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# Understanding properties of the Dirac eigenspectrum in QCD and approach to thermalization

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In this work, we calculate the eigenvalues of the probe (overlap) Dirac operator on thermal gauge ensembles of 2 + 1 flavor QCD generated using domain wall fermions as well as pure $SU(3)$ gauge theory on the lattice. Focusing on the infrared part of the eigenspectrum that lies within the non-perturbative magnetic scale, we propose suitable observables that allow us to categorize different regions of the eigenspectrum unambiguously. While most of these eigenmodes are completely delocalized and chaotic, i.e. their nearest-neighbor level spacing fluctuations are similar to random matrices of a Gaussian unitary ensemble (GUE), we showed that a classical non-thermal state of SU(3) gauge theory consisting of magnetic gluons is also chaotic, thus a non-trivial realization of the Bohigas-Giannoni-Schmit conjecture. This allowed us to estimate an upper bound on the thermalization time $\sim$ 1.44 fm/c of magnetic gluons by matching the magnetic scales in these two regimes. Furthermore we also observe the appearance of a few eigenmodes in deep-infrared part of the spectrum near and above the chiral crossover temperature, whose fractal dimensions might carry information about the universality class of the chiral transition in QCD.

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## 1. Introduction

The microscopic origin of thermalization in an isolated quantum system remains an open and profound question. A central challenge is to understand thermalization in terms of microscopic quantities, such as the properties of the eigenvalue spectrum. The properties of the eigenvalue spectrum of Hamiltonians describing interacting quantum systems are not very well understood. In this context, Bohigas, Giannoni, and Schmit (BGS) [1] conjectured that quantum spectral fluctuations of quantum systems whose classical counterparts exhibit chaotic dynamics can be described by one of the three Wigner Dyson universality classes of random matrix theory (RMT) subject to constraints imposed by symmetry. In classical systems, it has been shown that chaos leads to thermalization [2]. Thus eigenstates with fluctuations described by RMT may provide a route towards explaining thermalization in (strongly) interacting isolated quantum systems. Note that eigenstates of a quantum system might have even more intricate features than an RMT, through the so-called eigenstate thermalization hypothesis (ETH) [3]. Signatures of ETH have been reported in systems such as quantum billiards, spin systems with different global and local symmetries, subject to caveats.

In this work, we discuss a nontrivial realization of the BGS conjecture in a strongly interacting system consisting of non-Abelian $SU(3)$ gauge fields described by Quantum Chromodynamics (QCD). For this we study a typical classical state of QCD, described by soft gluons whose momentum modes below a certain magnetic scale are over-occupied, that exhibits chaos. We also show that in a typical thermal configuration of gluons, all the eigenvalues of the massless Dirac operator that are within the thermal magnetic scale $g^2T/\pi$ follow level-spacing ratios that are consistent with RMT belonging to the Gaussian Unitary Ensemble (GUE). Showing the validity of this conjecture, we proceed to motivate how we can estimate a thermalization time of these soft gluons. We further show the Dirac eigenvalue spectrum has additional features just above the chiral crossover transition, where the near-zero eigenmodes carry information about the criticality of the phase transition, even though the chiral condensate shows a smooth behavior. The microscopic origin of such eigenmodes and their role in the effective restoration of $U_A(1)$ is discussed in detail in our earlier works [4, 5]. This presentation is in part based on our works whose further details are in Refs. [5, 6].

## 2. Details of our lattice simulations

In order to study the 1-particle quantum states of $SU(3)$ in thermal equilibrium with or without dynamical quarks, we measure the eigenspectrum of the massless Dirac operator on different choices of gauge configurations. The gauge configurations describing QCD with two light and one heavier strange quark were generated using domain wall discretization for fermions, details of which are mentioned in Ref. [7]. The lattice spacing $a$ is set keeping the temperature fixed using $T = 1/(aN_\tau)$ for a range $T$ = 149-195 MeV. The spatial extent of the lattice are chosen to be large enough ~ 4 fm. We use the massless overlap Dirac operator [8, 9] as a probe to measure its eigenspectrum on the QCD configurations, since it has an exact chiral symmetry on the lattice. We have also generated thermal $SU(3)$ gauge configurations without dynamical quarks on a $32^3 \times 8$ lattice, but at a higher temperature $T \simeq 624$ MeV, about twice the deconfinement temperature. This was done since at high temperatures the gluon degrees of freedom are significantly larger compared to the quarks.

The details about the number of eigenvalues computed and configurations used in this study are given in Ref. [6]. To demonstrate BGS conjecture, we have generated gauge configurations with a non-thermal phase space distribution consisting of over-occupied and thus classical soft gluons without dynamical quarks. We obtain these classical gauge configurations on a three dimensional lattice through a Hamiltonian evolution in the temporal-axial gauge. For this we choose a lattice with $N = 64$ sites along each spatial direction, and the lattice spacing to be $Qa_s = 0.25$ in units of the gluon saturation scale $Q \sim 1.5$ GeV. The spatial extent in physical units is $\sim 2.1$ fm. This ensures the presence of sufficient number of gauge modes in the deep-infrared part of its momentum distribution.

## 3. Spectral properties of thermal states in QCD using HISQ Dirac operator

In our previous work [5], we have studied the properties of the eigenspectrum of the massless Highly Improved Staggered Quark (HISQ) Dirac operator on HISQ gauge ensembles just above the crossover transition. The histograms of the eigenvalue density in fine bins at three different temperatures above the chiral crossover temperature are shown in Fig. 1. A peak-like structure of near-zero eigenmodes is observed in our finest lattices, at all the temperatures we have studied. Whereas the linearly rising part of the spectrum that follows level spacing fluctuations is consistent with RMT belonging to the Gaussian Unitary ensemble (GUE) are due to the bulk eigenmodes.

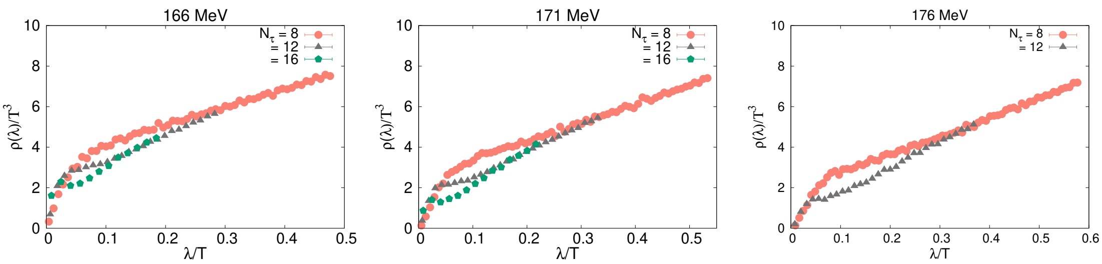


**Figure 1:** Eigenvalue spectrum for the HISQ Dirac operator at $T = 166, 171$ and $176$ MeV (left, centre and right respectively) shown for three different lattice spacings. The figure is taken from Ref. [5].

In order to study whether singlet $U_A(1)$ part of the chiral symmetry is simultaneously (effectively) restored when its $SU_A(2)$ counterpart, we study whether the integrated two point meson correlators defined as $\chi_\pi = \int d^4x \, \langle \pi^i(x)\pi^i(0) \rangle$ and $\chi_\delta = \int d^4x \, \langle \delta^i(x)\delta^i(0) \rangle$ are degenerate above the crossover temperature. The $\chi_\pi - \chi_\delta$ can further be written in terms of the eigenvalues of the massless Dirac operator. The continuum extrapolated results for $(\chi_\pi - \chi_\delta)/T^2$ for different temperatures are shown in Fig. 2. A finite value of this quantity at each of these temperatures indicates that the anomalous singlet subgroup of chiral symmetry is not effectively restored simultaneously with its non-singlet subgroup. We also observe that for our finest lattice, i.e., $N_\tau = 16$, the major contribution $\sim 99\%$ to $(\chi_\pi - \chi_\delta)/T^2$ arises due to the near-zero eigenvalues. Next, in order to estimate when does this contribution to $(\chi_\pi - \chi_\delta)/T^2$ due to near-zero eigenvalues vanishes, we fit the continuum data with an ansatz $a + b/T^2$, We have found that at $T \sim 1.15\, T_c$, this major contribution to the $U_A(1)$ breaking vanishes.

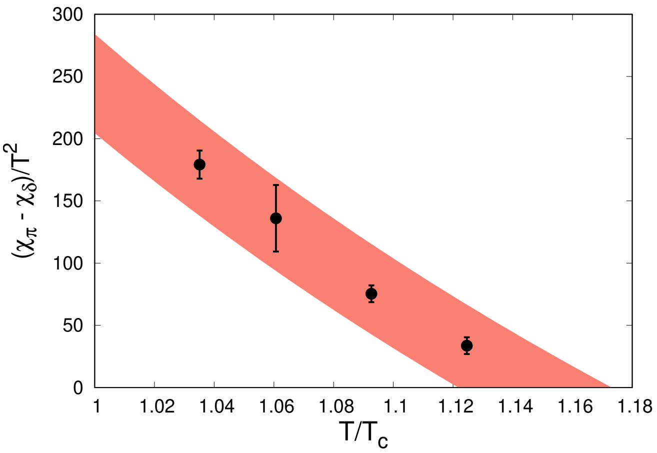


**Figure 2:** Continuum extrapolated values of $(\chi_\pi - \chi_\delta)/T^2$ as a function of temperature, shown as black points are fitted to a functional form $a + b/T^2$, shown as a band. The figure is taken from Ref. [5].

## 4. Spectral properties of thermal states in QCD using chiral fermions

To study the spectral properties of 1-particle quark states in thermal QCD, we compute two robust observables: $\langle \tilde{r} \rangle$ and $P(r)$, which are largely insensitive to systematic effects related to unfolding. The quantity $\tilde{r}_n = \min\left(r_n, \frac{1}{r_n}\right)$ [10] denotes normalized spacing ratios, where $r_n = \frac{s_{n+1}}{s_n}$ and $s_n = \lambda_{n+1} - \lambda_n$ are spacings between consecutive eigenvalues $\lambda_n$ of the massless Dirac operator. We compute the ensemble average $\langle \tilde{r} \rangle$ in bins of $\lambda/T$ to characterize different regions of the QCD Dirac spectrum. The temperature dependence of $\langle \tilde{r} \rangle$ for the intermediate and bulk eigenmodes is shown in Fig. 3. Bulk eigenmodes (red points) correspond to bins where $\langle \tilde{r} \rangle \approx 0.6026$, consistent with the prediction from a GUE. Intermediate eigenmodes (blue points) lie in a range where $\langle \tilde{r} \rangle$ has values between those of an uncorrelated ensemble (0.386) and a GUE. Our procedure is mentioned in detail in Refs. [4, 6]. We also study the probability distribution of nearest-neighbor spacing ratios [11], which is independent of unfolding and can be directly compared with the GUE prediction $P(r) = \frac{11.16(r+r^2)^2}{(1+r+r^2)^4}$. From the right panel of Fig. 3, we find that the bulk eigenmodes, identified using the above criterion, indeed follow the GUE prediction at all temperatures.

The localization properties of the Dirac eigenstates $\psi(\mathbf{x})$ in a three dimensional volume can be studied using the generalized Renyi entropies defined as,

$$R_\alpha = \frac{1}{1-\alpha} \ln \sum_{\mathbf{x}} p_{\mathbf{x}}^\alpha \ , \ 1 \le \alpha < \infty \ . \tag{1}$$

Here the probability density for an eigenstate at a spatial point $\mathbf{x}$ is $p_{\mathbf{x}} = |\psi(\mathbf{x})|^2$ after performing an averaging along the temporal direction $\tau$. From the values of the first Renyi entropy $R_1$ shown in Fig. 4, it is evident that for $SU(3)$ gauge theory without dynamical quarks at $T \sim 624$ MeV as well as in QCD at $T < T_c$, all the eigenmodes have similar values which are close to unity. This implies that all eigenvalues are almost completely delocalized over space and are ergodic, containing equivalent thermal information. This is an indirect manifestation of the ETH [12], which has been demonstrated also in non-Abelian theories in 2+1 D [13, 14]. Close to $T_c$ at $T = 177, 186, 195$

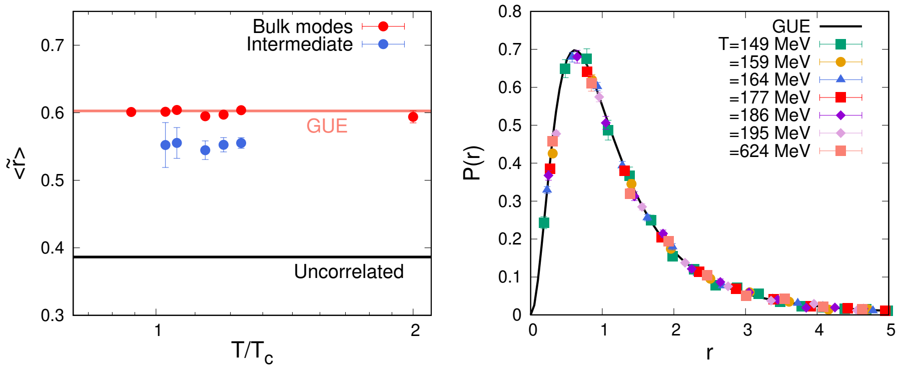


**Figure 3:** The $\langle\tilde{r}\rangle$ for different regimes of the eigenvalue spectrum of the massless Dirac operator at different $T$ and its comparison to the predictions from a GUE and an uncorrelated distribution denoted by red and black lines respectively (left panel). The probability distribution of ratios of consecutive level spacings $P(r)$ for bulk eigenmodes at different temperatures with and without dynamical quarks (right panel). The figure is taken from Ref. [6].

MeV the first Rényi entropy for intermediate eigenmodes is significantly below its values for bulk eigenmodes, signaling these eigenmodes are delocalized but not fully ergodic.

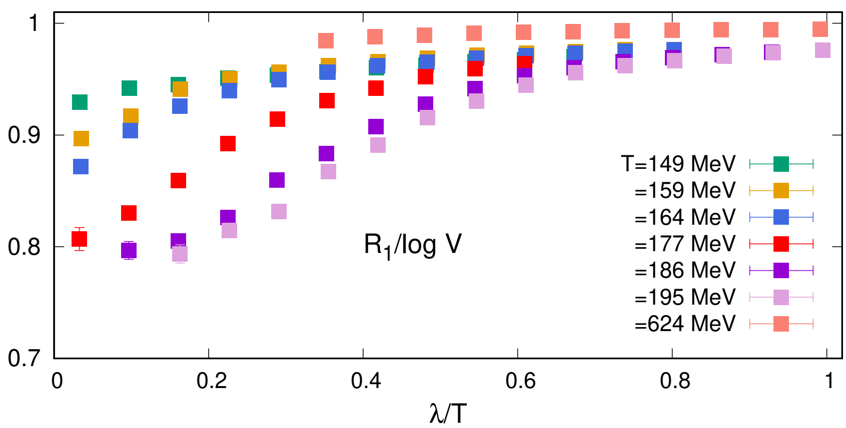


**Figure 4:** Variation of the first Renyi entropy for different regions of the Dirac eigenspectrum binned in $\lambda/T$ shown for different temperatures. The figure is taken from Ref. [6].

The fractal dimensions $D_f$ of the Dirac eigenvectors were calculated using the box-counting method [15], see Ref. [6] for details. The distributions of $D_f$ for intermediate and bulk eigenmodes are shown in Fig. 5. The median value of $D_f$ is $\sim 2.8$ for bulk and $\sim 2.5$ for intermediate eigenmodes. The latter value is consistent with the relation $D_f = 3-\beta/\nu$ [16], when we use the critical exponents $\beta, \nu$ of the three-dimensional $O(4)$ spin model [17]. These fractal-like intermediate eigenmodes thus carry information about the critical (universal) features which is probably $O(4)$ [5, 18, 19] even though chiral symmetry restoration in QCD is a smooth crossover transition. .

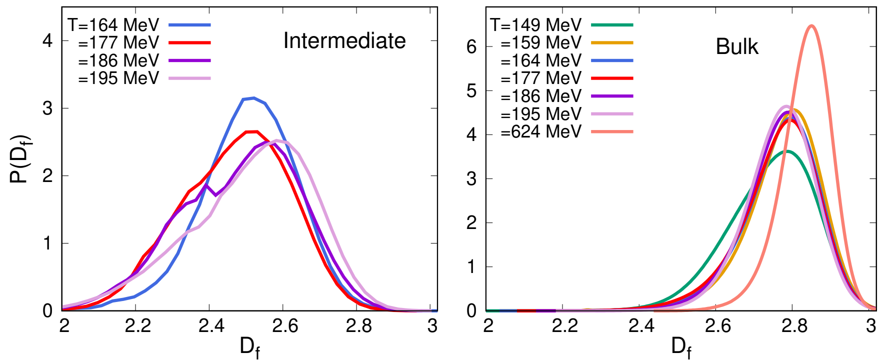


**Figure 5:** Probability distribution of the fractal dimension $D_f$ for intermediate(left) and bulk (right) eigenmodes at different temperatures, taken from Ref. [6].

## 5. Chaotic nature of a classical non-thermal state of QCD

Since the high-temperature thermal state of QCD exhibits spectral properties consistent with a RMT, we investigate whether there exists a classical state in QCD that exhibits chaos. Motivated by the Color Glass Condensate effective theory [20], where typical gluon momenta are of order the saturation scale $Q$ with $Q \gg \Lambda_{QCD}$, we sample an initial classical SU(3) gauge field configuration with a phase-space distribution $g^2 f_g(|\mathbf{p}|) = n_0 \frac{Q}{|\mathbf{p}|} e^{-\frac{|\mathbf{p}|^2}{2Q^2}}$. This non-thermal state has over-occupied infrared gluons with occupation numbers $\sim n_0/g^2$, which are non-perturbatively large and hence classical for $g \ll 1$ [21]. The gauge links $U_{i,\mathbf{x}}$ and electric fields $E^i_{a,\mathbf{x}}$ are then evolved using classical Hamilton's equations. Such a system rapidly loses memory of the initial conditions [22, 23] and enter into a self-similar scaling regime where $g^2 f_g(|\mathbf{p}|, t) = (Qt)^{-4/7} f_s \left[(Qt)^{-1/7} \frac{|\mathbf{p}|}{Q}\right]$, characteristic of a non-thermal fixed point. In this regime a separation between the different scales emerges [24], similar to high-temperature gauge theories in thermal equilibrium. How such a system thermalizes however remains an open question.

To test whether such a classical system exhibits chaos, we monitor the gauge-invariant distance [25] $D(U_l, U'_l, t) = \frac{1}{N_P} \sum_P \frac{1}{N_c} \left|\mathrm{tr} U_P - \mathrm{tr} U'_P\right|$ as a function of time, defined in terms of plaquettes $U_P$ and $U'_P$ at time $t$. At $t = 0$ one typically starts with infinitesimally close initial conditions, $n_0$ and $n_0 + \Delta n_0$ with $\Delta n_0 = 0.001$. The distance $D(t)$ for different initial values of $n_0/g^2$ are shown in the inset of Fig. 6. The $D(t)$ grows exponentially as a function of time indicating inherent chaotic nature and eventually saturates at late times due to finite gauge-space volume. Fitting the early-time behavior as $D(t) = D_0 \exp(\gamma t)$ yields a positive Lyapunov exponent $\gamma$ over a wide range of energy densities $\varepsilon/Q^4$ (Fig. 6), where $\varepsilon \propto n_0/g^2$.

## 6. Estimating an upper bound on the thermalization time for soft gluons

Having established that a particular classical state of SU(3) gauge theory is chaotic and a thermal state exists with level ratios consistent with a GUE, allows us to demonstrate a non-trivial realization of the BGS conjecture. This has profound implications for understanding thermalization of the soft magnetic gluons in a non-Abelian gauge theory. In the self-similar regime, it is

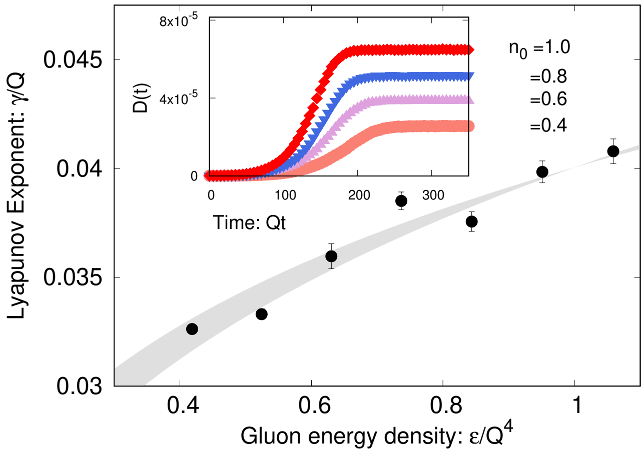


**Figure 6:** The Lyapunov exponent $\gamma$ in a $SU(3)$ plasma in the self-similar scaling regime shown as a function of energy density $\varepsilon$. The inset shows the exponential growth of the distance $D(t)$ between two classical trajectories for different initial gluon densities. Figure is taken from Ref. [6].

known that the magnetic ($\sqrt{\sigma}$), electric ($m_D$), and hard ($\Lambda$) scales evolve as $\sqrt{\sigma}(t) \sim Q(Qt)^{-3/10}$, $m_D(t) \sim Q(Qt)^{-1/7}$, and $\Lambda(t) \sim Q(Qt)^{1/7}$ [26]. Using this information we can estimate an upper bound on the thermalization time of these soft-gluons below the magnetic scale since these are over-occupied and chaotic in both classical and thermal regimes. A thermal state at a typical temperature $T \simeq 624$ MeV, is characterized by a magnetic scale $\sim 1.19T$. Matching the magnetic scales in these thermal and non-thermal states, i.e. $Q(Q\tau_{\text{th}})^{-3/10} = 1.19T$ gives us an upper bound on the thermalization time of these soft gluons, $\tau_{\text{th}} \approx 1.44$ fm/c. It has ben argued that dynamical fermions do not affect the universality of spectral properties below the magnetic scale [6], but fermion production phenomenon being intrinsically quantum, might drive the system from the non-thermal to a thermal fixed point.

## 7. Outlook

In this proceedings, we demonstrate a non-trivial realization of the BGS conjecture in a non-Abelian SU(3) gauge theory. A classical non-thermal state of SU(3) gauge theory is shown to be chaotic, while eigenmodes of the massless Dirac operator in presence of background gauge fields that represent a high-temperature thermal state at $\simeq 624$ MeV have spectral level ratios similar to that belonging to a GUE. The presence of dynamical fermions adds further interesting features to the eigenvalue spectrum near the chiral crossover. Although a smooth crossover, the fractal dimensions of the *intermediate* eigenmodes near $T_c$ has a median value that can be explained in terms of critical exponents of the $O(4)$ universality class. We also demonstrate a rapid thermalization of the non-perturbative, classical, infrared (magnetic) gluons in SU(3) gauge theory by matching the scales below which these are defined in thermal and non-thermal states respectively. Verifying this picture through quantum simulations with dynamical quarks and gauge interactions, probably in lower dimensions, would be a future direction of interest.

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