lyapunov exponent under both in and out-of-equilibrium conditions to understand how chaos develops and influences the approach to thermalization. In thermal equilibrium at very high temperatures, the hard, electric, and magnetic scales are well separated, and one can write an effective theory for the soft gauge fields (magnetic modes) which are classical due to their large occupation number. In the vicinity of the critical temperature T_c, $SU(2)$ gauge theory falls within the same universality class as Z_2 scalar field theory. Exploiting this correspondence, we determine the Lyapunov exponent in the scalar theory through out-of-time-ordered correlators (OTOC). To probe non-equilibrium dynamics, we prepare the gauge fields in a highly occupied, non-thermal initial state. Under classical Yang–Mills evolution, the system evolves toward a self-similar scaling regime characterized by a scale separation similar to that of the thermal case. Within this regime, we again compute the Lyapunov exponent and compare it to its equilibrium counterpart at similar energy densities. This comparison allows us to quantify the timescale over which the system approaches thermal equilibrium.

The 42nd International Symposium on Lattice Field Theory
TIFR, Mumbai, India
Nov 2-8, 2025

*Speaker

ISSN 1824-8039. Published by SISSA Medialab.

PoS(LATTICE2025)096

1. Introduction

Understanding thermalization in strongly interacting systems remains a central problem in modern physics. It is relevant for phenomena such as the dynamics of the early universe and the formation of quark–gluon plasma in heavy-ion collisions [1]. Thermalization is associated with ergodic behavior, where the system explores its accessible phase space through dynamical mixing. A useful quantity to characterize this process is the Kolmogorov–Sinai (KS) entropy rate, which is given by the sum of all positive Lyapunov exponents under certain conditions [2].

In this proceeding, we discuss the thermalization of the soft (infrared) gluons in a non-Abelian SU(2) gauge theory and estimate a characteristic equilibration time. In order to do so we use first-principles lattice simulations, to calculate the KS entropy rate in terms of the positive Lyapunov exponents in SU(2). Our analysis covers two distinct dynamical regimes: a far-from-equilibrium classical attractor configuration [3, 4], and a thermal ensemble deep in the deconfined phase at high temperature. Furthermore, we investigate the temperature dependence of the maximal Lyapunov exponent for the critical modes of SU(2) near the deconfinement transition by studying out-of-time-ordered correlators (OTOCs) in a Z_2 scalar field theory, which belongs to the same universality class as SU(2) gauge theory. The results presented in this talk are based on our recent work [5].

2. Chaos in non-Abelian gauge theory

Thermalization in isolated quantum systems is closely connected to chaos and ergodicity [6]. In a classical Hamiltonian system, signature of chaos involves exponential sensitivity to the initial conditions. This instability is quantified by a Lyapunov exponent defined as,

$$\lambda_L = \lim_{t\to\infty} \frac{1}{t} \ln \frac{d(t)}{d(0)}, \tag{1}$$

where $d(t)$ denotes the separation between trajectories at time t starting from an infinitesimally small separation $d(0)$ at $t = 0$.

In non-Abelian gauge theories at sufficiently high temperature, soft gluons with momenta $|\mathbf{p}| < g^2T/\pi$ are highly occupied and their dynamics can be described classically in terms of an effective Hamiltonian [7, 8]. The gauge links $U_\mu(\mathbf{x})$ may then be interpreted as coordinates in the classical configuration space, allowing for a direct estimation of Lyapunov exponents to gauge theories. A more general way to characterize chaotic dynamics that can be extended for quantum systems is in terms of out-of-time-ordered correlators (OTOCs) [9]. In the semiclassical limit, for **gauge fields** the OTOC is defined as

$$C(\mathbf{x}, t) = \langle \{\phi(\mathbf{x}, t), \phi(\mathbf{0}, 0)\}^2 \rangle = \left\langle \left(\frac{\delta\phi(\mathbf{x}, t)}{\delta\pi(\mathbf{0}, 0)} \right)^2 \right\rangle. \tag{2}$$

Exponential growth of $C(\mathbf{0}, t)$ implies chaos, and the maximal Lyapunov exponent can be extracted through

$$\lambda_L = \lim_{t\to\infty} \frac{1}{2t} \ln C(\mathbf{0}, t). \tag{3}$$

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OTOCs typically spread ballistically in space within the light cone. Along trajectories of fixed velocity $v = x/t$, their growth are governed by velocity-dependent Lyapunov exponents [10],

$$C(\mathbf{x}, t) \sim e^{2\lambda(v)t}, \qquad \lambda(v) = \lambda_L \left[1 - \left(\frac{v}{v_B}\right)^{\nu}\right], \tag{4}$$

where v_B is the butterfly velocity defined by $\lambda(v_B) = 0$. The spatial spread can be characterized in terms of a diffusion coefficient [11],

$$D = \frac{v_B^2}{\lambda_L}. \tag{5}$$

In the following sections, we describe how the spectrum of positive Lyapunov exponents and the diffusion coefficient are extracted in SU(2) gauge theory, both in thermal equilibrium at different temperatures and in a non-equilibrium case.

3. Discussion on the lattice techniques and algorithms

At high temperatures, SU(2) gauge theory exhibits a natural separation between the hard scale πT, electric scale gT, and the magnetic scale g^2T/π [12]. When these scales are well separated, one can integrate out the hard and electric gluon momentum modes to obtain an effective Hamiltonian describing the dynamics of soft (magnetic) modes [7]. In this effective theory, the color-electric fields $E^i_{\mathbf{x}}$ evolve according to a Langevin equation,

$$-\partial_t E^{ia}_{\mathbf{x}} + [D_j, F^{ji}(\mathbf{x})]^a = \sigma E^{ia}_{\mathbf{x}} + \zeta^{ia}_{\mathbf{x}}(t), \tag{6}$$

where $a = 1, 2, 3$ represents the color index, σ is the color conductivity and $\zeta^{ia}_{\mathbf{x}}(t)$ are stochastic color forces satisfying the fluctuation-dissipation relation. In our simulations, the lattice spacing is fixed by requiring that the energy density in this effective theory matches the Stefan-Boltzmann value at a given temperature, resulting in $Ta = (30/\pi^2)^{1/3}$. Thermalized configurations obtained at certain Langevin-times are then evolved in real time by switching off the noise and damping terms. The gauge-invariant distance $d(t)$ between infinitesimally separated configurations in real time is defined as,

$$d(t) = \frac{1}{2N_P} \sum_P \left| \mathrm{tr} U_P(t) - \mathrm{tr} U'_P(t) \right| . \tag{7}$$

The maximal Lyapunov exponent can be extracted from those distance measure which follow $d(t) \sim e^{\lambda_L t}$, [5].

A typical non-equilibrium state in SU(2) gauge theory is realized by initializing the gauge fields with a highly occupied momentum distribution for the soft modes,

$$\tilde{f}(\mathbf{p}) = g^2 f(\mathbf{p}) = n_0 \frac{Q_s}{|\mathbf{p}|} \exp\left[- \frac{|\mathbf{p}|^2}{2Q_s^2} \right], \tag{8}$$

where Q_s is the gluon-saturation scale (typically 1–2 GeV, we choose $Q_s = 1.5$ GeV for our work), g is the gauge coupling, and n_0 sets the initial infrared occupation number. Since initial gluon occupations are large, quantum fluctuations are subdominant, and the fields can be evolved classically using Hamilton's equations. The gauge links $U^i_{\mathbf{x}}(t)$ and electric fields $E^{ib}_{\mathbf{x}}(t)$ (with

$i = 1, 2, 3$ and $b = 1, 2, 3$) are evolved numerically until the system reaches a self-similar scaling regime [3, 4], where the hard, electric, and magnetic scales in this non-thermal plasma are well separated. A small perturbation $\delta n_0 = 10^{-5}$ is added to the initial occupation, and the gauge-invariant distance $d(t)$ between the perturbed and unperturbed trajectories calculated in terms of the plaquettes is measured in order to extract the Lyapunov exponent λ_L within the self-similar regime.

However as the temperature is lowered towards the deconfinement transition temperature T_c, the three scales coincide, and the above effective thermal field theory description is no longer applicable. However, SU(2) gauge theory belongs to the same universality class as the 3D Ising model [13], hence its long-wavelength critical modes can be studied using a relativistic Z_2 scalar field theory. On the lattice, the Hamiltonian describing such a system is denoted as

$$H = \sum_{\mathbf{x}} \left[\frac{\pi_{\mathbf{x}}^2}{2} - \frac{1}{2} \sum_{\mathbf{y}\sim\mathbf{x}} \phi_{\mathbf{x}}\phi_{\mathbf{y}} + \left(\frac{m^2 a^2}{2} + 3 \right) \phi_{\mathbf{x}}^2 + \frac{\lambda_s}{4!} \phi_{\mathbf{x}}^4 \right], \tag{9}$$

where $\sum_{\mathbf{y}\sim\mathbf{x}}$ runs over nearest neighbors. In our simulations, we have chosen $m^2 a^2 = -1$ and $\lambda_s = 1$. Thermal ensembles are generated using Langevin dynamics, followed by real-time classical evolution switching off the noise and damping terms as we have done for the SU(2) case. Infinitesimal perturbations are introduced as $\delta\phi_{\mathbf{x}}(0) = 0$ and $\delta\pi_{\mathbf{x}}(0) = r\delta_{\mathbf{x0}}$, where r is drawn from a Gaussian distribution with $|r| < 0.01$. These fluctuations evolve according to

$$\partial_t^2 \delta\phi_{\mathbf{x}}(t) = \nabla^2 \delta\phi_{\mathbf{x}}(t) - m^2 \delta\phi_{\mathbf{x}}(t) - \frac{\lambda_s}{2} \phi_{\mathbf{x}}^2(t) \delta\phi_{\mathbf{x}}(t), \tag{10}$$

and $\partial_t \delta\phi_{\mathbf{x}}(t) = \delta\pi_{\mathbf{x}}(t)$. Using the OTOC defined as

$$C(\mathbf{x}, t) = \langle \{\phi(\mathbf{x}, t), \phi(\mathbf{0}, 0)\}^2 \rangle = \left\langle \left(\frac{\delta\phi(\mathbf{x}, t)}{\delta\pi(\mathbf{0}, 0)} \right)^2 \right\rangle. \tag{11}$$

one can extract the maximal Lyapunov exponent for the critical mode.

4. Results

First, we have calculated the OTOC of Z_2 scalar field theory at various temperatures near the Ising transition. In the inset of Fig. 1, we have shown the exponential growth of OTOC at different temperatures from which we have extracted the maximal Lyapunov exponents. The exponent exhibits a peak at T_c, similar to behavior observed in spin systems [14]. Further, in the inset Fig 2, we show the heat-map of the OTOC at $T = T_c$ in a 2-dimensional x-t plane as a function of time t and one of the spatial coordinates x, performing a sum over the other spatial coordinates. The OTOC is observed to spread ballistically within the light cone from an initial small perturbation at the origin, evident from the color gradient of the heat-map. We have also calculated the butterfly velocity v_B by performing a fit of the OTOC $C(\mathbf{x}, t)$ to the ansatz given in Eq. 4. The values of the diffusion coefficient extracted out of v_B are shown in Fig 2. Their values remain constant below T_c but drop significantly just above T_c as a power-law as a function of the temperature.

Next, for thermal SU(2) gauge configurations deep in the deconfined phase, the maximal Lyapunov exponent λ_L was observed to grow approximately linearly with temperature in the range

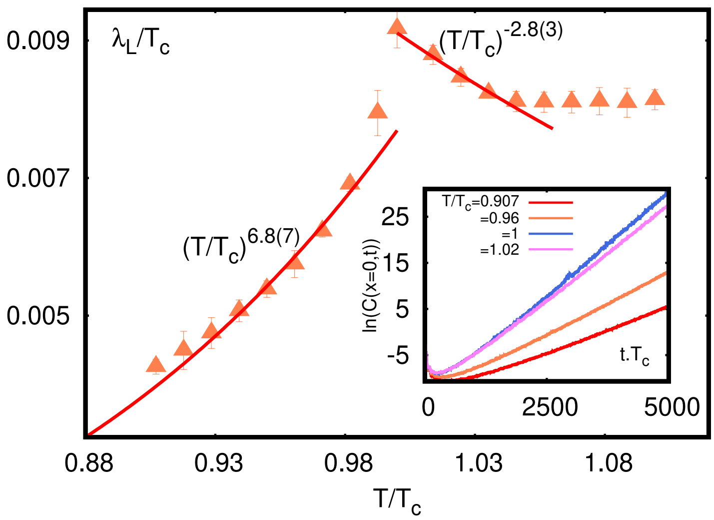

Figure 1: Maximal Lyapunov exponent versus T/T_c in the Z_2 scalar field theory, serving as an universal description of the critical long-distance dynamics of SU(2) near T_c. The figure is from Ref. [5].

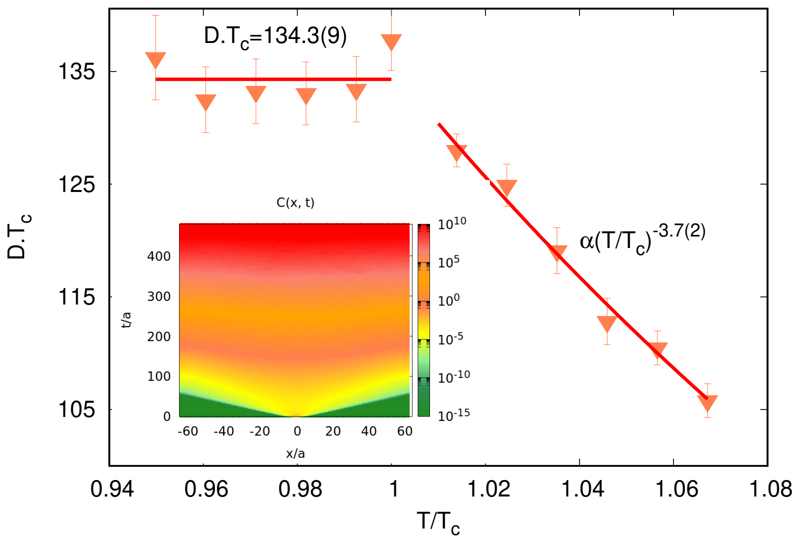

Figure 2: Diffusion coefficient of critical Z_2 modes as a function of T/T_c, serving as an universal description of the long-wavelength critical dynamics of SU(2). The figure is from Ref. [5].

0.6–3 GeV, with a slope $\lambda_L/T \sim 0.52$ as shown in Fig. 3. This is consistent with the Maldacena-Shenker-Stanford bound $\lambda_L \leq 2\pi T$ [9]. The diffusion coefficient D, extracted from the relation $D = v_B^2/\lambda_L$, decreases with temperature approximately as $(T/T_c)^{-0.95(3)}$ as shown in Fig. 4. This is consistent with perturbative expectations where the color conductivity σ grows linearly with T, while $\chi \sim T^2$ sets the relation $D = \sigma/\chi \sim 1/T$ [15–17].

The Lyapunov exponents extracted for non-thermal SU(2) states are also shown in Fig. 3 as blue points, represented in terms of an effective temperature $T_{\rm eff}$ determined from the energy

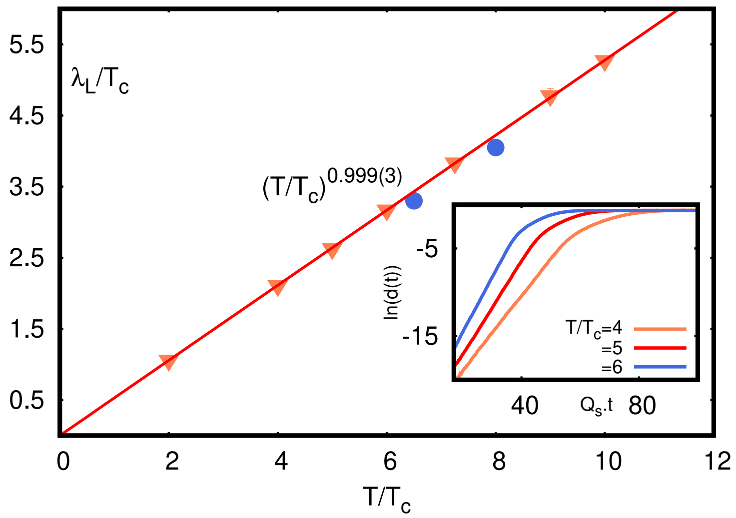

Figure 3: Temperature dependence of the maximal Lyapunov exponent, expressed as a function of T/T_c, for high-temperature thermal states in SU(2) gauge theory. The figure is from Ref. [5].

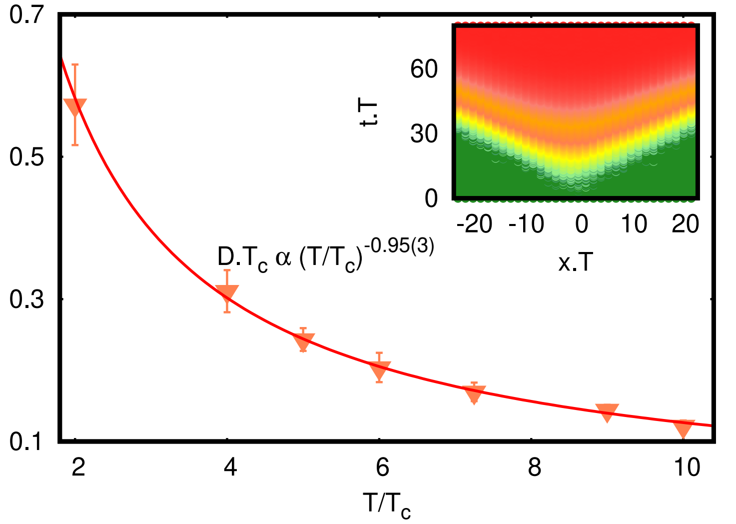

Figure 4: Temperature dependence of the diffusion coefficient, expressed as a function of T/T_c, for high-temperature thermal states in SU(2) gauge theory. The inset showns the spread of the chaotic trajectory and the butterfly velocity. The figure is from Ref. [5].

density ε, keeping the later same for the thermal as well as non-thermal states. We note here that the Lyapunov exponents of non-thermal states match closely with the thermal results at similar energy densities, demonstrating the universality of chaotic dynamics in both far-from-equilibrium and thermalized regimes. In the non-thermal self-similar regime attained with a higher initial gluon density ($\sqrt{n_0} \gtrsim 4$), the extracted λ_L was also ovserved to scale linearly with T_{eff}, similar to a thermal plasma for which the plasmon damping rate grows linearly with temperature [18].

5. Estimating the thermalization time

Next, we discuss the implications of our results for estimating the thermalization time of soft modes for non-Abelian SU(2) gluon fields. Thermalization proceeds through entropy maximization, and the entropy production rate can be quantified in terms of the Kolmogorov-Sinai (KS) entropy, defined as the sum of all positive Lyapunov exponents λ_i,

$$\dot{s}_{\text{KS}} a^4 = -\frac{\sum_i \lambda_i a}{N^3} \sim -4\lambda_L a \; . \tag{12}$$

Given the entropy density remains the same, the thermalization time t_{th} can be estimated by integrating the KS entropy rate by specifying the initial non-thermal and the final thermal states. For a thermal SU(2) state at $T = 600$ MeV, the entropy density is obtained from the thermodynamic relation $s_{\text{th}} T = \varepsilon + p$, such that $s_{\text{th}} a^3 = \frac{4}{3}\frac{\varepsilon a^4}{Ta}$. For the non-thermal self-similar gluon distribution, the entropy density was calculated numerically such that $s_{\text{non-th}} a^3 = -\int \tilde{f}(|\mathbf{p}|) \ln \tilde{f}(|\mathbf{p}|)\, d^3(a\mathbf{p}) \simeq 4.8$. Thus starting from a highly occupied non-thermal state, a thermal state at $T \simeq 600$ (450) MeV can be reached within $t_{\text{th}} \sim 0.50$ (0.7) fm/c. These temperatures correspond to typical initial fireball temperatures measured at ALICE and RHIC [19, 20], confirming a rapid thermalization scenario observed in central heavy-ion collisions.

6. Sphaleron Rate in gauge theory

Given this rapid thermalization of soft modes, it is natural to ask what happens for the ultra-soft gluons that drive topological transitions. The QCD vacua are infinitely degenerate, and each vacua are topologically distinct, characterized by an integer Chern-Simons number N_{CS}. These vacua thus cannot be smoothly deformed into one another. Quantum mechanically, tunneling between neighboring vacua occurs via instantons, while at finite temperature real-time transitions over the barrier are induced due to thermal fluctuations and are mediated by sphalerons. The sphaleron rate Γ_{sph} characterizes the rate of change of the Chern-Simons number N_{CS}, which governs anomalous processes such as axial charge production in non-Abelian plasmas. This quantity is defined in terms of the the long-time autocorrelation between the Chern-Simons number defined as

$$\Gamma_{\text{sph}} = \lim_{V\to\infty, \Delta t\to\infty} \frac{\langle (N_{\text{CS}}(t+\Delta t) - N_{\text{CS}}(t))^2 \rangle}{V \Delta t} \; , \tag{13}$$

where V is the spatial volume and Δt the evolution time. In order to extract the infrared contribution to the rate reliably, the UV fluctuations have been removed by a calibrated cooling of the gauge links along a fictitious cooling time τ denoted in terms of the following equations:

$$\partial_\tau U(t, \mathbf{x}; \tau) = -iga\tau^a E^{\mu,a}_{\text{cool}}(t, \mathbf{x}; \tau) U(t, \mathbf{x}; \tau), \quad E^{\mu,a}_{\text{cool}}(t, \mathbf{x}) = -\frac{\delta H}{\delta A^a_\mu(\mathbf{x})} \; . \tag{14}$$

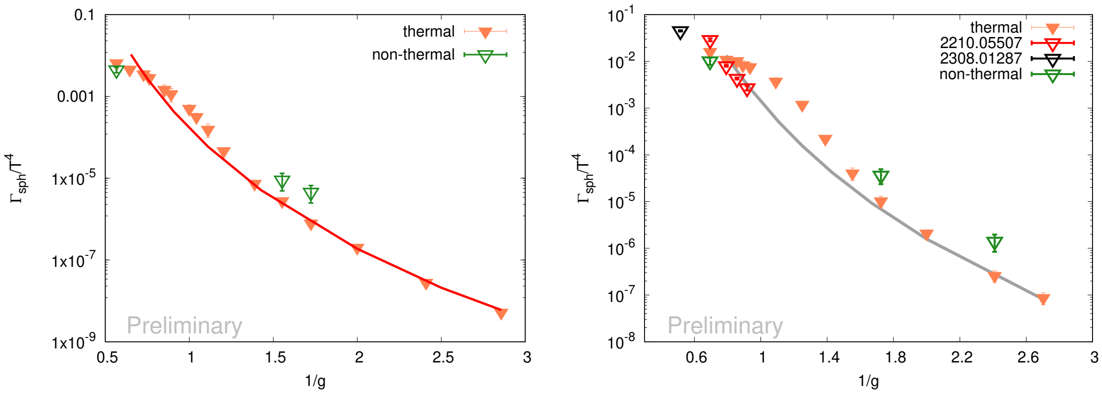

Figure 5: The dependence of the sphaleron rate with strong coupling g for SU(2) (left) and SU(3) (right) gauge theory respectively for both thermal and non-thermal plasma. The leading order perturbative prediction are shown in red lines. The sphaleron rates in SU(3) gauge theory shown as red points from Ref. [22] and in 2+1 flavor QCD from Ref. [23] are also shown in the right panel.

As shown in Fig 5, the sphaleron rate in a thermal non-Abelian plasma reproduces the perturbative scaling $\Gamma_{\mathrm{sph}} \sim \alpha_s^5 T^4$ denoted by the solid line only above the electro-weak scale. In far-from-equilibrium self-similar regime corresponding to very high effective temperatures, the sphaleron rate scales with time as $(Q_s t)^{-1.2}$, corresponding to the fourth power of the dynamically generated magnetic scale extracted from the spatial Wilson loops. This demonstrates that topological transition rate is non-perturbatively large due to the contribution from ultra-soft magnetic gluons. A comparison between the sphaleron rate in the self-similar non-equilibrium regime and in thermal equilibrium allows us to extract the characteristic equilibration time of these ultra-soft topological gluon modes. This timescale is especially relevant in reheating scenarios in the early universe, discussed in our forthcoming publication [21].

7. Conclusion

In this proceeding, we discuss the chaotic dynamics in non-Abelian gauge theories across both thermal and far-from-equilibrium regimes. By computing the gauge invariant distance measure, we have extracted the maximal Lyapunov exponent λ_L and demonstrated its linear scaling with temperature in the deconfined phase, consistent with the conjectured chaos bound. We observe that at similar energy densities, a typical non-thermal state in the self-similar scaling regime has similar chaoticity as the thermal case. Thus calculating the Kolmogorov–Sinai entropy rate, determined from the sum of positive Lyapunov exponents, we estimated a characteristic thermalization time of $t_{\mathrm{th}} \sim 0.5 fm/c$ for soft gluons at 600 MeV. Interestingly, our estimate of the thermalization time in this simple non-expanding system of overoccupied soft gluon modes is consistent with the early switch-on time for hydrodynamics necessary to describe flow observables in a heavy-ion collision event.

We have further analyzed the sphaleron rate, which quantifies topological transitions through Chern–Simons number diffusion. We showed that in thermal equilibrium at high temperature, the sphaleron rate follows the perturbative scaling $\Gamma_{\mathrm{sph}} \sim \alpha_s^5 T^4$, while in the self-similar non-thermal regime it scales with the fourth power of the dynamically generated magnetic scale. Matching the non-equilibrium sphaleron rates to its thermal values, at high temperatures, provides an independent estimate of the equilibration time for these ultra-soft gluons.

Acknowledgments

We are grateful to Sumilan Banerjee, Rajiv Gavai, Sören Schlichting and Ravi Shanker for discussions related to this work. The authors acknowledge the computing time provided by the High Performance Computing Center at the Institute of Mathematical Sciences.

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